6

Fraïssé Limits by the Pound

Fraïssé sequences and their limits are universal constructions whose impact on functional analysis and Banach space theory is not yet well appreciated. There are very good expositions in which one can find the many subtleties and applications of Fraïssé constructions: an introduction to the basic algebraic theory dealing exclusively with countable structures is in Hodges' treatise [214, Chapter 7], but even Pestov [385, Section 6.5] can serve that purpose; Kubiś paper [308] develops a wide variety of examples in various areas, including universal algebra, continuum theory and general topology; Lupini's paper [342] has a more functional analysis orientation. Our rather pedestrian approach is aimed to the construction and study of two concrete examples: the p-Gurariy space G_p, a separable p-Banach space of almost universal disposition, and the p-Kadec space K_p, a separable p-Banach space of almost universal complemented disposition with a 1-FDD. In a sense, they are the same object in different categories: G_p is the Fraïssé limit in the category of finite-dimensional p-Banach spaces and isometric embeddings, and K_p is the Fraïssé limit in a related category whose morphisms are pairs of maps (a contractive embedding and a projection) between finite-dimensional p-Banach spaces whose 'separable' objects (those arising as inductive limits of sequences of finite-dimensional ones) are spaces with 1-FDD. Let us present a comparison table of their similarities and different structural properties, even if we are well aware that some entries might be unintelligible at this moment:

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G_p	K_p
AUD	AUCD
Isometrically unique	Isometrically unique
Fraïssé limit of isometries	Fraïssé limit of contractive pairs
Trivial dual if $p < 1$	Separating dual for all p
Locally injective in $p\mathbf{B}$	No
$\mathscr{L}_\infty$-space when $p = 1$	Never
BAP only if $p = 1$	1-FDD for all p
Almost isotropic for all p	Only if $p = 1$
Universal for separable p-Banach spaces	Complementably universal for separable p-Banach with BAP

6.1 Fraïssé Classes and Fraïssé Sequences

A category $\mathbf{C}$ has the amalgamation property if each diagram of the form

$$\begin{array}{ccc} & & B \\ & \nearrow & \\ A & & \\ & \searrow & \\ & & C \end{array} \qquad \text{fits into a commutative diagram} \qquad \begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ A & & & & D \\ & \searrow & & \nearrow & \\ & & C & & \end{array}$$

and has the joint embedding property if, given two objects A, B, there is $C \in \mathbf{C}$ such that both A and B have morphisms into C:

$$\begin{array}{ccc} A & & \\ & \searrow & \\ & & C \\ & \nearrow & \\ B & & \end{array}$$

An object of $\mathbf{C}$ is initial if there is a unique morphism from it to any other object in $\mathbf{C}$. Any category with an initial object I and the amalgamation property has the joint embedding property: just amalgamate the morphisms $I \longrightarrow A$ and $I \longrightarrow B$. It is clear the categories $p\mathbf{B}$ and $\mathbf{Q}$ have the joint embedding and amalgamation properties since direct sums and pushouts can be used to construct the required diagrams. Much more relevant for the purposes of this chapter is that the same is true, for each $0 < p \leq 1$, for the 'isometric' subcategory of $p\mathbf{B}$ in which arrows are isometries and for the contractive subcategory $p\mathbf{B}_1$, as Lemma 2.5.2 says. The space 0 is initial in all these categories.

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Proposition 6.1.1 *Let* **C** *be a countable category (countable objects, countable arrows) having the amalgamation and joint embedding properties. Then there is a sequence of morphisms* $u_n : C_n \longrightarrow C_{n+1}$ *such that*

(a) *if* A *is an object of* **C** *then there is* n *such that* $\operatorname{Hom}(A, C_n) \neq \emptyset$*;*

(b) *if* $v : C_n \longrightarrow A$ *is a morphism of* **C** *then there is* $m > n$ *and a morphism* $w : A \longrightarrow U_m$ *such that* $w \circ v$ *is the bonding morphism* $U_n \longrightarrow U_m$*.*

Proof Since there are only countable many morphisms in **C**, we can take a sequence (f_n, k_n) passing through all the pairs of the form (f, k), where f is a morphism of **C** and $k \in \mathbb{N}$ is a 'control number', in such a way that each (f, k) appears infinitely many times. The sequence (u_n) is constructed by induction, starting with any morphism. If **C** has an initial object, choose any morphism whose domain is the initial object to start. Having defined $u_{n-1} : U_{n-1} \longrightarrow U_n$, we take a look at (f_n, k_n), with $f_n : A \longrightarrow B$ and control number k_n. If either $k_n \geq n$ or the 'domain' of f_n (the object A) is not U_{k_n} just wait: set $U_{n+1} = U_n$ and take u_n as the identity of U_n. Otherwise, $k_n < n$ and the domain f_n is U_{k_n}. Thus we have two morphisms with domain $A = U_{k_n}$, namely the 'bonding morphism' $\iota_{(k_n,n)} : U_{k_n} \longrightarrow U_{k_n+1} \longrightarrow \cdots \longrightarrow U_n$ and f_n itself. Since **C** has the amalgamation property, these fit into a commutative diagram

$$\begin{array}{ccc} U_{k_n} = A & \xrightarrow{\quad f_n \quad} & B \\ {\scriptstyle \text{bonding morphism}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{whatever}} \\ U_n & \xrightarrow{\quad f'_n \quad} & C \end{array}$$

Then, setting $U_{n+1} = C$ and $u_n = f'_n$ completes the induction step. Let us check that the resulting sequence $(u_n)_{n \geq 1}$ has the required properties. It is clear that (a) follows from (b) and the joint embedding property, so let us prove (b). Let $f : U_n \longrightarrow A$ be a morphism. Take $m > n$ such that $(f_m, k_m) = (f, n)$. Then the $(m-1)$th morphism of the sequence $(u_n)_{n \geq 1}$ arose from the amalgamation diagram

$$\begin{array}{ccc} U_{k_m} = U_n & \xrightarrow{\quad f \quad} & A \\ {\scriptstyle \iota_{(n,m-1)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \iota_{(n,m-1)'}} \\ U_{m-1} & \xrightarrow{\quad \iota_{m-1} = f' \quad} & U_m \end{array}$$

It follows that $\iota'_{(n,m-1)} \circ f = \iota_{(n,m-1)} \circ \iota_{m-1} = \iota_{(n,m)}$ is the bonding morphism $U_n \longrightarrow U_m$. □

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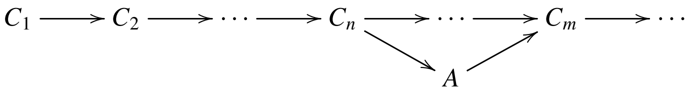
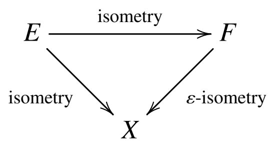
A sequence of morphisms satisfying the conditions of the proposition is called a Fraïssé sequence. The diagram



illustrates the relevant property of Fraïssé sequences.

6.2 Almost Universal Disposition

Fix $p \in (0, 1]$ once and for all. All reasoning that follows is independent of the actual value of p, but it is required that p be the same everywhere. A p-Banach space X is said to be of *almost universal disposition* (AUD) if, given finite-dimensional p-normed spaces E, F and isometries $u \colon E \longrightarrow X, v \colon E \longrightarrow F$, for each $\varepsilon > 0$, there is an ε-isometry $w \colon F \longrightarrow X$ such that $u = wv$. Diagramatically,



(6.1)

To be precise, one should speak of spaces of almost universal disposition for finite-dimensional p-Banach spaces, but let it stand. It is clear that assuming either that E is a subspace of X (and u is plain inclusion) or that E is a subspace of F (and v the inclusion) leads to equivalent formulations, a fact that will be used without further mention. The property of AUD was first considered for Banach spaces by Gurariy, who constructed the separable Banach space G that bears his name in 1966. Its general p-version is:

Theorem 6.2.1 *For each $p \in (0, 1]$ there exists a unique, up to isometries, separable p-Banach space of almost universal disposition.*

This space will be constructed, according to the general plan of the chapter, as the limit of a Fraïssé sequence of isometries between finite-dimensional spaces. It can also be constructed using the Device to obtain a 'countable and finite-dimensional' version of 2.13.1. Spaces of (almost) universal disposition will be encountered again in Section 7.3 and Note 7.5.4.

From Rational p-Norms to Allowable Isometries

We define now a countable category admitting amalgamations and whose morphisms are a family of isometries that is 'dense' among all isometries.

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A point $x \in \mathbb{K}^n$ is said to be rational if all its coordinates are rational. When $\mathbb{K} = \mathbb{C}$, this means that both the real and imaginary parts are rational numbers. A linear map $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is said to be rational if it carries rational points into rational points. A rational p-norm on $\mathbb{K}^n$ is one whose unit ball is the p-convex hull of a finite set of rational points. Thus, a rational p-norm is given by the formula

$$|x| = \inf \left\{ \left(\sum_i |\lambda_i|^p \right)^{1/p} : x = \sum_i \lambda_i x_i \right\}$$

for some finite set $x_1, \ldots, x_n$ of rational points. For each $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the set of all p-norms on $\mathbb{K}^n$, where $\mathbb{K}^0$ is understood as 0, and set $\mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}_n$. We recursively define a class of p-norms which, in the absence of an awe-inspiring name, we call 'allowed p-norms' (formally, a subset of $\mathcal{N}$), as follows:

(a) Each rational p-norm is allowed.
(b) If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is rational and injective and $| \cdot |$ is an allowed p-norm on $\mathbb{K}^m$ then $\|x\| = |f(x)|$ is an allowed p-norm on $\mathbb{K}^n$.
(c) If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is rational and surjective and $| \cdot |$ is an allowed p-norm on $\mathbb{K}^n$ then $\|y\| = \inf |x| : y = f(x)$ is allowed on $\mathbb{K}^m$.
(d) If $| \cdot |_1$ and $| \cdot |_2$ are allowed p-norms on $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively, then the p-sum $\|(x, y)\| = \left(|x|_1^p + |y|_2^p\right)^{1/p}$ is allowed on $\mathbb{K}^{n+m}$.
(e) If $| \cdot |_1$ and $| \cdot |_2$ are allowed p-norms on $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively, then the direct product $\|(x, y)\| = \max(|x|_1, |y|_2)$ is an allowed p-norm on $\mathbb{K}^{n+m}$.
(f) If $| \cdot |_1$ and $| \cdot |_2$ are allowed p-norms on $\mathbb{K}^n$ and $\mathbb{K}^m$, respectively, and $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is a rational map then the following p-norm is allowed on $\mathbb{K}^m$ for every rational number $\varepsilon > 0$:

$$\|y\| = \inf \left\{ \left(|x|_1^p + (1 + \varepsilon)^p |z|_2^p\right)^{1/p} : y = f(x) + z, x \in \mathbb{K}^n, g \in \mathbb{K}^m \right\}.$$

An allowed space is just the direct product of finitely many copies of the ground field furnished with an allowed p-norm. Finally, we declare an isometry $u : E \longrightarrow F$ allowable if E and F are allowed p-normed spaces and u is rational. Conditions (a) to (f) enable us to perform the basic categorical constructions within the allowable category, as we will see along this chapter.

Lemma 6.2.2 *There is a Fraïssé sequence of allowable isometries.*

Proof We need only check that the countable category of allowable isometries with initial object 0 admits amalgamations. The proof offers a good opportunity to review the pushout construction. Let $f : E \longrightarrow F$ and $g : E \longrightarrow G$ be allowable isometries. This means that E, F, G are $\mathbb{K}^k, \mathbb{K}^n, \mathbb{K}^m$ equipped with allowed p-norms and with both f and g rational. Condition (d) implies that $F \oplus_p G$ is an allowed space and the map $(f, -g)\colon E \longrightarrow F \oplus_p G$ is rational and injective. Let $(e_i)_{1\leq i\leq k}$ be the unit basis of E and let

$$(f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) \quad \text{and} \quad (g_1, \ldots, g_k, g_{k+1}, \ldots, g_m)$$

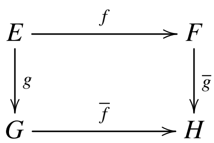
be rational bases of F and G with $f_i = f(e_i)$ and $g_i = g(e_i)$ for $1 \leq i \leq k$. Clearly,

$$(f_1 - g_1, \ldots, f_k - g_k, f_1 + g_1, \ldots, f_k + g_k f_{k+1}, \ldots, f_n, , g_{k+1}, \ldots, g_m)$$

is a rational basis of $F \oplus_p G = \mathbb{K}^{n+m}$ which we relabel as $(v_1, \ldots, v_k, \ldots, v_{n+m})$. We define a rational map $h\colon F \oplus_p G \longrightarrow \mathbb{K}^{n+m-k}$ by $h\left(\sum_{1\leq i\leq n+m} c_i v_i\right) = (c_{k+1}, \ldots, c_{n+m})$. Let H be $\mathbb{K}^{n+m-k}$ equipped with the p-norm

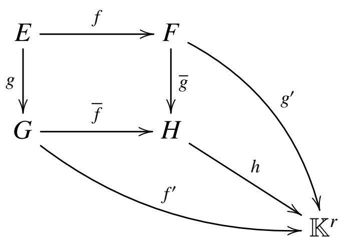
$$\|x\| = \inf\left\{\|y\|\colon\ x = h(y), y \in F \oplus_p G\right\},$$

which is allowed by (c). One has the commutative diagram



with $\overline{f}$ and $\overline{g}$ allowable since they are the inclusions of G and F into $F \oplus_p G$ followed by h. □

The isometric pushout diagram just constructed has the following additional property: for every pair of rational maps $g'\colon F \longrightarrow \mathbb{K}^r$ and $f'\colon G \longrightarrow \mathbb{K}^r$ such that $g'f = f'g$, there is a unique rational map $h\colon H \longrightarrow \mathbb{K}^r$ such that $h\overline{g} = h\overline{f}$:



Thus, the allowable category has both amalgamations and pushouts.

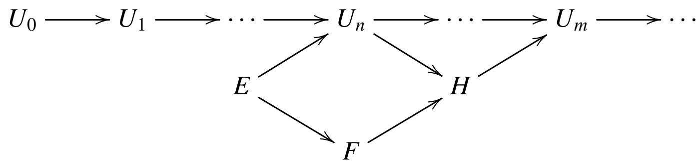
Proof of Theorem 6.2.1: Existence

Let us fix a Fraïssé sequence of allowable isometries

$$U_1 \longrightarrow U_2 \longrightarrow \cdots \longrightarrow U_n \longrightarrow U_{n+1} \longrightarrow \cdots \tag{6.2}$$

and prove that the direct limit U of that sequence in $p\mathbf{B}$ is a space of AUD. We can identify each U_k with its image in U so that one can assume $U = \overline{\bigcup_{k\geq 1} U_k}$.

To understand why the Fraïssé character of the sequence (6.2) entails the AUD of its limit, pick an isometry $E \longrightarrow F$ between finite-dimensional spaces and an isometry $E \longrightarrow U$. Assume first that we have been so lucky that $v\colon E \longrightarrow F$ is allowable and $E \longrightarrow U$ is the composition of an allowable isometry $u\colon E \longrightarrow U_n$ and the inclusion $U_n \longrightarrow U$. Since (6.2) is Fraïssé, amalgamating u and v within the allowable category yields a commutative diagram



so that the required extension of u is even an isometry in this case. Before passing to the general case, let us perform a couple of mathematical asanas to gain some flexibility. The first one is just to relax the commutativity of Diagram 6.1:

Lemma 6.2.3 *Let E be a finite-dimensional subspace of a p-Banach space X, and let F be a finite-dimensional p-Banach space. Assume that for every $\varepsilon > 0$ and every isometry $v\colon E \longrightarrow F$, there is an ε-isometry $w\colon F \longrightarrow X$ such that $\|w(v(x)) - x\| \le \varepsilon\|x\|$ for all $x \in X$. Then X is of almost universal disposition.*

Proof This obviously follows from the fact that if $\mathscr{H}$ is a basis of E then for every $\varepsilon > 0$ there is δ (depending on ε and $\mathscr{H}$) such that if $u\colon E \longrightarrow X$ is a linear map with $\|u(b)\| \le \delta$ for every $b \in \mathscr{H}$ then $\|u\| \le \varepsilon$. □

The second one is to open the no-brainer chakra: allowable isometries are 'dense' among all isometries between finite-dimensional spaces.

Lemma 6.2.4 *Let $u\colon E \longrightarrow F$ be an isometry where E is an allowed space and F is a finite-dimensional p-normed space. For each $\varepsilon > 0$, there is an allowable isometry $u_0\colon E \longrightarrow F_0$ and a surjective ε-isometry $g\colon F \longrightarrow F_0$ such that $u_0 = g\,u$.*

Proof We may assume that ε is rational. Let $(e_i)_{1\le i\le n}$ be the unit basis of $E = \mathbb{K}^n$ and pick $(f_j)_{1\le j\le m}$ such that $\{u(e_1), \dots, u(e_n), f_1, \dots, f_m\}$ is a basis of F. Let $g\colon F \longrightarrow \mathbb{K}^{n+m}$ be the isomorphism associated to that basis and take a rational p-norm $|\cdot|_0$ on $\mathbb{K}^{n+m}$ making g an ε-isometry such that $(1+\varepsilon)^{-1}\|y\| \le |g(y)|_0 \le (1+\varepsilon)\|y\|$. Then $u_0 = g\,u$ is a rational ε-isometry: in fact, $u_0(x) = (x, 0)$. We define a new p-norm on $\mathbb{K}^{n+m}$ by the formula

$$|y| = \inf\left\{\left(\|x\|^p + (1+\varepsilon)^p|z|_0^p\right)^{1/p} : y = u_0(x) + z, x \in \mathbb{K}^n, z \in \mathbb{K}^{n+m}\right\}.$$

Note that the unit ball of $|\cdot|$ is just the p-convex hull of the union of the unit ball of $\|\cdot\|$ and the ball of radius $(1+\varepsilon)^{-1}$ of $\|\cdot\|_0$. This p-norm satisfies the

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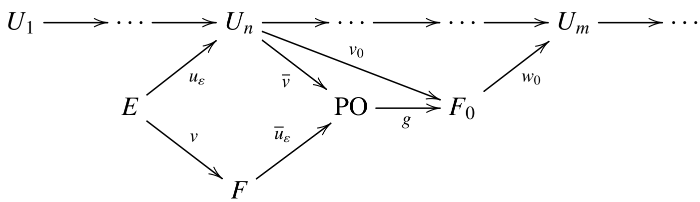
estimate $(1+\varepsilon)^{-1}|y|_0 \le |y| \le (1+\varepsilon)|y|_0$ for $y \in \mathbb{K}^{n+m}$, has to be allowed on $\mathbb{K}^{n+m}$ (by the last allowance rule) and makes u_0 into an isometry, which is therefore allowable. Hence, if F_0 is $\mathbb{K}^{n+m}$ equipped with $|\cdot|$, we have $(1+\varepsilon)^{-2}\|y\| \le |g(y)| \le (1+\varepsilon)^2\|y\|$ for $y \in F$. □

We are now ready to handle the general case and show that U satisfies the hypothesis of Lemma 6.2.3. Let F be a finite-dimensional p-Banach space, $v\colon E \longrightarrow F$ an isometry and E a subspace of U. Fix $\varepsilon > 0$. Since $\bigcup_k U_k$ is dense in U, there is a contractive ε-isometry $u_\varepsilon\colon E \longrightarrow U_n$, for n sufficiently large, such that $\|u_\varepsilon(x) - x\| \le \varepsilon\|x\|$ for all $x \in E$. Form the pushout in $p\mathbf{B}$,

$$\begin{array}{ccc} E & \xrightarrow{u_\varepsilon} & U_n \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \overline{v}} \\ F & \xrightarrow{\overline{u}_\varepsilon} & \mathrm{PO} \end{array}$$

so that $\overline{u}_\varepsilon$ is again a contractive ε-isometry, while $\overline{v}$ is an isometry to which Lemma 6.2.4 can be applied to find an allowable isometry $v_0\colon U_n \longrightarrow F_0$ together with a surjective ε-isometry $g\colon \mathrm{PO} \longrightarrow F_0$ such that $v_0 = g\overline{v}$. Finally, the Fraïssé character of (6.2) guarantees that for some $m > n$, there is an allowable $w_0\colon F_0 \longrightarrow U_m$ such that $w_0 v_0$ is the inclusion of U_n into U_m. The full picture appears in the commutative diagram



By letting $w = w_0 g \overline{u}_\varepsilon$, we obtain a contractive 3ε-isometry extending u_ε and so $\|w(x) - x\| \le \varepsilon\|x\|$ for $x \in E$.

Proof of Theorem 6.2.1: Uniqueness

Is it not almost obvious that any two separable p-Banach spaces of almost universal disposition are almost isometric? That is, that for each $\varepsilon > 0$, there is a surjective ε-isometry between them. Proposition 6.2.10 provides an explicit proof, just in case it is not clear. Much more surprising is that they are actually isometric, which we are going to prove now. Our approach to isometric properties of spaces of AUD depends one way or another on the following pair of lemmas:

Lemma 6.2.5 *Fix $\varepsilon \in (0,1)$. Let X and Y be p-normed spaces and let $\imath\colon X \longrightarrow X \oplus Y$ and $\jmath\colon Y \longrightarrow X \oplus Y$ be the canonical inclusions. If $f\colon X \longrightarrow Y$ is an ε-isometry then there is a p-norm on $X \oplus Y$ for which $\imath$ and $\imath$ are isometries such that $\|\jmath f - \imath\| \le \varepsilon$.*

Proof The p-norm that does the trick is

$$\|(x,y)\| = \inf\left\{\left(\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p\right)^{1/p} : (x,y) = \left(x_0 + x_2, y_1 - f(x_2)\right)\right\}.$$

We must check that $\|(x,0)\| = \|x\|_X$ for all $x \in X$. The inequality $\|(x,0)\| \le \|x\|_X$ is obvious. For the converse, suppose $x = x_0 + x_2$ and $y_1 = f(x_2)$. Then

$$\begin{aligned}
\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p &= \|x_0\|_X^p + \|f(x_2)\|_Y^p + \varepsilon^p\|x_2\|_X^p \\
&\ge \|x_0\|_X^p + (1-\varepsilon)^p\|x_2\|_X^p + \varepsilon^p\|x_2\|_X^p \\
&= \|x_0\|_X^p + \|(1-\varepsilon)x_2\|_X^p + \|\varepsilon x_2\|_X^p \\
&\ge \|x\|_X^p.
\end{aligned}$$

Next we prove that $\|(0,y)\| = \|y\|_Y$ for every $y \in Y$. That $\|(0,y)\| \le \|y\|_Y$ is again obvious. To prove the reversed inequality, assume $x_0 + x_2 = 0$ and $y = y_1 - f(x_2)$. As $t \to t^p$ is subadditive on $\mathbb{R}_+$ for $p \in (0,1]$, we have

$$\begin{aligned}
\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p &= \|x_2\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p \\
&= \|y_1\|_Y^p + (1+\varepsilon^p)\|x_2\|_X^p \\
&\ge \|y_1\|_Y^p + (1+\varepsilon)^p\|x_2\|_X^p \\
&\ge \|y_1\|_Y^p + \|f(x_2)\|_Y^p \\
&\ge \|y\|_Y^p.
\end{aligned}$$

Finally, $\|\jmath f - \imath\| = \sup_{\|x\|\le 1}\|\jmath(f(x)) - \imath(x)\| = \sup_{\|x\|\le 1}\|(-x, f(x))\| \le \varepsilon$. □

The indulgent reader will forgive us if, for the remainder of the chapter, we use the notation $X \boxplus_f^\varepsilon Y$ for the space $X \oplus Y$ endowed with the quasinorm defined in the preceding proof. This quasinorm depends on f and ε and also on p, but this should cause no confusion. A linear operator $f\colon X \longrightarrow Y$ will be called a strict ε-isometry if $(1+\varepsilon)^{-1}\|x\|_X < \|f(x)\|_Y < (1+\varepsilon)\|x\|_X$ for $0 < \varepsilon < 1$ and every non-zero $x \in X$. When X is finite-dimensional, every strict ε-isometry is an η-isometry for some $\eta < \varepsilon$.

Lemma 6.2.6 *Let U be a p-Banach space of almost universal disposition. Let Y be a finite-dimensional p-Banach space, X a subspace of U and $\varepsilon \in (0,1)$. If $f\colon X \longrightarrow Y$ is a strict ε-isometry then for each $\delta > 0$, there exists a δ-isometry $g\colon Y \longrightarrow U$ such that $\|g(f(x)) - x\| < \varepsilon\|x\|$ for every non-zero $x \in X$.*

Proof Choose $0 < \eta < \varepsilon$ for which f is an η-isometry. Reducing δ if necessary, we may assume that $\delta^p + (1+\delta)^p\eta^p < \varepsilon^p$. Form the space $X \boxplus_f^\eta Y$ and let $\imath\colon X \longrightarrow X \boxplus_f^\eta Y$ and $\jmath\colon Y \longrightarrow X \boxplus_f^\eta Y$ denote the canonical inclusions so that $\|\jmath f - \imath\| \le \eta$. If $h\colon X \boxplus_f^\eta Y \longrightarrow U$ is a δ-isometry such that $\|h(\imath(x)) - x\| \le \delta\|x\|$ for $x \in X$ then $g = h\jmath$ is a δ-isometry from Y into U and

$$\begin{aligned}\|x - g(f(x))\|^p &\le \|x - h(\imath(x))\|^p + \|h(\imath(x)) - h(\jmath(f(x))\|^p \\ &\le \delta^p\|x\|^p + (1+\delta)^p\|\imath(x) - \jmath(f(x))\|^p \\ &\le (\delta^p + (1+\delta)^p\eta^p)\|x\|^p < \varepsilon^p\|x\|^p.\end{aligned}$$ □

We need a technique to 'paste' operators defined on a chain of subspaces. Let A and B be p-Banach spaces and (A_n) a chain of subspaces whose union is dense in A. Let $a_n\colon A_n \longrightarrow B$ be a sequence of operators such that $\|a_{n+1}|_{A_n} - a_n\| \le \varepsilon_n$, where $\sum_n \varepsilon_n^p < \infty$, with $\sup_n \|a_n\| < \infty$. For each $x \in A_k$, the Cauchy sequence $(a_n(x))_{n\ge k}$ converges in B so there is a unique operator $a\colon A \longrightarrow B$ such that $a(x) = \lim_{n\ge k} a_n(x)$ whenever x is in some A_k. This operator shall be referred to as the *pointwise limit* of the sequence (a_n). The following remarkable result completes the proof of Theorem 6.2.1.

Proposition 6.2.7 *Fix $\varepsilon \in (0,1)$. Let U, V be separable p-Banach spaces of almost universal disposition, and let X be a finite-dimensional subspace of U. If $f\colon X \longrightarrow V$ is a strict ε-isometry then there exists a bijective isometry $h\colon U \longrightarrow V$ such that $\|h(x) - f(x)\| \le \varepsilon\|x\|$ for every $x \in X$. In particular, U and V are isometric.*

Proof Fix $0 < \varepsilon_0 < \varepsilon$ such that f is an ε_0-isometry. Let $(\varepsilon_n)_{n\ge1}$ be any decreasing sequence of positive numbers with $\varepsilon_1 < \varepsilon_0$. We inductively define sequences of linear operators $(f_n), (g_n)$ and finite-dimensional subspaces (X_n), (Y_n) of U and V, respectively, such that the following conditions are satisfied for every $n \ge 0$:

(0) $X_0 = X$, $Y_0 = f[X]$, and $f_0 = f$;
(1) $f_n\colon X_n \longrightarrow Y_n$ is an ε_n-isometry;
(2) $g_n\colon Y_n \longrightarrow X_{n+1}$ is an ε_{n+1}-isometry;
(3) $\|g_n f_n(x) - x\| < \varepsilon_n\|x\|$ for every non-zero $x \in X_n$;
(4) $\|f_{n+1}g_n(y) - y\| < \varepsilon_{n+1}\|y\|$ for every non-zero $y \in Y_n$;
(5) $X_n \subset X_{n+1}$, $Y_n \subset Y_{n+1}$, $\bigcup_n X_n$ and $\bigcup_n Y_n$ are dense in U and V, respectively.

We use (0) to start the inductive construction. Suppose that f_i, X_i, Y_i, for $i \le n$, and g_i for $i < n$, have been constructed. Applying Lemma 6.2.6 twice, we find g_n, X_{n+1}, f_{n+1} and Y_{n+1}. To guarantee that (5) holds, we may start by choosing sequences (x_n) and (y_n) dense in U and V, respectively, and then require first that X_{n+1} contain both x_n and $g_n[Y_n]$ and then that Y_{n+1} contain both y_n and $f_{n+1}[X_{n+1}]$. After that, fix $n \geq 0$ and $x \in X_n$ with $\|x\| = 1$. Using (4) and (1), we get $\|f_{n+1}g_nf_n(x) - f_n(x)\| < \varepsilon_{n+1}\|f_n(x)\| \leq \varepsilon_{n+1}(1 + \varepsilon_n)$, while (3) and (2) yield $\|f_{n+1}g_nf_n(x) - f_{n+1}(x)\| \leq \|f_{n+1}\|\,\|g_nf_n(x) - x\| < (1 + \varepsilon_{n+1})\varepsilon_n$ Combining,

$$\begin{aligned}\|f_n(x) - f_{n+1}(x)\|^p &\leq \|f_{n+1}g_nf_n(x) - f_n(x)\|^p + \|f_{n+1}g_nf_n(x) - f_{n+1}(x)\|^p \\ &\leq \varepsilon_{n+1}^p(1 + \varepsilon_n)^p + (1 + \varepsilon_{n+1})^p\varepsilon_n^p. \end{aligned} \tag{6.3}$$

If we agree that $(\varepsilon_n)_{n\geq 1}$ was chosen so that

$$\sum_{n\geq 0}\left(\varepsilon_{n+1}^p(1 + \varepsilon_n)^p + (1 + \varepsilon_{n+1})^p\varepsilon_n^p\right) < \varepsilon^p, \tag{6.4}$$

then $(f_m(x))_{m\geq n}$ is a Cauchy sequence. We define $h(x) = \lim_{m\geq n} f_m(x)$ for $x \in \bigcup_n X_n$. This h is an isometry since it is an ε_n-isometry for every n. Consequently, it extends to an isometry $U \longrightarrow V$, which we do not relabel. Furthermore, (6.3) and (6.4) imply $\|f(x) - h(x)\|^p \leq \sum_{n=0}^{\infty} \|f_n(x) - f_{n+1}(x)\|^p \leq \varepsilon^p\|x\|^p$ for $x \in X$. It remains to see that h is a bijection. To this end, we check as before that $(g_n(y))_{n\geq m}$ is a Cauchy sequence for every $y \in Y_m$. Once this is done, we obtain an isometry $g\colon V \longrightarrow U$. Conditions (3) and (4) inform us that $gh = \mathbf{1}_U$ and $hg = \mathbf{1}_V$. □

Let us denote (the isometric type of) this unique space G_p and call it the p-Gurariy space; when $p = 1$, we obtain the original Gurariy space, denoted G. Proposition 6.2.7 establishes that the spaces G_p are *almost isotropic*, in the sense that given $x, y \in \mathsf{G}_p$ with $\|x\| = \|y\| = 1$ and $\varepsilon > 0$, there is a bijective isometry f of G_p such that $\|y - f(x)\| \leq \varepsilon$. The next section uncovers some additional properties that G_p shares with all spaces of AUD.

Extension of Operators and Automorphisms

The second lesson we will learn in the forthcoming Section 7.1 is that extending operators to operators does not mean extending isomorphisms to isomorphisms. Even so, the first lesson is that extending isometries means extending operators. Thus, the AUD notion, which is more demanding than local injectivity or the forthcoming UFO (Definition 7.1.3), imposes severe restrictions on the extension of both operators and automorphisms.

Proposition 6.2.8 *Every p-Banach space U of AUD:*

(a) *is locally* 1^+*-injective; for* $p = 1$*, this means that it is a Lindenstrauss space,*
(b) *contains an isometric copy of each separable p-Banach space.*
(c) *Moreover, if* $p < q \leq 1$ *then* $\mathfrak{L}(U, Y) = 0$ *for all q-Banach spaces Y; in particular, U has trivial dual.*

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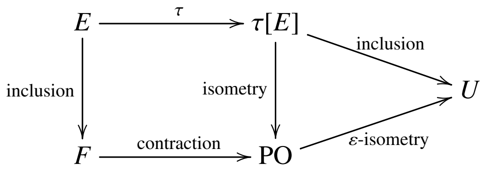
Proof Part (a) is a dirty pushout trick. Assume $\tau\colon E \longrightarrow U$ is contractive and that U is of almost universal disposition. Look at the diagram



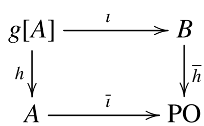
and draw your own conclusions. Part (b) can be derived by iteratively applying Proposition 6.2.6: let X be a separable p-Banach space, and let $(X_n)_{n\geq 1}$ be a chain of finite-dimensional subspaces whose union is dense in X. Then there is a sequence $f_n\colon X_n \longrightarrow U$ in which f_n is a strict 2^{-n}-isometry such that $\|f_{n+1}|_{X_n} - f_n\| < 2^{-n}$. The pointwise limit of these operators is an isometry of X into U. To prove (c), we first prove that, given a normalised $x \in \mathsf{G}_p$ and $\varepsilon > 0$, there are $x'_\varepsilon, x''_\varepsilon \in \mathsf{G}_p$ such that $x = x'_\varepsilon + x''_\varepsilon$ with $\|x'_\varepsilon\|, \|x''_\varepsilon\| \leq (1+\varepsilon)2^{-1/p}$. Indeed, consider the isometry $u\colon [x] \longrightarrow \mathsf{G}_p$ given by plain inclusion and the isometry $v\colon [x] \longrightarrow \ell_p^2$ given by $v(x) = 2^{-1/p}(1,1)$. Let $w\colon \ell_p^2 \longrightarrow \mathsf{G}_p$ be any ε-isometry extending u, and set $x'_\varepsilon = 2^{-1/p}w(1,0)$ and $x''_\varepsilon = 2^{-1/p}w(0,1)$. That done, the proof goes as in the L_p case in 1.1.5: if $u\colon \mathsf{G}_p \longrightarrow Y$ is an operator and $\|x\| = 1$, then taking $\varepsilon > 0$ and $x'_\varepsilon, x''_\varepsilon \in \mathsf{G}_p$ as before, we have

$$\|ux\| \leq \left(\|ux'_\varepsilon\|^q + \|ux''_\varepsilon\|^q\right)^{1/q} \leq (1+\varepsilon)2^{1-q/p}\|u\|.$$

Since x and ε are arbitrary, $\|u\| \leq 2^{1-q/p}\|u\|$, which is only possible if $u = 0$. □

Lemma 6.2.9 *Let A be a finite-dimensional subspace of a space U of AUD and let B be finite-dimensional. If $g\colon A \longrightarrow B$ is an embedding then for each $\varepsilon > 0$, there is an embedding $f\colon B \longrightarrow U$ such that $f(g(a)) = a$ for every $a \in A$ with $\|f\| \leq (1+\varepsilon)\|g^{-1}\|$ and $\|f^{-1}\| \leq (1+\varepsilon)\|g\|$.*

Proof We use an even dirtier trick than before. In less than no time, the reader will realise that one can assume $\|g^{-1}\| = 1$. To ease notation, we will write $h = g^{-1}$. Let us take the pushout with the inclusion $g[A] \longrightarrow B$ as follows:



It is clear that $\imath$ is an isometry and $\|h^{-1}\| = \|g\| \geq 1$. By Lemma 2.5.2, $\bar\imath$ is an isometry and $\overline{h}$ is an embedding with $\|\overline{h}\| \leq 1$ and $\|(\overline{h})^{-1}\| \leq \|h^{-1}\| = \|g\|$. Now let $w\colon \mathrm{PO} \longrightarrow U$ be an ε-isometry such that $w\bar\imath(a) = a$ for all $a \in A$. Then $f = w\overline{h}$ is an embedding which obviously satisfies $fg(a) = a$, for all $a \in A$, and $\|f\| \leq (1+\varepsilon)$. Moreover, $\|f^{-1}\| \leq \|(\overline{h})^{-1}\|\|w^{-1}\| \leq \|g\|(1+\varepsilon)$. □

Proposition 6.2.10 *Let U and V be separable spaces of AUD. Let $A \subset U$ and $B \subset V$ be finite-dimensional subspaces. If $\varphi_0 \colon A \longrightarrow B$ is an isomorphism then, for each $\varepsilon > 0$, there is an isomorphism $\varphi \colon U \longrightarrow V$ extending φ_0 and such that $\|\varphi\| \leq (1+\varepsilon)\|\varphi_0\|$ and $\|\varphi^{-1}\| \leq (1+\varepsilon)\|\varphi_0^{-1}\|$.*

Proof The result follows from Lemma 6.2.9 and a simple back-and-forth argument. Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive numbers such that $\prod_n (1+\varepsilon_n) \leq 1+\varepsilon$, and write $U = \overline{\bigcup_n U_n}$, where (U_n) is an increasing sequence of finite-dimensional subspaces of U beginning with $U_0 = A$. Moreover, let (V_n) be an increasing sequence of finite-dimensional subspaces of V such that $V = \overline{\bigcup_n V_n}$, with $V_0 = B$. Let $\varphi_0 \colon A \longrightarrow B$ be an isomorphism. By Lemma 6.2.9, let $\psi_1 \colon V_1 \longrightarrow U$ be an extension of $\varphi_0^{-1} \colon \varphi_0[U_0] \longrightarrow U$, with $\|\psi_1\| \leq (1+\varepsilon_1)\|\varphi_0^{-1}\|$ and $\|\psi_1^{-1}\| \leq (1+\varepsilon_1)\|\varphi_0\|$. Then let $\varphi_2 \colon \psi_1[V_1] + U_2 \longrightarrow V$ be an extension of $\psi_1^{-1} \colon \psi_1[V_1] \longrightarrow V$ such that $\|\varphi_2\| \leq (1+\varepsilon_2)\|\psi_1^{-1}\|$ and $\|\varphi_2^{-1}\| \leq (1+\varepsilon_2)\|\psi_1\|$ provided by Lemma 6.2.9. Continuing in this way, one obtains a pair of operators φ, ψ such that $\psi\varphi = \mathbf{1}_U, \varphi\psi = \mathbf{1}_V$, with $\|\varphi\| \leq (1+\varepsilon)\|\varphi_0\|$ and $\|\psi\| \leq (1+\varepsilon)\|\varphi_0^{-1}\|$ and $\varphi|_A = \varphi_0$. □

6.3 Almost Universal Complemented Disposition

The following notion is a kind of almost universal disposition focused only on 1-complemented subspaces; another possibility, considered in [116], is to additionally require that the projections be, in some sense, 'compatible'.

[Ə] If F is a finite-dimensional p-normed space, E is a 1-complemented subspace of F and $u \colon E \longrightarrow X$ is an isometry with 1-complemented range, then for every $\varepsilon > 0$, there is an ε-isometry $F \longrightarrow X$ with $(1+\varepsilon)$-complemented range extending u.

To properly frame it, we will consider the structure of embedding and projection as a whole.

Categories of Pairs

We will find it convenient to use the notation $u \colon E \rightleftarrows F$ for pairs $u = \langle u^\flat, u^\sharp \rangle$ consisting of operators $u^\flat \colon E \longrightarrow F$ and $u^\sharp \colon F \longrightarrow E$ such that $u^\sharp u^\flat = \mathbf{1}_E$. Thus, $u^\flat$ is an embedding of E into F and $u^\sharp$ is a projection along $u^\flat$. It is to be understood that the 'solid' arrow represents the embedding part $u^\flat$ and the 'dotted' arrow is the projection part $u^\sharp$, so that the space E is the 'domain' of u and F is the 'codomain'. Our explanation for this musical

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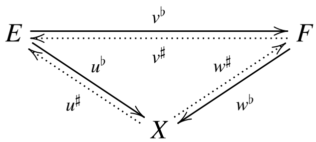
notation is that the reader should think of flat and sharp keys on a piano as modulations of the same note (in this case, the arrow). The composition of $u\colon E \rightleftarrows F$ and $v\colon F \rightleftarrows G$ is, as one would expect, $v \circ u = \langle v^\flat u^\flat, u^\sharp v^\sharp \rangle$. We measure the 'size' of a pair by taking $\|u\| = \max\left(\|u^\flat\|, \|u^\sharp\|\right)$. Note that $\|u\| \geq 1$ (unless $E = 0$) and that $\|u\| \leq 1 + \varepsilon$ implies that $u^\flat$ is an ε-isometry. If $\|u\| = 1$ (or $u = 0$), we say that u is contractive. Finally, we declare a contractive pair $u\colon E \rightleftarrows F$ to be allowable if E and F are allowed p-normed spaces and both $u^\flat$ and $u^\sharp$ are rational maps. Clearly, the allowable pairs form a countable category.

Definition 6.3.1 A p-normed space X is said to be of almost universal complemented disposition (AUCD) if, for all contractive pairs $u\colon E \rightleftarrows X$ and $v\colon E \rightleftarrows F$ with F a finite-dimensional p-normed space, and every $\varepsilon > 0$, there exists a pair $w\colon F \rightleftarrows X$ such that $u = w \circ v$ and $\|w\| \leq 1 + \varepsilon$.

The situation is illustrated by the following diagram in which both the solid arrows (embeddings) and the dotted arrows (projections) commute:



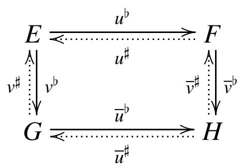
(6.5)

Hence, the AUCD property is formally stronger than $[\Game]$. Note that, according to our definitions, the 'null pair' $0 \rightleftarrows F$ is contractive. Thus, spaces with trivial dual are excluded from Definition 6.3.1 and do not have property $[\Game]$.

Amalgamating Pairs

We now establish that pairs have the amalgamation property.

Lemma 6.3.2 *Given pairs $u\colon E \rightleftarrows F$ and $v\colon E \rightleftarrows G$ there are pairs $\overline{u} = \langle \overline{u}^\flat, \overline{u}^\sharp \rangle$ and $\overline{v} = \langle \overline{v}^\flat, \overline{v}^\sharp \rangle$ such that the following diagram commutes:*



Moreover,

- *if u and v are contractive then so are $\overline{u}$ and $\overline{v}$;*
- *if u is contractive and $\|v^\flat\| \leq 1$ then $\overline{u}$ is contractive and $\|\overline{v}^\sharp\| \leq \|v^\sharp\|$;*

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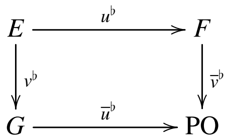
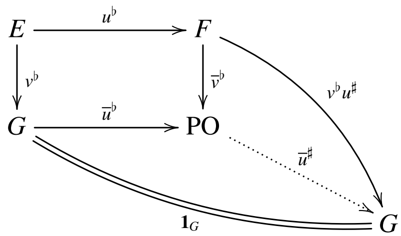
- *if u and v are allowable pairs then $\overline{u}$ and $\overline{v}$ can be taken to be allowable.*

Proof The proof is based on the isometric properties of the pushout construction presented in Section 2.5. We start with $u^\flat$ and $v^\flat$ so that $H = \text{PO}$ is their pushout space and obtain the commutative diagram



(6.6)

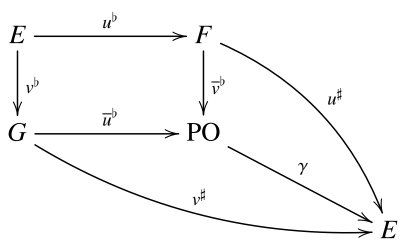
The projection $\overline{u}^\sharp$ is provided by the universal property of the pushout applied to the operators $\mathbf{1}_G, v^\flat u^\sharp$:



while $\overline{v}^\sharp$ is obtained from $\mathbf{1}_F$ and $u^\flat v^\sharp$. We have (see Lemma 2.5.2) that

- $\|\overline{u}^\flat\|, \|\overline{v}^\flat\| \leq 1$,
- $\|\overline{u}^\sharp\| \leq \|v^\sharp\| \, \|u^\flat\|$,
- $\|\overline{v}^\sharp\| \leq \|u^\sharp\| \, \|v^\flat\|$,
- $\overline{u}^\sharp \overline{u}^\flat = \mathbf{1}_G$, that is, $\overline{u} = \langle \overline{u}^\flat, \overline{u}^\sharp \rangle$ is a pair,
- $\overline{v}^\sharp \overline{u}^\flat = u^\flat v^\sharp$,
- $\overline{v}^\sharp \overline{v}^\flat = \mathbf{1}_F$, that is, $\overline{v} = \langle \overline{v}^\flat, \overline{v}^\sharp \rangle$ is a pair,
- $\overline{u}^\sharp \overline{v}^\flat = v^\flat u^\sharp$.

It only remains to check that the projections commute, that is, $u^\sharp \overline{v}^\sharp = v^\sharp \overline{u}^\sharp$. This follows from the uniqueness part of the universal property of the pushout construction: since $u^\sharp u^\flat = v^\sharp v^\flat$ (they are the identity on E), there must be a unique operator $\gamma \colon \text{PO} \longrightarrow E$ making the following diagram commute:



Since both $u^\sharp\overline{v}^\sharp$ and $v^\sharp\overline{u}^\sharp$ can be chosen for γ, they agree. This also proves the first two 'moreover' statements. To prove the third one, just use the allowable version of the pushout that appears in Lemma 6.2.2. □

Correction and Approximation

Before putting Fraïssé to work, let us state and prove three useful correction and approximation techniques that greatly simplify the manipulation of pairs. Before even that, we make the simple observation that every isomorphism $f\colon X \longrightarrow Y$ can be understood as part of a pair $\langle f, f^{-1}\rangle\colon\ X \rightleftarrows Y$.

Lemma 6.3.3 *Let E be a finite-dimensional subspace that is complemented by a projection P in a p-Banach space X, and let $e_1, \dots, e_k$ be a normalised basis of E. For every $\varepsilon > 0$, there is $\delta > 0$, depending on ε, $\|P\|$ and the chosen basis, such that if $x_i \in X$ satisfy $\|e_i - x_i\| < \delta$ for $1 \le i \le k$ then the linear map $f\colon X \longrightarrow X$ given by*

$$f(x) = \begin{cases} x_i & \text{if } x = e_i \text{ for } 1 \le i \le k \\ x & \text{if } x \in \ker P \end{cases}$$

satisfies $\|f - \mathbf{1}_X\| < \varepsilon$.

Proof Take K so large that $\left(\sum_i |\lambda_i|^p\right)^{1/p} \le K \|\sum_i \lambda_i e_i\|$. Pick $x \in X$ and write $x = y + z$ with $y = Px$ and then $y = \sum_i \lambda_i e_i$. Then, since $z \in \ker P$, one has

$$\|fx - x\| = \|fy - y\| = \left\|\sum_i \lambda_i(x_i - e_i)\right\| \le \delta \left(\sum_i |\lambda_i|^p\right)^{1/p} \le \delta K\|y\| \le \delta K\|P\|\|x\|.$$

Hence $\delta = \varepsilon/(K\|P\|)$ suffices. □

In particular, f is an automorphism. The hypothesis that E is complemented is necessary: in a rigid space (where the only endomorphisms are the scalar multiples of the identity), such an f cannot exist.

Lemma 6.3.4 *If $u\colon\ E \rightleftarrows F$ is a pair with $\|u\| \le 1 + \varepsilon$ then there is a p-norm $|\cdot|$ on F such that, for all $f \in F$, one has*

$$(1+\varepsilon)^{-1}\|f\| \le |f| \le (1+\varepsilon)\|f\| \tag{6.7}$$

and u becomes contractive when the original p-norm of F is replaced by $|\cdot|$.

Proof The hypotheses imply that $u^\flat$ is an ε-isometry. The unit ball of the new p-norm of F has to be the p-convex hull of the set

$$u^\flat[B_E] \cup (1+\varepsilon)^{-1} B_F.$$

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We thus define $|f| = \inf\{(\|x\|^p + (1+\varepsilon)^p\|g\|^p)^{1/p} : f = u^\flat(x) + g, x \in E, g \in F\}$ and check that everything works with this p-norm. First, taking $x = 0$ and $g = f$, we have $|f| \le (1+\varepsilon)\|f\|$. The other inequality of (6.7) is as follows: if $f = u^\flat(x) + g$, then $|f| \ge (1+\varepsilon)^{-1}\|f\|$ since

$$\|x\|^p + (1+\varepsilon)^p\|g\|^p = \|x\|^p + (1+\varepsilon)^p\|f - u^\flat x\|^p \ge \frac{\|u^\flat x\|^p + \|f - u^\flat x\|^p}{(1+\varepsilon)^p} \ge \frac{\|f\|^p}{(1+\varepsilon)^p}.$$

Let us compute the 'new' quasinorms of $u^\flat$ and $u^\sharp$. Given $x \in E$, one has $|u^\flat x|^p \le \|x\|^p$, so the quasinorm of $u^\flat$ is at most 1. Actually, it is clear that $|u^\flat x| = \|x\|$ for all $x \in E$. Indeed, we have

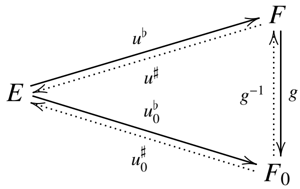
$$\begin{aligned} |u^\flat x|^p &= \inf\left\{\|y\|^p + (1+\varepsilon)^p\|g\|^p : u^\flat x = u^\flat(y) + g, y \in E, g \in F\right\} \\ &= \inf\left\{\|y\|^p + (1+\varepsilon)^p\|u^\flat(x-y)\|^p : y \in E\right\} \\ &\ge \inf\{\|y\|^p + \|x-y\|^p : y \in E\} \\ &= \|x\|^p. \end{aligned}$$

Finally, we check that $|u^\sharp| = \sup_{|f|\le 1}\|u^\sharp f\| = \sup_{|f|<1}\|u^\sharp f\| \le 1$. If $|f| < 1$, we can write $f = u^\flat(x) + g$, with $\|x\|^p + (1+\varepsilon)^p\|g\|^p < 1$. Hence

$$\begin{aligned} \|u^\sharp f\| &= \|u^\sharp(u^\flat x + g)\| = \|x + u^\sharp g\| \\ &\le \left(\|x\|^p + \|u^\sharp g\|^p\right)^{1/p} \le (\|x\|^p + (1+\varepsilon)^p\|g\|^p)^{1/p} < 1. \end{aligned}$$ □

The following is a version of Lemma 6.2.4 for pairs.

Lemma 6.3.5 *Given a contractive pair $u\colon E \rightleftarrows F$, with allowed domain E, and $\varepsilon > 0$, there is an allowable pair $u_0\colon E \rightleftarrows F_0$ and an ε-isometry $g\colon F \longrightarrow F_0$ making the following diagram commute:*



Proof Assume that ε is rational. Let $(e_i)_{1\le i\le n}$ be the unit basis of $E = \mathbb{K}^n$, and let $(f_j)_{1\le j\le m}$ be a basis of $\ker u^\sharp$. Then $\{u^\flat(e_1), \dots, u^\flat(e_n), f_1, \dots, f_m\}$ is a basis of F. Let $g\colon F \longrightarrow \mathbb{K}^{n+m}$ be the induced isomorphism. Take a rational p-norm $|\cdot|_0$ on $\mathbb{K}^{n+m}$ such that $(1+\varepsilon)^{-1}\|y\| \le |g(y)|_0 \le (1+\varepsilon)\|y\|$ for every $y \in F$. Now consider the pair $u_0 = \langle g, g^{-1}\rangle \circ u$. Then u_0 is rational (we have $u_0^\flat(x) = (x, 0)$ and $u_0^\sharp(x,y) = x$) and $\left\| u_0\colon E \rightleftarrows (\mathbb{K}^{n+m}, |\cdot|_0) \right\| \le 1 + \varepsilon$. Finally, we define a new p-norm on $\mathbb{K}^{n+m}$ by the formula

$$|y| = \inf\left\{\left(\|x\|^p + (1+\varepsilon)^p|z|_0^p\right)^{1/p} : y = u_0^\flat(x) + z, x \in \mathbb{K}^n, z \in \mathbb{K}^{n+m}\right\}.$$

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This p-norm has to be allowed on $\mathbb{K}^{n+m}$ (by the last condition of the list), satisfies the estimate $(1+\varepsilon)^{-1}|\cdot|_0 \leq |\cdot| \leq (1+\varepsilon)|\cdot|_0$ and makes u_0 into a contractive pair (see the proof of Lemma 6.3.4) which is therefore allowable. Hence, if F_0 is $\mathbb{K}^{n+m}$ equipped with $|\cdot|$ then for every $y \in F$, we have

$$(1+\varepsilon)^{-2}\|y\| \leq |g(y)| \leq (1+\varepsilon)^2\|y\|. \qquad \square$$

A Space of Almost Universal Complemented Disposition

The allowable pairs form a countable category that admits amalgamations (Lemma 6.3.2) and has an initial object. It follows from Proposition 6.1.1 that there exists a Fraïssé sequence

$$U_1 \rightleftarrows U_2 \rightleftarrows \cdots \rightleftarrows U_n \rightleftarrows \cdots \rightleftarrows U_m \rightleftarrows \cdots$$

Define the p-Kadec space K_p to be the direct limit of the inductive system formed by the $(u_n^\flat)$:

$$U_1 \xrightarrow{u_1^\flat} U_2 \longrightarrow \cdots \longrightarrow U_n \xrightarrow{u_n^\flat} U_{n+1} \longrightarrow \cdots$$

Theorem 6.3.6 K_p *is a space of almost universal complemented disposition.*

Proof We identify each U_n with its image in K_p. Let $u\colon E \rightleftarrows \mathsf{K}_p$ and $v\colon E \rightleftarrows F$ be contractive pairs, where F is a finite-dimensional p-normed space, and let $0 < \varepsilon < 1$. We recommend that the reader work out the case in which both u and v are allowable pairs, using amalgamation and the properties of Fraïssé sequences. In the general case, we first push u into some U_n even if this spoils the isometric character of the embedding and the projection is no longer contractive. To this end, note that since the union of the subspaces U_n is dense in K_p, a straighforward application of Lemma 6.3.3 provides an integer n and an automorphism f of K_p such that $f[u^\flat[E]] \subset U_n$ with $\|f - \mathbf{1}_{\mathsf{K}_p}\| < \varepsilon$ and $\max\left(\|f\|, \|f^{-1}\|\right) < 1+\varepsilon$. After dividing f by $\|f\|$ and multiplying f^{-1} by $\|f\|$, we may assume that $\|f\| = 1$ and $\|f^{-1}\| < (1+\varepsilon)^2$. Then $\langle f, f^{-1}\rangle \circ u$ is a pair from E to K_p that 'factors' through the natural pair $\imath_n\colon U_n \rightleftarrows \mathsf{K}_p$ in the sense that $\langle f, f^{-1}\rangle \circ u = \imath_n \circ u_\varepsilon$, where $u_\varepsilon\colon E \rightleftarrows U_n$ is defined as

$$u_\varepsilon^\flat = \imath_n^\sharp f u^\flat, \qquad u_\varepsilon^\sharp = u^\sharp f^{-1} \imath_n^\flat.$$

Indeed $u_\varepsilon^\sharp$ is a projection along $u_\varepsilon^\flat$ since $u_\varepsilon^\sharp u_\varepsilon^\flat = u^\sharp f^{-1}\imath_n^\flat \imath_n^\sharp f u^\flat = \mathbf{1}_E$. Now we work with this u_ε and return to u at the end of the proof. Let us amalgamate u_ε and v in the pushout diagram

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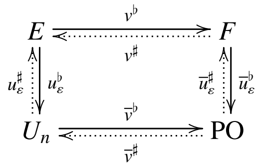
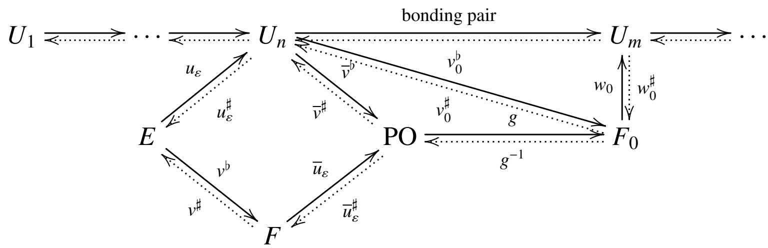


Note that since $\|u_\varepsilon^\flat\| \leq 1$, the lower pair $\overline{v} = \langle \overline{v}^\flat, \overline{v}^\sharp \rangle$ is contractive. Then we apply Lemma 6.3.5 to $\overline{v}$ to obtain an allowed space F_0 together with an ε-isometry $g\colon \mathrm{PO} \longrightarrow F_0$ such that $\overline{v}_0 = \langle g, g^{-1} \rangle \circ \overline{v}$ is an allowable pair. Finally, as the sequence $(u_n)_{n\geq 1}$ is Fraïssé, there is $m > n$ and an allowable pair $w_0\colon\ F_0 \rightleftarrows U_m$ such that $w_0 \circ \overline{v}_0$ is the bonding pair $U_n \rightleftarrows U_m$, so we have the following commutative diagram of pairs:



In particular, we have $w_0 \circ \langle g, g^{-1} \rangle \circ \overline{u} \circ v = u_\varepsilon = \langle f, f^{-1} \rangle \circ u$, and letting $w = \langle f^{-1}, f \rangle \circ w_0 \circ \langle g, g^{-1} \rangle \circ \overline{u}_\varepsilon$, we are done, since $w \circ v = u$ and

$$\|w\| \leq \|\langle f^{-1}, f \rangle\| \, \|w_0\| \, \|\langle g, g^{-1} \rangle\| \, \|\overline{u}_*\| \leq (1+\varepsilon)^3 < 1 + 7\varepsilon. \qquad \square$$

Further Properties of K_p

Next we study isometric properties of K_p: universality and uniqueness. There is a key fact that allows us to recover ‘approximate pairs’ (pairs of operators $f^\dagger\colon X \longrightarrow Y$ and $f^\ddagger\colon Y \longrightarrow X$ whose composition is close to the identity of X) as a composition of the arrows of two pairs with a common, ad hoc codomain.

Lemma 6.3.7 *Let $f^\dagger\colon X \longrightarrow U$ and $f^\ddagger\colon U \longrightarrow X$ be contractive operators such that $\|f^\ddagger f^\dagger - \mathbf{1}_X\| \leq \varepsilon$. There are contractive pairs $\alpha\colon\ X \rightleftarrows X \boxplus_{f^\dagger}^{\varepsilon} U$ and $\beta\colon\ U \rightleftarrows X \boxplus_{f^\dagger}^{\varepsilon} U$ such that $f^\dagger = \beta^\sharp \alpha^\flat$, $f^\ddagger = \alpha^\sharp \beta^\flat$ and $\|\alpha^\flat - \beta^\flat f^\dagger\| \leq \varepsilon$.*

The relevant diagram is

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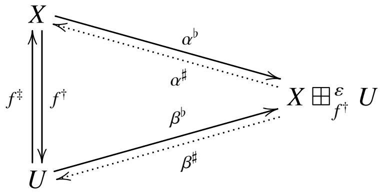

Proof We know from the proof of Lemma 6.2.5 that $\|(x,0)\| = \|x\|$ and $\|(0,y)\| = \|y\|$ for every $x \in X$ and every $y \in U$. Thus, letting $\alpha^\flat(x) = (x,0)$ and $\beta^\flat(y) = (0,y)$, we see that $\|\alpha^\flat - \beta^\flat f^\dagger\| \leq \varepsilon$. As for the projections, we are forced to define $\alpha^\sharp(x,y) = x + f^\ddagger(y)$ and $\beta^\sharp(x,y) = y + f^\dagger(x)$. It is then clear that

$$\alpha^\sharp\alpha^\flat = \mathbf{1}_X, \quad \beta^\sharp\beta^\flat = \mathbf{1}_U, \quad f^\dagger = \beta^\sharp\alpha^\flat, \quad f^\ddagger = \alpha^\sharp\beta^\flat.$$

To see that $\alpha^\sharp$ and $\beta^\sharp$ are contractive, pick $(x,y) \in X \boxplus^{\varepsilon}_{f^\dagger} U$ and assume

$$(x,y) = (x_0 + x_2, y_1 - f^\dagger x_2) = (x_0, 0) + (0, y_1) + (x_2, -f^\dagger(x_2)).$$

We then have

$$\|\alpha^\sharp(x,y)\| = \|x_0 + x_2 + f^\ddagger(y_1) - f^\ddagger f^\dagger(x_2)\| \leq (\|x_0\|^p + \|y_1\|^p + \varepsilon^p\|x_2\|^p)^{1/p},$$
$$\|\beta^\sharp(x,y)\| = \|f^\dagger(x_0) + y_1\| \leq (\|x_0\|^p + \|y_1\|^p)^{1/p} \leq (\|x_0\|^p + \|y_1\|^p + \varepsilon^p\|x_2\|^p)^{1/p}$$

and, since $\|(x,y)\|$ is the infimum of the numbers that might appear in the right-hand side, we have $\|\alpha^\sharp\|, \|\beta^\sharp\| \leq 1$. □

Universality

A *skeleton* in a quasi-Banach space X is an increasing chain $(X_n)_{n\geq 1}$ of finite-dimensional subspaces of X whose union is dense in X and such that each X_n is 1-complemented in X_{n+1}. Those inclusions and projections can be arranged into a sequence of contractive pairs $X_1 \rightleftarrows X_2 \rightleftarrows \cdots$ A quasi-Banach space has a skeleton if and only if it is the direct limit of a sequence of contractive pairs, and 'skeleton' is just a transcription of 1-FDD: if $(Y_n)_{n\geq 1}$ is a 1-FDD of X then defining $X_n = Y_1 + \cdots + Y_n$, we obtain a skeleton; conversely, if $(X_n)_{n\geq 1}$ is a skeleton then fixing contractive projections $\pi_n \colon X_{n+1} \longrightarrow X_n$ and letting $Y_1 = X_1$ and $Y_{n+1} = \ker \pi_n$, we get a 1-FDD.

Proposition 6.3.8 *Every p-Banach space with a skeleton is isometric to a 1-complemented subspace of* K_p.

Proof Let $(X_n)_{n\geq 1}$ be a skeleton of X. For each integer $n \geq 1$, we denote the 'bonding' pair $X_n \rightleftarrows X_{n+1}$ by ξ_n, that is, $\xi_n^\flat$ is the inclusion of X_n into

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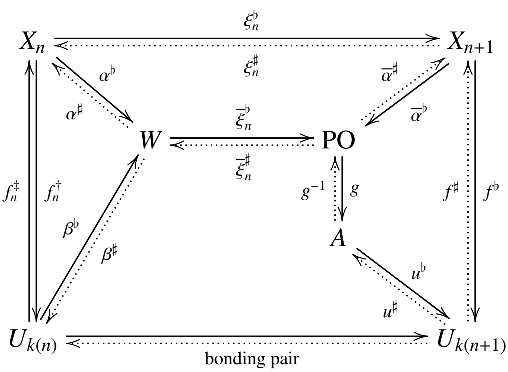
X_{n+1} and $\xi_n^\sharp\colon X_{n+1} \longrightarrow X_n$ is a fixed contractive projection. Considering the spaces U_n as subspaces of K_p, we shall construct an increasing sequence of integers $(k(n))_{n\geq 0}$ and a system of contractive operators $f_n^\dagger\colon X_n \longrightarrow U_{k(n)}$ and $f_n^\ddagger\colon U_{k(n)} \longrightarrow X_n$ such that

(1) $\|f_n^\ddagger f_n^\dagger - \mathbf{1}_{X_n}\| < 2^{-n}$,
(2) $\|f_{n+1}^\dagger|_{X_n} - f_n^\dagger\| < 2^{-n}$,
(3) $\|f_{n+1}^\ddagger|_{U_{k(n)}} - f_n^\ddagger\| < 2^{-n}$.

Since $\sum_n 2^{-np} < \infty$, the pointwise limits of the sequences $(f_n^\dagger)$ and $(f_n^\ddagger)$ provide a contractive pair $X \rightleftarrows \mathsf{K}_p$, which will complete the proof. The required sequence is constructed by induction. We can assume $X_1 = 0$ so that $f_1^\dagger = 0$ and $f_1^\ddagger = 0$. Now suppose that $f_n^\dagger\colon X_n \longrightarrow U_{k(n)}$ and $f_n^\ddagger\colon U_{k(n)} \longrightarrow X_n$ have already been constructed and let us see how to get $k(n+1)$ and the maps $f_{n+1}^\dagger\colon X_{n+1} \longrightarrow U_{k(n+1)}$ and $f_{n+1}^\ddagger\colon U_{k(n+1)} \longrightarrow X_{n+1}$. We suggest that the reader fetch a pencil and some paper for a bit of scribbling.



Set $\varepsilon = \|f_n^\ddagger f_n^\dagger - \mathbf{1}_{X_n}\| < 2^{-n}$ and reserve a small $\delta > 0$ of room. The precise value of δ will be specified later.

• First, we apply Lemma 6.3.7 to $f_n^\dagger$ and $f_n^\ddagger$. By doing so, we obtain the space W and the left triangle in the preceding diagram. Note that $\alpha = \langle \alpha^\flat, \alpha^\sharp\rangle$ and $\beta = \langle \beta^\flat, \beta^\sharp\rangle$ are contractive pairs such that

$$\|\beta^\flat f_n^\dagger - \alpha^\flat\| \leq \varepsilon, \qquad f_n^\dagger = \beta^\sharp \alpha^\flat, \qquad f_n^\ddagger = \alpha^\sharp \beta^\flat.$$

• Then, we amalgamate ξ_n and α using Lemma 6.3.2, which yields the upper commutative trapezoid.

• Next, we apply Lemma 6.3.5 to the composition $\overline{\xi}_n \circ \beta$ (which is a contractive pair), thus obtaining a surjective δ-isometry $g\colon \mathrm{PO} \longrightarrow A$ in such a way that the composition $\langle g, g^{-1}\rangle \circ \overline{\xi}_n \circ \beta$ turns out to be an allowable pair.

• Since the sequence of pairs (u_n) is Fraïssé, there must be some $k(n+1) > k(n)$ and an allowable pair $u\colon A \rightleftarrows U_{k(n+1)}$ such that $u \circ \langle g, g^{-1}\rangle \circ \overline{\xi}_n \circ \beta$ is the bonding pair $U_{k(n)} \rightleftarrows U_{k(n+1)}$.

• Now, look at the pair $f = u \circ \langle g, g^{-1}\rangle \circ \overline{\alpha}$. Note that f need not be contractive, as we only have the bound $\|f\| \le \|\langle g, g^{-1}\rangle\| \le 1 + \delta$.

One has:

(4) $f^\sharp f^\flat = \mathbf{1}_{X_{n+1}}$,
(5) $\|f^\flat|_{X_n} - f_n^\dagger\| \le (1+\delta)\varepsilon$,
(6) $f^\sharp|_{U_{k(n)}} = \xi_n^\flat f_n^\ddagger$.

The first identity is trivial. As for (5), note that $f^\flat|_{X_n} = u^\flat\, g\, \overline{\xi}_n^\flat\, \alpha^\flat$, hence

$$\|f^\flat|_{X_n} - f_n^\dagger\| = \|u^\flat\, g\, \overline{\xi}_n^\flat\, \alpha^\flat - \underbrace{u^\flat\, g\, \overline{\xi}_n^\flat \beta^\flat}_{\text{inclusion}}\, f_n^\dagger\| \le \|g\|\|\beta^\flat\, f_n^\dagger - \alpha^\flat\| \le (1+\delta)\varepsilon.$$

To check (6), observe that the inclusion of $U_{k(n)}$ into $U_{k(n+1)}$ can be written as $u^\flat\, g\, \overline{\xi}_n^\flat \beta^\flat$. Besides, $f^\sharp = \overline{\alpha}^\sharp\, g^{-1} u^\sharp$, so, recalling that $\overline{\alpha}^\sharp\, \overline{\xi}_n^\flat = \xi^\flat\, \alpha^\sharp$, we have

$$f^\sharp|_{U_{k(n)}} = \overline{\alpha}^\sharp\, g^{-1} u^\sharp\, u^\flat\, g\, \overline{\xi}_n^\flat \beta^\flat = \overline{\alpha}^\sharp\, \overline{\xi}_n^\flat \beta^\flat = \xi_n^\flat \alpha^\sharp \beta^\flat = \xi_n^\flat f_n^\ddagger.$$

As a final touch to render the maps contractive, set $f_{n+1}^\dagger = \frac{f^\flat}{1+\delta}$ and $f_{n+1}^\ddagger = \frac{f^\sharp}{1+\delta}$. Then $f_{n+1}^\ddagger f_{n+1}^\dagger = (1+\delta)^{-2}\mathbf{1}_{X_{n+1}}$. Hence, using (4) for small δ, we get

$$\|f_{n+1}^\ddagger f_{n+1}^\dagger - \mathbf{1}_{X_{n+1}}\| \le 1 - \frac{1}{(1+\delta)^2} < \frac{1}{2^{n+1}}.$$

And also for δ sufficiently small, we get from (5) and (6) that

$$\|f_{n+1}^\dagger|_{X_n} - f_n^\dagger\|^p \le \|f_{n+1}^\dagger - f^\flat\|^p + \|f^\flat|_{X_n} - f_n^\dagger\|^p \le \delta^p + (1+\delta)^p\varepsilon^p < 2^{-pn},$$
$$\|f_{n+1}^\ddagger - \xi_n^\flat f_n^\ddagger\| = \|f_{n+1}^\ddagger - f^\sharp\| \le \delta < 2^{-n}.$$ □

To deduce now that K_p is complementably universal for the spaces with the BAP, we need only firmly grab the trolley of Proposition 6.3.8 and push it resolutely towards Lemma 2.2.20's Platform 9 & 3/4: that we can freely assume that the space Y has a 1-FDD, and actually a skeleton, instead of a mere BAP. Do it without hesitation:

Corollary 6.3.9 *Every separable p-Banach space with the BAP is isomorphic to a complemented subspace of* K_p.

It is difficult to imagine a space peskier than K_p. Indeed, the following spaces are all isomorphic to K_p:

- Products $\mathsf{K}_p \times X$, when X is a separable p-Banach space with the BAP.
- Spaces of K_p-valued sequences $X(\mathsf{K}_p)$, when X is a p-Banach sequence space – in particular, this includes $\ell_q(\mathsf{K}_p)$ for $p \leq q < \infty$ and $c_0(\mathsf{K}_p)$.
- The p-convex envelope of K_q for $0 < q < p$ (see Corollary 6.3.12) and the space $C(\Delta, \mathsf{K}_p)$.

In contrast, if $0 < p < 1$, the space $\mathsf{K}_p \oplus_p L_p$ is of almost universal complemented disposition and not isomorphic to K_p. This assertion will later on be complemented by Propositions 6.3.13 and 10.7.2.

Uniqueness

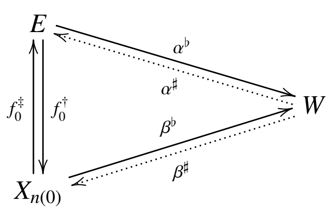
We now address the uniqueness of K_p. The peak result here is Proposition 6.3.11, the 1-complemented companion of Proposition 6.2.7. We are pleased to make the reader aware that the skeleton assumption is quite natural: it corresponds, in the category of contractive pairs, to standard separability in $p\mathbf{B}$. The route to the proof is now based on a stability result that is interesting in its own right:

Proposition 6.3.10 *Let E be a finite-dimensional p-Banach and $\varepsilon > 0$. Let X be a p-Banach space with a skeleton and that satisfies $[\partial]$. If $f^\dagger\colon E \longrightarrow X$ and $f^\ddagger\colon X \longrightarrow E$ are contractive operators such that $\|f^\ddagger f^\dagger - \mathbf{1}_E\| < \varepsilon$ then there is an isometry $f^\flat\colon E \longrightarrow X$ with 1-complemented range and such that $\|f^\dagger - f^\flat\| < \varepsilon$.*

Proof We fix a skeleton (X_n) of X, and we denote the corresponding pairs of operators by $\xi_n\colon X_n \rightleftarrows X$ and $\xi_{(n,k)}\colon X_n \rightleftarrows X_k$. We also fix a sequence $(\varepsilon_n)_{n\geq 0}$ of positive numbers with $\varepsilon_1 < \varepsilon$ such that $\|f^\ddagger f^\dagger - \mathbf{1}_E\| < \varepsilon_1$ and $\sum_{n\geq 0} \varepsilon_n^p < \varepsilon^p$. Note that we must first choose ε_1 and then the other ε_n. Using a small perturbation of the identity of X, we can obtain $n(0)$ and contractive operators $f_0^\dagger\colon E \longrightarrow X_{n(0)}$ and $f_0^\ddagger\colon X_{n(0)} \longrightarrow E$ such that

$$\|f^\dagger - f_0^\dagger\| < \varepsilon_0 \qquad \text{and} \qquad \|f_0^\ddagger f_0^\dagger - \mathbf{1}_E\| \leq \varepsilon_1.$$

Applying Lemma 6.3.7 to $f_0^\dagger, f_0^\ddagger$ and ε_1, we obtain the diagram



in which α and β are contractive pairs and

$$f_0^\dagger = \beta^\sharp\alpha^\flat, \quad f_0^\ddagger = \alpha^\sharp\beta^\flat, \quad \|\alpha^\flat - \beta^\flat f_0^\ddagger\| \leq \varepsilon_1.$$

Since X has property [⅁], after normalising a suitable almost-isometry $W \longrightarrow X$ and the corresponding projection, we obtain $n(1) > n(0)$ and contractive operators $\gamma^\dagger : W \longrightarrow X_{n(1)}$ and $\gamma^\ddagger : X_{n(1)} \longrightarrow W$ satisfying

$$\|\gamma^\ddagger\gamma^\dagger - \mathbf{1}_W\| < \varepsilon_2 \qquad \text{and} \qquad \|\gamma^\dagger\beta^\flat - \xi^\flat_{(n(0),n(1))}\| < \varepsilon_2.$$

Letting $f_1^\dagger = \gamma^\dagger\alpha^\flat$ and $f_1^\ddagger = \alpha^\sharp\gamma^\ddagger$, we have $\|f_1^\ddagger f_1^\dagger - \mathbf{1}_E\| < \varepsilon_2$ and

$$\begin{aligned}\|f_1^\dagger - f_0^\dagger\|^p &= \|\gamma^\dagger\alpha^\flat - \gamma^\dagger\beta^\flat f_0^\dagger + \gamma^\dagger\beta^\flat f_0^\dagger - f_0^\dagger\|^p \\ &\leq \|\alpha^\flat - \beta^\flat f_0^\dagger\|^p + \|\gamma^\dagger\beta - \xi^\flat_{(n(0),n(1))}\|^p \\ &< \varepsilon_1^p + \varepsilon_2^p.\end{aligned}$$

Continuing in this way, we obtain an increasing sequence $(n(k))_{k\geq 0}$ and contractive operators $f_k^\dagger : E \longrightarrow X_{n(k)}$ and $f_k^\ddagger : X_{n(k)} \longrightarrow E$ satisfying

- $\|f_k^\ddagger f_k^\dagger - \mathbf{1}_E\| \leq \varepsilon_{k+1}$,
- $\|f_{k+1}^\dagger - f_k^\dagger\| < \left(\varepsilon_{k+1}^p + \varepsilon_{k+2}^p\right)^{1/p}$.

The second estimate implies that $(f_k^\dagger)_k$ is a Cauchy sequence in $\mathfrak{L}(E, X)$ since

$$\|f_{k+m}^\dagger - f_k^\dagger\| \leq \left(\sum_{i=0}^{m-1} \|f_{k+i+1}^\dagger - f_{k+i}^\dagger\|^p\right)^{1/p} \leq \left(\sum_{i=0}^{m-1} \varepsilon_{k+i+2}^p + \varepsilon_{k+i+1}^p\right)^{1/p}.$$

The first estimate then implies that the double sequence $(f_k^\ddagger f_n^\dagger)_{k,n}$ converges to the identity of E in the sense that for every $\delta > 0$ there is m such that $\|f_k^\ddagger f_n^\dagger - \mathbf{1}_E\| < \delta$ whenever $k, n \geq m$. Define $f^\flat : E \longrightarrow X$ as the pointwise limit of the sequence $(f_k^\dagger)_k$. To obtain a suitable projection along $f^\flat$, we can use the local compactness of E: pick a non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and set $f^\sharp(x) = \lim_{\mathcal{U}(n)} f_n^\ddagger(x)$ for $x \in \bigcup_k X_k$, and extend by continuity to all of X. It is clear that $f^\flat$ and $f^\sharp$ are contractive. Finally, given $y \in E$, we have

$$f^\sharp f^\flat y = \lim_{\mathcal{U}(n)} f_n^\ddagger\left(f^\flat y\right) = \lim_{\mathcal{U}(n)} f_n^\ddagger\left(\lim_k f_k^\dagger y\right) = \lim_{\mathcal{U}(n)}\left(\lim_k f_n^\ddagger f_k^\dagger y\right) = \lim_{k,n} f_n^\ddagger f_k^\dagger y = y.$$

This shows that $f^\flat$ is an isometry whose range is 1-complemented in X. □

Proposition 6.3.11 *Let X, Y be p-Banach spaces with skeletons and that satisfy* [⅁], *let A be a* 1*-complemented subspace of X and let B be a* 1*-complemented subspace of Y. If $f_0 : A \longrightarrow B$ is a surjective isometry then for every $\varepsilon > 0$, there is a surjective isometry $f : X \longrightarrow Y$ such that $\|f|_A - f_0\| < \varepsilon$.*

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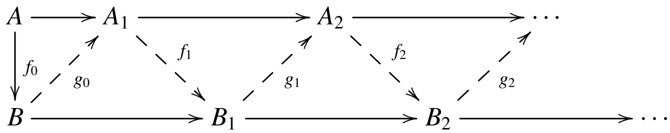
Proof The proof is a typical back-and-forth argument, oiled by Proposition 6.3.10. Fix a sequence of positive real numbers $(\varepsilon_n)_{n\geq 0}$ such that $\sum_{n\geq 0}\varepsilon_n^p < \varepsilon^p$. Let (X_n) and (Y_n) be chains of finite-dimensional 1-complemented subspaces of X and Y, respectively, with dense union. Set $A_1 = A + X_1$. Then f_0^{-1} embeds isometrically B into A_1, as a 1-complemented subspace. Since Y has property [Ↄ], for each $\delta > 0$, there is a δ-isometry $f_{1/2}\colon A_1 \longrightarrow Y$ whose range is $(1+\delta)$-complemented, extending the inclusion of B. Apply Proposition 6.3.10 to $f_{1/2}$ with δ small enough to obtain an isometry $f_1\colon A_1 \longrightarrow Y$ with 1-complemented range such that $\|f_1(f_0^{-1}(y)) - y\| < \varepsilon_1\|y\|$ for all non-zero $y \in B$. Set $B_1 = f_1[A_1] + B + Y_1$ and apply the same argument to obtain an isometry $g_1\colon B_1 \longrightarrow X$ with 1-complemented range with $\|g_1(f_1(x)) - x\| < \varepsilon_1\|x\|$ for all non-zero $x \in A_1$. Now set $A_2 = g_1[B_1] + A_1 + X_2$ and let $f_2\colon A_2 \longrightarrow Y$ be an isometry with 1-complemented range such that $\|f_2(g_1(y)) - y\| < \varepsilon_2\|y\|$ for all non-zero $y \in B_1$, and so on. Continuing in this way, we obtain increasing sequences $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ of finite-dimensional 1-complemented subspaces of X and Y, respectively, with dense union, where $A_0 = A$ and $B_0 = B$ together with isometries $f_n\colon A_n \longrightarrow B_n$ and $g_n\colon B_n \longrightarrow A_{n+1}$ satisfying

(1) $\|g_n(f_n(x)) - x\| < \varepsilon_n\|x\|$ for all non-zero $x \in A_n$,
(2) $\|f_{n+1}(g_n(y)) - y\| < \varepsilon_n\|y\|$ for all non-zero $y \in B_n$,

where $g_0 = f_0^{-1}$. The situation is illustrated in the following ('almost commutative') diagram



We define an operator $f\colon X \longrightarrow Y$ as follows: if $x \in A_k$, set $f(x) = \lim_{n\geq k} f_n(x)$. The definition makes sense because $(f_n(x))_{n\geq k}$ is a Cauchy sequence. Indeed, for $x \in A_n$, we have

$$\begin{aligned}\|f_{n+1}(x) - f_n(x)\|^p &\leq \|f_{n+1}(x) - f_{n+1}(g_n(f_n(x)))\|^p + \|f_{n+1}(g_n(f_n(x))) - f_n(x)\|^p\\ &\leq \|f_{n+1}\|^p\|x - g_n(f_n(x))\|^p + \varepsilon_n^p\|f_n(x)\|^p\\ &\leq 2\varepsilon_n^p\|x\|^p.\end{aligned}$$

Since $\sum_{n\geq 0}\varepsilon_n^p$ is finite, we see that f is well defined on $\bigcup_n A_n$, and so it extends to a contractive operator on X which we also call f. Besides, for x normalised in $A = A_0$, we have

$$\|f(x) - f_0(x)\| \leq \left(\sum_{n\geq 0}\|f_{n+1}(x) - f_n(x)\|^p\right)^{1/p} \leq \left(\sum_{n\geq 0} 2\varepsilon_n^p\right)^{1/p} \leq 2^{1/p}\varepsilon.$$

Proceeding analogously with the sequence (g_n), we obtain a contractive operator $g\colon Y \longrightarrow X$ given by $g(y) = \lim_{n\geq k} g_n(y)$ for $y \in B_k$. It follows from (1) and (2) that $gf = \mathbf{1}_X$ and $fg = \mathbf{1}_Y$. □

Now that Proposition 6.3.11 is complete, let us stop and smell the roses it has brought to blossom. One result with a fine scent is that any p-Banach space with a skeleton and property $[\Game]$ is isometric to K_p and, therefore, of almost universal complemented disposition. (Intrigued by the role of the skeleton? Please move to the next section.) Another fragrant one is:

Corollary 6.3.12 *If* $0 < p < q \leq 1$ *then the* q*-Banach envelope of* K_p *is* K_q*.*

Proof We only sketch the proof. Fix $0 < p < q \leq 1$. The key point is that a contractive pair $u\colon E \rightleftarrows F$ between finite-dimensional p-normed spaces is also a contractive pair $u\colon E_{(q)} \rightleftarrows F_{(q)}$. Thus, taking the pairs between finite-dimensional q-normed spaces that arise from the q-Banach envelopes of the allowable pairs of p-Banach spaces, the class of which we will momentarily call $\mathfrak{A}$, we obtain a Fraïssé class $\mathfrak{A}_{(q)}$ since the amalgamation property is inherited from $\mathfrak{A}$. These pairs are 'dense' among the contractive pairs of finite-dimensional q-normed spaces because each q-norm is also a p-norm. Moreover, if $U^1 \rightleftarrows U^2 \rightleftarrows \ldots$ is the Fraïssé sequence used to define K_p, it is clear that the q-Banach envelope of K_p arises from the sequence $U^1_{(q)} \rightleftarrows U^2_{(q)} \rightleftarrows \ldots$, which is easily seen to be a Fraïssé sequence for $\mathfrak{A}_{(q)}$. Therefore, its limit is isometric to K_q. □

The Banach space K_1 is almost isotropic too. This follows from Proposition 6.3.11 and the fact that all lines are 1-complemented. It is not isotropic (almost isotropic with $\varepsilon = 0$) since the unit sphere of K_1 contains points where the norm is smooth and points where it is not (think of an isometric copy of, say ℓ_∞^2), while a surjective isometry should preserve each class. In sharp contrast, there is no equivalent p-norm rendering K_p almost isotropic when $p < 1$: if X is almost isotropic and isomorphic to K_p then the functional $|x| = \|x\| + \sup_{\|x^*\|\leq 1} |x^*(x)|$ is another p-norm which must be preserved by every isometry of the original p-norm of X. It quickly follows (see the complete argument in [57, Theorem 3.3]) that $|\cdot| = 2\|\cdot\|$. Thus, $\|x\| = \sup_{\|x^*\|\leq 1} |x^*(x)|$ and X would be locally convex, which is not the case.

Other Spaces of Kadec Type

Let us examine now what occurs when dropping the skeleton assumption:

Proposition 6.3.13 *Every separable p-Banach space is isometric to a* 1*-complemented subspace of a separable p-Banach space with property* [⅁].

The proof is based on a slight weakening of our notion of pair introduced in Section 6.3.

Definition 6.3.14 A λ-pair $u = \langle u^\flat, u^\sharp\rangle$ consists of two contractive operators $u^\flat\colon E \longrightarrow F$ and $u^\sharp\colon F \longrightarrow E$ such that $u^\sharp u^\flat = \lambda \mathbf{1}_E$, where $\lambda > 0$. A $\bullet$-pair is a λ-pair for some unspecified λ.

Thus, 1-pairs are the former contractive pairs. Note that if $u = \langle u^\flat, u^\sharp\rangle$ is a λ-pair then $\langle u^\flat, \lambda^{-1}u^\sharp\rangle$ is a pair which is not contractive in general and $u^\flat$ is a contractive ε-isometry, where $\varepsilon = \lambda^{-1} - 1$. Also, if $u = \langle u^\flat, u^\sharp\rangle$ is a pair then the normalisation $\langle u^\flat/\|u^\flat\|, u^\sharp/\|u^\sharp\|\rangle$ is a λ-pair, where $\lambda = (\|u^\flat\|\|u^\sharp\|)^{-1}$. We extend the use of the notation $u\colon E \rightleftarrows F$ to $\bullet$-pairs as well as most of the conventions of Section 6.3. If $u\colon E \rightleftarrows F$ is a λ-pair and $v\colon F \rightleftarrows G$ is a μ-pair, then $v \circ u = \langle v^\flat u^\flat, u^\sharp v^\sharp\rangle$ is a $\lambda\mu$-pair. The distance between two $\bullet$-pairs u, v between the same spaces is defined as $\|u - v\| = \max\left(\|u^\flat - v^\flat\|, \|u^\sharp - v^\sharp\|\right)$.

Lemma 6.3.15 *Let X be a p-Banach space and I an index set. For each $i \in I$, let $u_i\colon E_i \rightleftarrows F_i$ be a* 1*-pair and $v_i\colon E_i \rightleftarrows X$ a λ_i-pair. Then there is a p-Banach space X', a* 1*-pair $\xi\colon X \rightleftarrows Y$ and, for each $i \in I$, a λ_i-pair $\overline{v}_i\colon F_i \rightleftarrows X'$ such that $\xi \circ v_i = \overline{v}_i \circ u_i$; i.e. the diagram*

$$\begin{array}{ccc} E_i & \overset{u_i^\flat}{\underset{u_i^\sharp}{\rightleftarrows}} & F_i \\ {\scriptstyle v_i^\sharp}\uparrow\downarrow{\scriptstyle v_i^\flat} & & {\scriptstyle \overline{v}_i^\sharp}\uparrow\downarrow{\scriptstyle \overline{v}_i^\flat} \\ X & \overset{\xi^\flat}{\underset{\xi^\sharp}{\rightleftarrows}} & X' \end{array}$$

is commutative. Moreover, if I is finite and each F_i is finite-dimensional then $X'/\xi^\flat[X]$ is finite-dimensional.

Proof This is a combination of the Device technique and Lemma 6.3.2. Consider the 1-pair $\Pi\colon \ell_p(I, E_i) \rightleftarrows \ell_p(I, F_i)$ given by $\Pi^\flat = \prod_i u_i^\flat, \Pi^\sharp = \prod_i u_i^\sharp$ and the operator $\Sigma = \bigoplus_i v_i^\flat\colon \ell_p(I, E_i) \longrightarrow X$ and form the pushout

$$\begin{array}{ccc} \ell_p(I, E_i) & \overset{\Pi^\flat}{\underset{\Pi^\sharp}{\rightleftarrows}} & \ell_p(I, F_i) \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \overline{\Sigma}} \\ X & \overset{\overline{\Pi^\flat}}{\underset{\overline{\Pi^\sharp}}{\rightleftarrows}} & \mathrm{PO}(\Pi^\flat, \Sigma) \end{array}$$

where $\overline{\Pi^\sharp}$ arises from the universal property of PO applied to the pair of operators $(\mathbf{1}_X, \Sigma\Pi^\sharp)$. This provides the 1-pair ξ. As for the $\bullet$-pairs $\overline{v}_j$ we first define $\overline{v}_j^\flat$ as the restriction of $\overline{\Sigma}$ to F_j. To get $\overline{v}_j^\sharp$, just consider the pair of operators $\lambda_j\pi_j\colon \ell_p(I, F_i) \longrightarrow F_j$ and $u_j^\flat v_j^\sharp\colon X \longrightarrow F_j$, where π_j sends $(x_i)_{i\in I}$ to x_j. Note that $u_j^\flat v_j^\sharp\Sigma = \lambda_j\pi_j\Pi^\flat$, as both send $(y_i)_{i\in I}$ to $\lambda_j y_j$. $\square$

Proof of Proposition 6.3.13 Let $\mathfrak{A}$ be the set of allowable pairs of p-normed spaces, and let $(a_n)_{n\geq 1}$ be an enumeration of $\mathfrak{A}$, where $a_1 = \mathbf{1}_{\mathbb{K}}$. We are going to construct a chain of contractive pairs

$$X_1 \underset{\xi_1^\sharp}{\overset{\xi_1^\flat}{\rightleftarrows}} X_2 \underset{\xi_2^\sharp}{\overset{\xi_2^\flat}{\rightleftarrows}} X_3 \rightleftarrows \cdots \tag{6.8}$$

together with a sequence of sets of $\bullet$-pairs $(D^n)_{n\geq 1}$ and enumerations $(u_k^n)_{k\geq 1}$ in such a way that

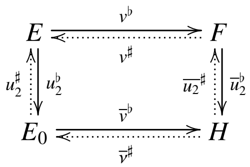
(1) $X_1 = X$ and $X_{n+1}/\xi_n^\flat[X_n]$ is finite-dimensional;
(2) the elements of D^n are $\bullet$-pairs $u\colon E \rightleftarrows X_n$ with allowed domain;
(3) D^n has the following density property: for every $\bullet$-pair $v : E \rightleftarrows X_n$ with allowed domain and each $\varepsilon > 0$ there is $u \in D^n$ such that $\|v^\flat - u^\flat\| < \varepsilon$ and $\|v^\sharp v^\flat - u^\sharp u^\flat\| < \varepsilon$ (note that we don't care about $\|v^\sharp - u^\sharp\|$);
(4) $D^n \subset D^{n+1}$ in the sense that if $u \in D^n$ then $\xi_n \circ u \in D^{n+1}$;
(5) if $a \in \mathfrak{A}_{\leq n}$ and $u \in \bigcup_{k\leq n} D^k_{\leq n}$ is a λ-pair with the same domain then there is a commutative diagram

$$\begin{array}{ccc} E & \underset{a^\sharp}{\overset{a^\flat}{\rightleftarrows}} & F \\ {\scriptstyle u^\sharp}\uparrow\downarrow{\scriptstyle u^\flat} & & {\scriptstyle \overline{u}^\sharp}\uparrow\downarrow{\scriptstyle \overline{u}^\flat} \\ X_n & \underset{\xi_n^\sharp}{\overset{\xi_n^\flat}{\rightleftarrows}} & X_{n+1} \end{array}$$

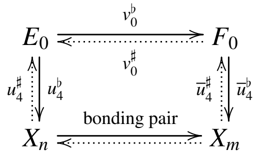
where $\overline{u}$ is a λ-pair, $\mathfrak{A}_{\leq n} = \{a_1, \ldots, a_n\}$ and similarly with $D^k_{\leq n}$.

For the initial step, we set $X_1 = X$ and choose D^1 as in (3). This can be done because each $\mathfrak{L}(E, X)$ is separable and there are countably many allowed spaces. Condition (5) is automatic because of our choice of a_1. For the inductive step, assume that one has constructed $X_1, \ldots, X_n$ together with $D^1, \ldots, D^n$ and the corresponding enumerations that satisfy (1)–(5). Then we apply the preceding lemma to the 1-pairs of $\mathfrak{A}_{\leq n}$ and the $\bullet$-pairs in $\bigcup_{k\leq n} D^k_{\leq n}$ that have the same domain, and we set $X_{n+1} = X_n'$ and $\xi_n = \xi$. Note that $X_{n+1}/\xi_n^\flat[X_n]$ is finite-dimensional. Finally, we choose a countable set of $\bullet$-pairs D^{n+1} 'containing' every $\bullet$-pair of the form $\xi_n \circ v$ for $v \in D^n$ and satisfying (3), and we enumerate it. Let $\eth(X)$ be the direct limit of the system (6.8).

Let us verify that $\reflectbox{$\partial$}(X)$ has property $[\reflectbox{$\partial$}]$. Assume E is 1-complemented in a finite-dimensional space F and that $u^\flat\colon E \longrightarrow \reflectbox{∂}(X)$ is an isometry with 1-complemented range. We consider 1-pairs $v\colon E \rightleftarrows F$ and $u\colon E \rightleftarrows X$ in which $v^\flat$ is the inclusion of E into F, $v^\sharp$ is a contractive projection onto E and $u^\sharp$ is a contractive projection along $u^\flat$. Fix $\varepsilon > 0$. Furthermore, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ will appear in the course of the proof, and the only thing that we care about is that $\varepsilon_{n+1} \to 0$ as $\varepsilon_n \to 0$. Pick $\delta > 0$. First, we use Lemma 6.3.3 to obtain a small automorphism f of $\reflectbox{$\partial$}(X)$ and n such that $fu^\flat[E] \subset X_n$. Let u_1 be the normalisation of $\langle fu^\flat, u^\sharp f^{-1}\rangle$. This can be done in such a way that $\|f - \mathbf{1}_{\reflectbox{∂}(X)}\| < \varepsilon_1$; $\|f\|, \|f^{-1}\| < 1 + \varepsilon_1$; and $\|u_1 - u\| < \varepsilon_1$. We can assume that $E = \mathbb{K}^k$ with some p-norm $\|\cdot\|$. Let $\|\cdot\|_0$ be a small allowed perturbation of the p-norm of E, and let $E_0 = (E, \|\cdot\|_0)$. Let u_2 be the normalisation of the formal identity $\langle \mathbf{1}, \mathbf{1}\rangle\colon E \rightleftarrows E_0$, and let



be provided by Lemma 6.3.2. As $\overline{v}$ is a 1-pair with allowed domain, we can immediately activate Lemma 6.2.4 to get an almost isometry $g\colon H \longrightarrow F_0$ such that $v_0 = \langle g, g^{-1}\rangle \circ \overline{v}\colon\ E_0 \rightleftarrows F_0$ is allowable. This can be done in such a way that (a) $\max\left(\|u_2^\sharp u_2^\flat - \mathbf{1}_E\|, \|\overline{u}_2^\sharp \overline{u}_2^\flat - \mathbf{1}_E\|\right) < \varepsilon_2$; (b) $\|g\|, \|g^{-1}\| < 1 + \varepsilon_2$ and (c) $\|x\|_0 \le \|x\| < (1 + \varepsilon_2)\|x\|_0$ for all non-zero $x \in E$. Let $u_3\colon E_0 \rightleftarrows \reflectbox{∂}(X)$ be the normalisation of u_1 with respect to the p-norm of E_0. This is clearly a λ-pair, with $1 - \lambda < \varepsilon_3$, provided ε_1 and ε_2 are sufficiently small. By (3), we can find a μ-pair $u_4 \in D^n$ such that $\|u_4^\flat - u_3^\flat\| < \varepsilon_3$, still with $1 - \mu < \varepsilon_3$. By (5), there is $m > n$ and a μ-pair $\overline{u}_4$, making the following diagram commute:



Let us consider the operator $U = \overline{u}_4^\flat g \overline{u}_2^\flat\colon F \longrightarrow X_m$. It should be obvious that if $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are sufficiently small,

- $\|U|_E - u\colon E \longrightarrow \reflectbox{∂}(X)\| < \delta$;
- $\|U\| < 1 + \delta$;
- if $P = \overline{u}_2^\sharp g^{-1} \overline{u}_4^\sharp$, then $PU = \eta \mathbf{1}_F$, with $|\eta - 1| < \delta$ and $\|P\| < 1 + \delta$.

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We now write $F = E \oplus \ker v^\sharp$ in order to then set $\tilde{U}(x,y) = u^\flat(x) + U(y)$, a true extension of $u^\flat$. If δ is sufficiently small, then $\|U - \tilde{U}\| < \varepsilon$ and $\|P\tilde{U} - \mathbf{1}_F\| < \varepsilon$. To obtain a small norm projection of $\eth(X)$ along $\tilde{U}$, we use:

6.3.16 Correcting a defective pair *Let F and Y be p-Banach spaces. Let $f^\dagger\colon F \longrightarrow Y$ and $f^\ddagger\colon Y \longrightarrow F$ be operators satisfying $\|f^\ddagger f^\dagger - \mathbf{1}_F\| \le \varepsilon$, where $\varepsilon < 1$. There is an automorphism a of F such that*

- $\|a - \mathbf{1}_F\| \le \varepsilon(1-\varepsilon^p)^{-1/p}$,
- $\|a\| \le (1-\varepsilon^p)^{-1/p}$,
- $\|a^{-1}\| \le (1+\varepsilon^p)^{1/p}$,
- $af^\ddagger f^\dagger = \mathbf{1}_F$.

The proof is straightforward: set $a = \sum_{n\ge 0}(\mathbf{1}_F - f^\ddagger f^\dagger)^n$ and check the required properties. □

The inexorable conclusion is that when X doesn't have the BAP, the space $\eth(X)$ cannot be isomorphic to K_p since it cannot have the BAP either. Therefore:

6.3.17 *For every $p \in (0,1]$, there exist non-isomorphic separable p-Banach spaces with property* $[\eth]$.

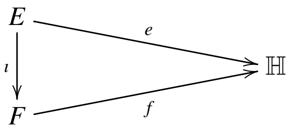
It is clear that if X has a skeleton then so does $\eth(X)$, and Proposition 6.3.13 provides an alternative construction of K_p *and* a new proof of Proposition 6.3.8. On the other hand, it is clear that the only reason $\eth(X)$ could fail the BAP (or any other approximation property) is because X already lacks it: X is 1-complemented in $\eth(X)$ and $\eth(X)/X$ has a 1-FDD.

6.4 A Universal Operator on G_p

Finding operators on a given quasi-Banach space can be a difficult task. Or an impossible one, since rigid spaces exist. The space G_p is not rigid: Propositions 6.2.7 and 6.2.10 say that it has plenty of automorphisms. Our aim is to construct a contraction $u \in \mathfrak{L}(\mathsf{G}_p)$ with $\ker u \approx \mathsf{G}_p$ and satisfying the following condition:

(♡) For every separable p-Banach space X and every contractive operator $s\colon X \longrightarrow \mathsf{G}_p$, there exists an isometry $e\colon X \longrightarrow \mathsf{G}_p$ such that $s = ue$.

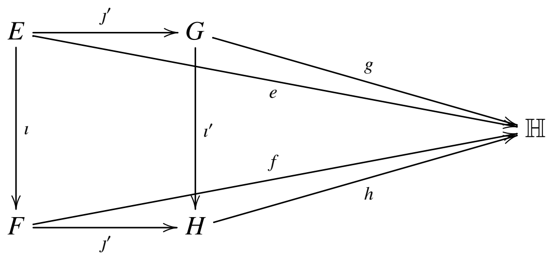
This will show that G_p has non-trivial projections since, taking as s the identity of G_p, one obtains an isometric embedding $e\colon \mathsf{G}_p \longrightarrow \mathsf{G}_p$ and eu is a projection on G_p with kernel and range isometric to G_p. To get the announced construction, we will fix a separable p-Banach space $\mathbb{H}$ and develop some abstract nonsense. Piece by piece, everything will make sense. We start by defining a special category **H** whose objects are contractive operators from finite-dimensional p-Banach spaces into $\mathbb{H}$; a morphism from $e\colon E \longrightarrow \mathbb{H}$ to $f\colon F \longrightarrow \mathbb{H}$ is an isometry $\imath\colon E \longrightarrow F$ such that $e = f\imath$:



Although this category is conceptually more complex than those used in the preceding sections, our treatment, based on purely formal properties, is similar. Our nonsense training begins with:

Lemma 6.4.1 *The category* **H** *admits amalgamations.*

What does this mean? It means that when one has three objects $e\colon E \longrightarrow \mathbb{H}$, $f\colon F \longrightarrow \mathbb{H}$, $g\colon G \longrightarrow \mathbb{H}$ in **H** and morphisms $\imath\colon e \longrightarrow f$, $\jmath\colon e \longrightarrow g$, there is another object $h\colon H \longrightarrow \mathbb{H}$ and morphisms $\imath'\colon g \longrightarrow h$, $\jmath'\colon f \longrightarrow h$ such that $\jmath' \circ \imath = \imath' \circ \jmath$. The point is, we *do* know that the lemma is true and *how* to prove it: just stare at the commutative diagram



(6.9)

and set $H = \mathrm{PO}$, the pushout of $\imath$ and $\jmath$. Great. We also need an '**H**-version' of Lemma 6.2.5:

Lemma 6.4.2 *Let $f\colon X \longrightarrow Y$ be an ε-isometry between finite-dimensional p-Banach spaces, and let $r\colon X \longrightarrow \mathbb{H}$ and $s\colon Y \longrightarrow \mathbb{H}$ be contractive operators such that $\|sf - r\| \le \varepsilon$. Let $\imath$ and $\jmath$ be the inclusions of X and Y, respectively, into $X \boxplus_f^\varepsilon Y$. The operator $r \oplus s\colon X \boxplus_f^\varepsilon Y \longrightarrow \mathbb{H}$ is contractive, and $(r \oplus s)\, i = r$, $(r \oplus s)\, j = s$. In particular, $\imath\colon r \longrightarrow (r \oplus s)$ and $\jmath\colon s \longrightarrow (r \oplus s)$ are morphisms in* **H**.

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Proof Fix $(x, y) \in X \oplus Y$ and assume $x = x_0 + x_2$, $y = y_1 - f(x_2)$. Then

$$\|(r \oplus s)(x, y)\|^p = \|r(x_0) + r(x_2) + s(y_1) - s(f(x_2))\|^p \le \|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p \|x_2\|_X^p.$$

As $\|(x, y)\|^p$ is the infimum of all expressions that can arise as the right-hand side of the preceding inequality, we see that $\|(r \oplus s)(x, y)\|^p \le \|(x, y)\|^p$. □

To return to Fraïssé's world, we need a countable 'dense' subcategory of **H** having amalgamations. Let D be a dense, countable, linearly independent subset of $\mathbb{H}$ and let $\mathbb{H}_0$ denote the dense subspace of all finite linear combinations of elements of D with rational coefficients. We define a subcategory $\mathbf{H}_0$ of **H** as follows:

- The objects of $\mathbf{H}_0$ are contractive operators $e\colon E \longrightarrow \mathbb{H}$ whose domain is allowed and that send the rational vectors of E into $\mathbb{H}_0$.
- Given objects $e\colon E \longrightarrow \mathbb{H}$ and $f\colon F \longrightarrow \mathbb{H}$ in $\mathbf{H}_0$, an $\mathbf{H}_0$-morphism $\imath\colon e \longrightarrow f$ is an **H**-morphism whose underlying isometry is allowable.

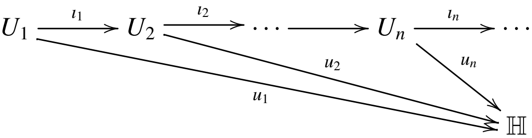
Lemma 6.4.3 $\mathbf{H}_0$ *has amalgamations.*

Proof (Proof of) Lemma 6.2.2 + Diagram 6.9. □

Proposition 6.1.1 says that $\mathbf{H}_0$ has a Fraïssé sequence

$$u_1 \xrightarrow{\ \imath_1\ } u_2 \xrightarrow{\ \imath_2\ } \cdots \tag{6.10}$$

Since each $u_n\colon U_n \longrightarrow \mathbb{H}$ is an object of $\mathbf{H}_0$ and the arrows $\imath_n$ are morphisms in $\mathbf{H}_0$, what one actually has is a commutative diagram



(6.11)

having the following property:

(†) Given a finite-dimensional p-Banach space V, an isometry $e\colon U_n \longrightarrow V$ and a contractive operator $v\colon V \longrightarrow \mathbb{H}$ such that $ve = u_n$, for each $\varepsilon > 0$, there exist $m > n$ and an ε-isometry $e'\colon V \longrightarrow U_m$ such that $\|e'e - \imath_{(n,m)}\| < \varepsilon$ and $\|u_m e' - v\| < \varepsilon$, where $\imath_{(n,m)} = \imath_{m-1} \cdots \imath_n$.

Of course, all this comes preloaded in the definition of a Fraïssé sequence when v and e are in $\mathbf{H}_0$, even with $\varepsilon = 0$. In the general case, we first apply Lemma 6.2.4 to e, thus obtaining an allowable $e_0\colon U_n \longrightarrow V_0$ and a surjective ε-isometry $g\colon V \longrightarrow V_0$ such that $e_0 = ge$. Although $u_n = vg^{-1}e_0$, which is Ok, we cannot apply the preceding case to vg^{-1} because we do not know that vg^{-1}

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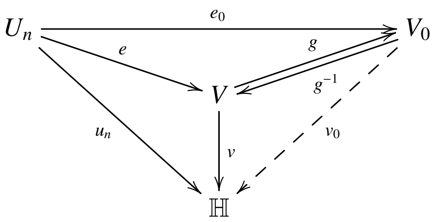
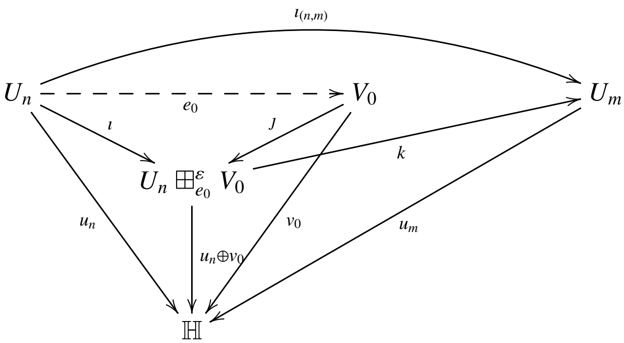
is contractive or that it takes rational vectors to $\mathbb{H}_0$. It is clear, however, that there is $v_0 \colon V_0 \longrightarrow \mathbb{H}_0$ in $\mathbf{H}_0$ such that $\|v_0 - vg^{-1}\| < \varepsilon$:



The dashed part of the diagram is there to remind us that it is merely *almost* commutative. Construct the space $U_n \boxplus_{e_0}^{\varepsilon} V_0$, equipped with the direct sum operator $u_n \oplus v_0$, and activate Lemmas 6.2.5 and 6.4.2: if we denote the inclusions of U_n and V_0 into $U_n \boxplus_{e_0}^{\varepsilon} V_0$ by $\imath$ and $\jmath$, we have $\|\jmath e_0 - \imath\| \leq \varepsilon$ by Lemma 6.2.5, and since $\|v_0 e_0 - u_n\| \leq \varepsilon$, we conclude that $\imath \colon u_n \longrightarrow u_n \oplus v_0$ and $\jmath \colon v_0 \longrightarrow u_n \oplus v_0$ are $\mathbf{H}$-morphisms. But $u_n \oplus v_0$ maps the rational vectors of $U_n \boxplus_{e_0}^{\varepsilon} V_0$ to $\mathbb{H}_0$, and $\imath \colon U_n \longrightarrow U_n \boxplus_{e_0}^{\varepsilon} V_0$ is allowable, by (f). It follows that $\imath \colon u_n \longrightarrow u_n \oplus v_0$ is actually in $\mathbf{H}_0$. Since (6.10) is Fraïssé for $\mathbf{H}_0$, there is $m > n$ and $k \colon u_n \oplus v_0 \longrightarrow u_m$ such that $k \circ \imath = \imath_{(n,m)}$, the bonding morphism $u_n \longrightarrow u_m$. Let us ignore V for a moment and depict the situation in the diagram



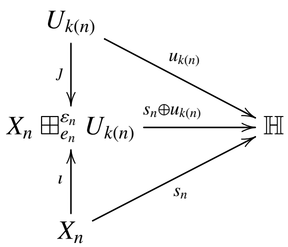
It is now clear that the required map is $e' = k\jmath g$:

- e' is an ε-isometry since g is and $\jmath, k$ are isometries.
- $\|e'e - \imath_{(n,m)}\| = \|k\jmath ge - \imath_{(n,m)}\| = \|k\jmath e_0 - \imath_{(n,m)}\| = \|k\jmath e_0 - k\imath + k\imath - \imath_{(n,m)}\| \leq \varepsilon$.
- As $\|v_0 - vg^{-1}\| < \varepsilon$, we have $\|v - v_0 g\| \leq \varepsilon\|g\| \leq \varepsilon(1+\varepsilon)$. But $u_m e' = u_m k\jmath g = v_0 g$, so $\|u_m e' - v\| \leq 2\varepsilon$.

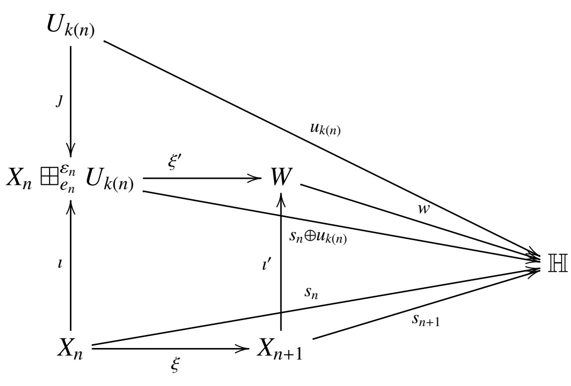
Consider the directed system of p-Banach spaces $U_0 \xrightarrow{\imath_1} U_1 \xrightarrow{\imath_2} Y_2 \longrightarrow \cdots$ underlying the sequence (6.11). Set $U = \varinjlim U_n$ and let $u \colon U \longrightarrow \mathbb{H}$ be the direct limit of the operators u_n. The main properties of these objects can be summarised as follows: we know now that U contains an isometric copy of X as long as X is separable and $\mathfrak{L}(X, \mathbb{H}) \neq 0$, and will know soon (Theorem 6.4.5) that U is isometric to G_p. In the meantime:

Proposition 6.4.4 *For every separable p-Banach space X and every contractive operator $s\colon X \longrightarrow \mathbb{H}$, there exists an isometry $e\colon X \longrightarrow U$ such that $s = ue$. In particular, u is surjective and right-invertible.*

Proof We identify each U_n with its image in U so that $u_n = u|_{U_n}$ and all the bonding maps are plain inclusions. Fix an operator $s\colon X \longrightarrow \mathbb{H}$ with $\|s\| \leq 1$. Let $(X_n)_{n\geq 1}$ be an increasing sequence of finite-dimensional subspaces whose union is dense in X, with $X_1 = 0$. Set $s_n = s|_{X_n}$ and $\varepsilon_n = 2^{-n/p}$. We shall inductively construct an increasing sequence $k\colon \mathbb{N} \longrightarrow \mathbb{N}$ and contractive ε_n-isometries $e_n\colon X_n \longrightarrow U_{k(n)}$ satisfying $\|u_{k(n)}e_n - s_n\| \leq \varepsilon_n$ and also $\|e_{n+1}|_{X_n} - e_n\| \leq (\varepsilon_n^p + \varepsilon_{n+1}^p)^{1/p}$. This clearly implies that the sequence (e_n) converges pointwise to an isometry $e\colon X \longrightarrow U$ such that $s = ue$. We set $k(1) = 1$ and $e_1 = 0$. Having defined $e_n\colon X_n \longrightarrow U_{k(n)}$ with $\|s_n - u_{k(n)}e_n\| \leq \varepsilon_n$, we may apply Lemma 6.4.2 with $f = e_n$ to get the commutative diagram



which shows that $\imath$ is a **H**-morphism from s_n to $s_n \oplus u_{k_n}$. On the other hand, the inclusion of X_n into X_{n+1}, which we momentarily denote by ξ, is clearly an **H**-morphism from s_n to s_{n+1}, and amalgamating $\imath$ and ξ, we arrive at the diagram



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Here, W is a finite-dimensional p-normed space and $\|w\| \le 1$. Applying (†) to the isometry $\xi' \jmath$, we find $k(n+1) > k(n)$ and obtain a contractive ε_{n+1}-isometry $e' : W \longrightarrow U_{k_{n+1}}$ such that

$$\|u_{k(n+1)}e' - w\| \le \varepsilon_{n+1} \quad \text{and} \quad \|e'\xi' \jmath - \imath_{(k(n),k(n+1))}\| \le \varepsilon_{n+1}. \tag{6.12}$$

Setting $e_{n+1} = e'\imath' : X_{n+1} \longrightarrow W \longrightarrow U_{k(n+1)}$, we complete the induction step. Indeed, e_{n+1} is a contractive ε_{n+1}-isometry. Moreover,

$$\|u_{k(n+1)}e_{n+1} - s_{n+1}\| = \|u_{k(n+1)}\varepsilon'\imath' - w\imath'\| \le \|u_{k(n+1)}e' - w\| \le \varepsilon_{n+1},$$

since $s_{n+1} = w\imath'$, while

$$\|e_n - e_{n+1}|_{X_n}\|^p = \|e_n - e'\imath'\xi\|^p = \|e_n - e'\xi'\imath\|^p \le \underbrace{\|e_n - e'\xi'\jmath e_n\|^p}_{(\star)} + \underbrace{\|e'\xi'(\imath - \jmath e_n)\|^p}_{(\star\star)}.$$

We have $(\star) = \|\imath_{(k(n),k(n+1))}e_n - e'\xi'\jmath e_n\| \le \|e'\xi'\jmath - \imath_{(k(n),k(n+1))}\| \le \varepsilon_{n+1}$ by (6.12) and $(\star\star) \le \|\imath - \jmath e_n\| \le \varepsilon_n$ by Lemma 6.2.5. This completes the induction step and the proof. □

Taking $s = \mathbf{1}_{\mathbb{H}}$, we see that u is surjective and right-invertible. Thus, we have a split exact sequence $0 \longrightarrow \ker u \longrightarrow U \xrightarrow{u} \mathbb{H} \longrightarrow 0$.

Theorem 6.4.5 *Whatever the space $\mathbb{H}$ could be,* $\ker u$ *is isometric to G_p and so U is isomorphic to $\mathsf{G}_p \times \mathbb{H}$. If, additionally, $\mathbb{H}$ is a locally 1^+-injective p-Banach space then also U is isometric to G_p and, therefore, G_p is isomorphic to $\mathsf{G}_p \times \mathbb{H}$.*

Proof To prove that $\ker u$ is isometric to G_p we first check that the operator $u : U \longrightarrow \mathbb{H}$ has the following additional property:

(‡) If E is a subspace of a finite-dimensional p-Banach space F, $g : F \longrightarrow \mathbb{H}$ is contractive and $e : E \longrightarrow U$ is an isometry such that $ue = g|_E$ then for each $\delta > 0$, there is a δ-isometry $f : F \longrightarrow U$ satisfying $\|f|_E - e\| < \delta$ and $\|uf - g\| < \delta$.

Indeed, after taking a small perturbation, we may assume that $e : E \longrightarrow U_n$ is an ε-isometry with $\|ue - g|_E\| < \varepsilon$. Apply Lemma 6.4.2 to $e : E \longrightarrow U_n$, $g : E \longrightarrow \mathbb{H}$ and $u_n : U_n \longrightarrow \mathbb{H}$ to obtain a commutative diagram

$$\begin{array}{ccc} U_n & & \\ \downarrow \jmath & \searrow^{u_n} & \\ E \boxplus_e^{\varepsilon} U_n & \xrightarrow{g \oplus u_n} & \mathbb{H} \\ \uparrow \imath & \nearrow_{g} & \\ E & & \end{array}$$

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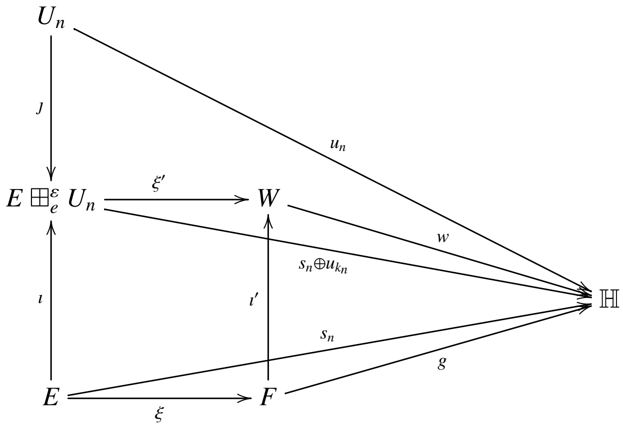
with $\|\jmath e - \imath\| \le \varepsilon$. Now, amalgamating $\imath\colon E \longrightarrow E \boxplus_e^\varepsilon U_n$, which is a morphism from $g\colon E \longrightarrow \mathbb{H}$ to $g \oplus u_n\colon E \boxplus_e^\varepsilon U_n \longrightarrow \mathbb{H}$, with the inclusion $\xi\colon E \longrightarrow F$ regarded as a morphism from $g\colon E \longrightarrow \mathbb{H}$ to $g\colon F \longrightarrow \mathbb{H}$, we obtain a finite-dimensional p-normed space W and a commutative diagram



with $\|w\| \le 1$. Now applying (†) to w and the embedding $\xi'\jmath$, we obtain $m > n$ and an almost isometry $w'\colon W \longrightarrow U_m$ such that $u_m w'$ is close to w and $w'\xi'\jmath$ is close to $\imath_{(n,m)}$. Finally, the composition

$$f : F \xrightarrow{\ \imath'\ } W \xrightarrow{\ w'\ } U_m \longrightarrow U$$

is the desired δ-isometry. Returning to $\ker u$, let F be a finite-dimensional p-normed space; $e\colon E \longrightarrow \ker u$ an isometry, where E is a subspace of F; and $\varepsilon > 0$. We shall construct an ε-isometry $f\colon F \longrightarrow \ker u$ such that $\|f(x) - e(x)\| \le \varepsilon\|x\|$ for every $x \in E$. This will show that $\ker u$ is of AUD, thus completing the proof. To do so, fix some small δ and apply (‡), taking g as the zero operator from F to $\mathbb{H}$ to get a δ-isometry $f'\colon F \longrightarrow U$ such that $\|f'|_E - e\| < \delta$ and $\|uf'\| < \delta$. Of course, we cannot guarantee that f' takes values in $\ker u$. To amend this, let $r\colon \mathbb{H} \longrightarrow U$ be a right-inverse for u, with $\|r\| \le 1$, and set $f = (\mathbf{1}_U - ru)f'$, that is, $f(x) = f'(x) - r(u(f'(x)))$. Then f takes values in $\ker u$ since $uf = 0$ and, moreover, $\|f - f'\| = \|ruf'\| \le \delta$. Thus, for δ sufficiently small, $f\colon F \longrightarrow \ker u$ is an ε-isometry with $\|f|_E - e\| < \varepsilon$ and we are done.

We now assume that $\mathbb{H}$ is locally 1^+-injective among p-Banach spaces and prove that U is isometric to G_p. It suffices to check it is of AUD by showing that it satisfies the hypothesis of Lemma 6.2.3. Let $v\colon E \longrightarrow F$ be an isometry, where E is a subspace of U and F a finite-dimensional p-normed space. Fix

$\delta > 0$, and pick a contractive δ-isometry $u\colon E \longrightarrow U_n$ such that $\|u(x) - x\| \leq \delta\|x\|$ for $x \in E$. Let us form the pushout square

$$\begin{array}{ccc} E & \xrightarrow{\ v\ } & F \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \overline{u}} \\ U_n & \xrightarrow{\ \overline{v}\ } & \mathrm{PO} \end{array}$$

Here $\overline{v}$ is an isometry and $\overline{u}$ is a contractive δ-isometry as in Lemma 2.5.2. Since $\mathbb{H}$ is locally 1^+-injective and PO is finite-dimensional, there is an operator $\tilde{u}_n\colon \mathrm{PO} \longrightarrow \mathbb{H}$ such that $u_n = \tilde{u}_n\overline{v}$, with $\|\tilde{u}_n\| \leq 1 + \delta$. Next we touch up the p-norm of PO to render $\tilde{u}_n$ contractive: for instance, we may take $|x| = \max(\|x\|_{\mathrm{PO}}, \|\widehat{u_n}(x)\|_{\mathbb{H}})$. If V denotes the space PO so p-normed then $\overline{v}\colon U_n \longrightarrow V$ is still isometric and $\|\tilde{u}_n\colon V \longrightarrow \mathbb{H}\| \leq 1$ and we may use (†) to get $m > n$ and a δ-isometry $v'\colon V \longrightarrow U_m$ such that $\|v'\overline{v} - \imath_{(n,m)}\| \leq \delta$. Finally, if $\delta > 0$ is sufficiently small, the composition

$$w : F \xrightarrow{\ \overline{u}\ } \mathrm{PO} \xrightarrow{\text{identity}} V \xrightarrow{\ v'\ } U_m \xrightarrow{\text{inclusion}} U$$

is an ε-isometry such that $\|w(v(x)) - x\| \leq \varepsilon\|x\|$ for every $x \in E$. This shows that U is isometric to G_p. □

Time for applications.

Corollary 6.4.6 $\mathsf{G}_p \simeq \mathsf{G}_p \times \mathsf{G}_p \simeq c_0(\mathbb{N}, \mathsf{G}_p) \simeq C(\Delta, \mathsf{G}_p)$.

Proof The theorem just proved yields that if $\mathbb{H}$ is a separable locally 1^+-injective p-Banach space then $\mathsf{G}_p \times \mathbb{H}$ is isomorphic to G_p. Pick $\mathbb{H} = \mathsf{G}_p$, which is locally 1^+-injective according to Proposition 6.2.8(a), to obtain that G_p is isomorphic to $\mathsf{G}_p \times \mathsf{G}_p$ and thus to any finite product $\mathsf{G}_p \times \cdots \times \mathsf{G}_p$. The spaces $c_0(\mathsf{G}_p)$ and $C(\Delta, \mathsf{G}_p)$ can be written as the limit of a chain of subspaces isometric to $\mathsf{G}_p \oplus_\infty \cdots \oplus_\infty \mathsf{G}_p$, and so they are locally 1^+-injective; since $\mathsf{G}_p \simeq \mathsf{G}_p \times c_0(\mathbb{N}, \mathsf{G}_p)$ and $\mathsf{G}_p \simeq \mathsf{G}_p \times C(\Delta, \mathsf{G}_p)$, the Pełczyński decomposition method applies. □

The applications of Theorem 6.4.5 are seriously limited by the scarcity of examples of locally injective p-Banach spaces for $p < 1$, which basically are reduced to ... G_p! When $p = 1$, all Lindenstrauss spaces are locally 1^+-injective Banach spaces, and Corollary 6.4.6 can be strengthened to:

Corollary 6.4.7 *Every separable Lindenstrauss space is isometric to a subspace of* G *that is complemented by a contractive projection whose kernel is isometric to* G.

Thus, if X is a separable Lindenstrauss space, then $\mathsf{G} \simeq X \times \mathsf{G} \simeq X \check{\otimes}_\varepsilon \mathsf{G}$. This does not mean that every copy of X is complemented in G. Also, Theorem 6.4.5 shows that some hyperplanes of the almost isotropic space G are isometric to the whole space, since when $p = 1$, the base field is 1-injective, and we can fix $\mathbb{H} = \mathbb{K}$. To the best of our knowledge, Hilbert spaces were the only previously known spaces combining both properties.

6.5 Notes and Remarks

6.5.1 What If $\varepsilon = 0$?

Upon moving ε from *here* to *there* in the definitions of Section 6.3 (and there are various heres and theres to choose), we obtain more or less equivalent variants of the definitions appearing in the text. Actually, the version of property [ᘐ] and the definition of AUCD we used do not match those of [183] or [116]. While 6.3.16 clearly shows that [ᘐ] is equivalent to Garbulińska's property (E) of [183], we cannot ensure that Definition 6.3.1 is equivalent to Definition 2.1 in [116]. And yet, as the following shows, an ε of room is necessary to stay in the separable world.

Proposition *Let X be a p-Banach space containing a 2-dimensional Euclidean subspace E and having the following property: for every 3-dimensional p-normed space F and every isometry $v\colon E \longrightarrow F$ with 1-complemented range, there is an isometry $w\colon F \longrightarrow X$ such that wv is the inclusion of E into X. Then the dimension of X is at least the continuum.*

Proof The proof uses an idea of Haydon, taken from [80; 75]. Let us follow it in the real case. Let E be the Euclidean plane and S the unit sphere of E. For each $u \in S$, we consider the p-norm

$$|(x,t)|_u = \max\Big(\|x\|_2, \|(\langle x,u\rangle, t)\|_p\Big)$$

on $E \times \mathbb{R}$ and let F_u denote the resulting 3-dimensional space. (The unit ball of F_u is the intersection of a 'vertical' right cylinder and a 'horizontal' right prism whose basis is the 2-dimensional ℓ_p-ball with 'peaks' at $(0,1)$ and $(u,0)$.) Note that $\|(x,0)\|_u = \|x\|_2$ (so E is isometric to a subspace of F_u) and that $|(x,t)|_u \geq \|x\|_2$ for each $(x,t) \in F_u$ (so the obvious projection is contractive). Now we consider E as a subspace of X and assume that for every $u \in S$, we can find an isometry $f_u\colon F_u \longrightarrow X$ such that $f_u(x,0) = x$. Clearly, f_u must have the form $f_u(x,t) = x + te_u$ for some fixed e_u in the unit sphere of X. Now let

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S_+ be the 'positive part' of S so that $0 < \langle u, v\rangle < 1$ for different $u, v \in S_+$. We claim that $\|e_u - e_v\| = 1$ for $u, v \in S_+$ unless $u = v$. Pick $\lambda > 0$; we have

$$\|e_u - e_v\|^p \geq \|e_u + \lambda u\|^p - \|e_v + \lambda u\|^p = |(\lambda u, 1)|_u^p - |(\lambda u, 1)|_v^p.$$

But $|(\lambda u, 1)|_u^p = 1 + \lambda^p$, while for large λ, $|(\lambda u, 1)|_v^p = \max(\lambda^p, 1 + \lambda^p \langle u, v\rangle^p) = \lambda^p$. Hence the dimension of X is, at least, the cardinality of S_+. □

Thus any space of 'universal (complemented) disposition for spaces of dimension up to 3' has dimension at least $\mathfrak{c}$.

6.5.2 Before G_p Spaces Fade Out

Shortly hereafter, Fraïssé constructions will fade away in the remainder of this volume, although spaces of (almost) universal (complemented) disposition will not. But first, a few remarks that, once you are told, become very noticeable. The headline is that very few things are known about operators on G_p when $p < 1$. Indeed, the behaviour of operators on G_p is puzzling. On one hand, $\mathfrak{L}(\mathsf{G}_p)$ contains a large number of automorphisms and isometries as well as some projections. It follows from Proposition 6.2.7 that if F is a finite-dimensional subspace of G_p then G_p/F depends only on the dimension of F, up to isomorphisms. Let us denote the isomorphism type of the quotient of G_p by an n-dimensional subspace by $\mathsf{G}_p/(n)$. Since G_p is isomorphic to its square, $\mathsf{G}_p/(n+m) \simeq \mathsf{G}_p/(n) \times \mathsf{G}_p/(m)$ and also $\mathsf{G}_p/(n) \simeq (\mathsf{G}_p/(1))^n$. The sequence $0 \longrightarrow \mathbb{K} \longrightarrow \mathsf{G}_p(1) \longrightarrow 0$ is not trivial because G_p has trivial dual and therefore $\mathsf{G}_p/(1)$ is not a $\mathscr{K}$-space. So, the prickly issue is whether G_p is a $\mathscr{K}$-space. If the answer were yes then G_p could not be isomorphic to $\mathsf{G}_p/(1)$ (something we do not know either). When $p = 1$, both questions have an affirmative answer: G is isomorphic to its hyperplanes and, as for any $\mathscr{L}_\infty$-space, it is a $\mathscr{K}$-space by 3.4.6. However, we do not know whether G_p is prime or primary when $0 < p < 1$ or how to find an uncomplemented copy of G_p in the whole space, which is quite irritating. And since we cannot discard the existence of non-zero separable and separably injective p-Banach spaces, G_p could actually be such a space. In any case, all such spaces must be complemented subspaces of G_p. Ironically, it is the abundance of operators with values in G_p that makes it very difficult to define operators on G_p:

Proposition *If X is a separable p-Banach space and Y is a topological vector space such that $\mathfrak{L}(X, Y) = 0$, then $\mathfrak{L}(\mathsf{G}_p, Y) = 0$.*

Indeed, assume $u\colon \mathsf{G}_p \longrightarrow Y$ is non-zero and take $g \in \mathsf{G}_p$ such that $u(g) \neq 0$. Let $v\colon X \longrightarrow \mathsf{G}_p$ be an embedding, pick $x \in X$ and let $w \in \mathfrak{L}(\mathsf{G}_p)$ be such that $w(v(x)) = g$. Then uwv is a non-zero operator in $\mathfrak{L}(X, Y)$, a contradiction.

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Thus, for instance, $\mathfrak{L}(\mathsf{G}_p, Y) = 0$ if Y is either an ultrasummand, by the Corollary in Note 1.8.3, or $Y = L_0$ since $\mathfrak{L}(L_p/H_p, L_0) = 0$ for exactly the same reasons that $\mathfrak{L}(L_p/H_p, L_p) = 0$ for $0 < p < 1$ (see Kalton [250, Theorem 7.2] or Aleksandrov [6, Corollary 4.4 on p. 49]). The same reasoning shows that every non-zero operator defined on G_p must be an isomorphism on some copy of ℓ_2 because this is what happens in L_p when $0 < p < 1$; see [283, Theorem 7.20] for perhaps the simplest proof.

6.5.3 Fraïssé Classes of Banach Spaces

Knowing that a given structure is a Fraïssé limit opens a door to a deeper appreciation of its properties. It is therefore not unproductive to ask which classes of finite-dimensional Banach spaces have the amalgamation property. Two obvious answers are ‘all finite-dimensional spaces’ (whose Fraïssé limit is G) and ‘the Euclidean ones’ (whose Fraïssé limit is the separable Hilbert space). To be true, what people knowledgeable about (continuous) Fraïssé structures work with are *separable* classes with *stable* versions of the amalgamation property. What is required is that, given δ-isometries $f : E \longrightarrow F$ and $g : E \longrightarrow G$, there exist some space H in the class and *isometries* $\overline{g} : F \longrightarrow H$, $\overline{f} : G \longrightarrow H$ such that $\|\overline{g}f - \overline{f}g\| < \varepsilon$, with ε depending on δ, and perhaps on $\dim E$. Our naive approach relies on Lemma 6.2.5 to guarantee stability. Very recently [170], the Banach spaces L_p for $p \neq 4, 6, 8 \ldots$ have gained access to the elite club of Fraïssé spaces, which means that the class of finite-dimensional subspaces of L_p has a certain (stable) amalgamation property for those values of p. Those amalgamations are not plain pushouts, though. What prevents L_p from being Fraïssé when $p = 4, 6 \ldots$ depends on the fact, proved by B. Randrianantoanina, that those L_p contain isometric copies of the same finite-dimensional spaces with very different projection constants [398], which is in turn connected with [345, Theorem 3] where, elaborating earlier work of Plotkin/Rudin, Lusky had shown that if $0 < p < \infty$ is not $4, 6, 8, \ldots$ then, given an isometry $\varphi_0 : E \longrightarrow L_p$ from a finite-dimensional subspace E of L_p and $\varepsilon > 0$, there is an automorphism $\varphi \in \mathfrak{L}(L_p)$ extending φ_0 such that $\|\varphi\|, \|\varphi^{-1}\| < 1 + \varepsilon$ (the reader can check this with Proposition 6.2.10).

The Kechris–Pestov–Todorcevic (KPT) correspondence [293] provides an unexpected connection between Fraïssé structures and topological dynamics. A topological group is extremely amenable if every continuous action on a compact set has a fixed point. The KPT correspondence states that, given a Fraïssé class **C**, the group of automorphisms of its Fraïssé limit is extremely amenable in the strong operator topology if and only if **C** has the approximate Ramsey property, something that has to do with continuous colorings; see

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[385, Section 6.6] for a very readable introduction. Neither implication in the KPT correspondence is trivial. One can count among the applications that the otherwise mysterious isometry group of G is extremely amenable, because the class of all finite-dimensional normed spaces has the approximate Ramsey property; see [32]. Moving in the opposite direction, it implies that finite-dimensional Euclidean spaces have the approximate Ramsey property since the isometry group of a separable Hilbert space is (a Lévy group and thus) extremely amenable; see [385, Section 2.2]. More information on extreme amenability and approximate Ramsey properties can be found in [385] and more examples of Fraïssé classes in functional analysis in [342].

Sources

This chapter's blueprints were drawn in [80; 75; 116], which, in turn, are based on ideas of [310; 183]. The spaces underpinning the chapter are very classical objects in Banach space theory. In [203] Gurariy constructed the space that bears his name, coined the term AUD and proved Proposition 6.2.10 and, in particular, that any two separable AUD Banach spaces are *almost* isometric. The prefix was eliminated by Lusky in [343], a fine paper (which goes without saying when talking about Lusky's papers) which contains the additional result that the isometry group of G acts transitively on the set of smooth points of the unit sphere. More information about G and related constructions can be found in [22, Section 3.4]. A new proof of the uniqueness of Gurariy space was given by Kubiś and Solecki in [310]: the proof basically consists in showing that any separable AUD Banach space is the Fraïssé limit of the class of finite-dimensional spaces and isometries. Given the potential target of their paper, a tactical move was not to pronounce the word 'Fraïssé'. The paper contains the Banach ancestor of the key Lemma 6.2.5 and has (perhaps shared with [308]) the unquestionable merit of introducing Fraïssé structures into the Banach space business. The construction of G_p in Section 6.2 just transplants Kubiś and Solecki's ideas to the soil of p-Banach spaces; the presentation in [80] is more akin to [22, Chapter 3] and uses the Device. A forerunner of G_p appears in [248, Theorem 4.3]. The construction of K_p, taken from the 'related issues' of [75], is an adaptation for quasi-Banach spaces of Garbulińska-Węgrzyn's [183], where the idea of regarding spaces of Kadec type as Fraïssé structures appears for the first time and property [$\Game$] is introduced as property (E). Categories of embedding-projection pairs had been defined and exploited in [308, Section 6]. Proposition 6.3.13 is the [$\Game$] version of [116, Theorem 4.1]; there, it is shown that if X is a Banach space with separable dual then the output $\Game(X)$ is a Banach space with the additional property that given contractive pairs

$u\colon E \rightleftarrows X$ and $v\colon E \rightleftarrows F$, where F is a finite-dimensional normed space, for every $\varepsilon > 0$, there exists a pair $w\colon F \rightleftarrows X$ such that $\|u - w \circ v\| < \varepsilon$ and $\|w\| < 1 + \varepsilon$. This property was called 'almost universal complemented disposition' in [116]. We are not as convinced today that it deserves that name, mainly because, as mentioned before, we cannot ensure it is equivalent to AUCD. The topic of complementably universal spaces for a class $\mathscr{A}$ (spaces in the class containing complemented copies of every space in $\mathscr{A}$) emerges in 1969 when Pełczyński [381] exhibits his celebrated 'universal basis' space: a complementably universal space for the class of Banach spaces with basis. In 1971, Kadec [240] obtains the first complementably universal member of the class of separable Banach spaces with the BAP. Back to back with it, the next article in the same issue of *Studia* is from Pełczyński and Wojtaszczyk [384] and shows the existence of a complementably universal space for FDD, necessarily isomorphic to Kadec's. Still in the same volume, Pełczyński proved [382] that every Banach space with the BAP is complemented in a space with a basis, thus making it clear that his own universal space was complementably universal for the BAP and thus isomorphic to Kadec space. Kalton (who else?) performs in [247] a study of universal and complementably universal F-spaces and mentions the existence a complementably universal p-Banach space for the BAP for fixed $0 < p < 1$. He just adds that 'it is easy to duplicate the results for Banach spaces'. It is clear from [247, Theorem 4.1 (b) and Corollary 7.2] that Kalton is alluding to Pełczyński's universal space. He concludes by remarking that 'there are a number of other existence and non-existence results known for other classes of separable spaces'. From the Pełczyński decomposition method, it follows that two separable complementably universal p-Banach spaces for the BAP are isomorphic, and thus it turns out that K_p is isomorphic to Kalton's space, while K_1 is just a renorming of the spaces of Pełczyński, Kadec and Pełczyński and Wojtaszczyk. Several questions can be posed about those spaces, but two especially burning ones are: Does Kadec's space have property [ᘐ] in its own norm? Are the isometry groups of the spaces K_p extremely (or otherwise) amenable in the SOT? It cannot go unmentioned that no separable complementably universal space exists for the class of separable Banach spaces [233]. Universal operators date back to Rota's celebrated 'model operator' on Hilbert space. The material of Section 6.4 is taken from [80]. The category **H** is a typical *slice* category; see [27, Section 1.6, Example 4]. Theorem 6.4.5 subsumes several results scattered in the literature.