Canad. J. Math. Vol. **58** (6), 2006 pp. 1144–1202

Partial $*$-Automorphisms, Normalizers, and Submodules in Monotone Complete C^*-Algebras

Masamichi Hamana

Abstract. For monotone complete C^*-algebras $A \subset B$ with A contained in B as a monotone closed C^*-subalgebra, the relation $X = AsA$ gives a bijection between the set of all monotone closed linear subspaces X of B such that $AX + XA \subset X$ and $XX^* + X^*X \subset A$ and a set of certain partial isometries s in the "normalizer" of A in B, and similarly for the map $s \mapsto \operatorname{Ad} s$ between the latter set and a set of certain "partial $*$-automorphisms" of A. We introduce natural inverse semigroup structures in the set of such X's and the set of partial $*$-automorphisms of A, modulo a certain relation, so that the composition of these maps induces an inverse semigroup homomorphism between them. For a large enough B the homomorphism becomes surjective and all the partial $*$-automorphisms of A are realized via partial isometries in B. In particular, the inverse semigroup associated with a type II_1 von Neumann factor, modulo the outer automorphism group, can be viewed as the fundamental group of the factor. We also consider the C^*-algebra version of these results.

1 Introduction

The notion of (Murray–von Neumann) equivalence of projections and that of Rieffel's (strong) Morita equivalence in the theory of operator algebras have a certain similarity, and the latter may be regarded as an immediate generalization of the former in the following sense. Two C^*-algebras A_1 and A_2 are strongly Morita equivalent if and only if there exist a C^*-algebra B containing A_1 and A_2 as C^*-subalgebras and a norm closed linear subspace X of B such that

$$XX^*X \subset X, \quad K_l(X) := \overline{\operatorname{lin}} XX^* = A_1, \quad K_r(X) := \overline{\operatorname{lin}} X^*X = A_2,$$

where we write $X^* := \{x^* : x \in X\}$, $XY := \{xy : x \in X, y \in Y\}$ for $X, Y \subset B$, and $\overline{\operatorname{lin}}$ denotes the norm closed linear span. Similarly two von Neumann algebras A_1 and A_2 are Morita equivalent if and only if there exist a von Neumann algebra B containing A_1 and A_2 as von Neumann subalgebras and a σ-weakly closed linear subspace X of B such that

$$XX^*X \subset X, \quad M_l(X) := \overline{\operatorname{lin}}^{\sigma} XX^* = A_1, \quad M_r(X) := \overline{\operatorname{lin}}^{\sigma} X^*X = A_2,$$

where $\overline{\operatorname{lin}}^{\sigma}$ denotes the σ-weakly closed linear span. We call a norm closed linear subspace X satisfying $XX^*X \subset X$ as above a *triple subsystem* of B following Youngson [35], and note that the condition $XX^*X \subset X$ implies necessarily $K_l(X)$ and $K_r(X)$

Received by the editors June 12, 2003; revised August 23, 2005.
AMS subject classification: Primary: 46L05, 46L08, 46L40; secondary: 20M18.
©Canadian Mathematical Society 2006.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(resp., $M_l(X)$ and $M_r(X)$) being C^*-subalgebras (resp., von Neumann subalgebras) of B. (Although these descriptions of (strong) Morita equivalences seem to be slightly nonstandard, the equivalences to the usual ones follow from the well-known linking C^*-algebra technique [3]. Some authors call our "triple system" a "ternary ring of operators" following Zettl [36].) If A_1 and A_2 are both 1-dimensional, then $A_j = \mathbb{C}p_j$, $j = 1, 2$, and $X = \mathbb{C}x$ for some projections p_j and partial isometry x such that $p_1 = xx^*$ and $p_2 = x^*x$. This precisely means that the projections p_1 and p_2 are equivalent in the C^*-algebra B.

Moreover, one notes a rather formal resemblance of "projection" and "C^*-subalgebra" with that of "partial isometry" and "triple subsystem". That is, for a norm closed linear subspace X of some C^*-algebra B, the condition of X being a C^*-subalgebra is written as $X^2 \subset X$ (X is closed under multiplication) and $X^* = X$ (X is self-adjoint), which resembles the defining property $p^2 = p = p^*$ of a projection p. The condition of X being a triple subsystem, *i.e.,* $XX^*X \subset X$ resembles the defining property $xx^*x = x$ of a partial isometry x [1, p. 5].

The main results of this paper show that these resemblances are not superficial, but reflect a much more down-to-earth relationship in a certain situation. Let A and B be von Neumann algebras with A contained in B as a von Neumann subalgebra, and consider a σ-weakly closed linear subspace X of B such that

$$(*) \qquad AX + XA \subset X, \quad XX^* + X^*X \subset A.$$

This condition for X means precisely that $\overline{\text{lin}}^{\sigma} XX^*$ and $\overline{\text{lin}}^{\sigma} X^*X$ are σ-weakly closed two-sided ideals (direct summands) of A and X is a σ-weakly closed triple subsystem of B which implements the Morita equivalence of them. Then such an X is written in the form $X = AsA$ for some partial isometry s in B. We characterize such partial isometries s as elements of the "normalizer" of A in B and relate them to "partial $*$-automorphisms" of A by sending s to $\operatorname{Ad} s\colon s^*sAs^*s \to ss^*Ass^*$, $x \mapsto (\operatorname{Ad} s)(x) := sxs^*$. Here we mean by the *normalizer* of A in B, the set $N_B(A) := \{x \in B : xAx^* \subset A, x^*Ax \subset A\}$, and by a *partial $*$-automorphism* of A, we mean a $*$-isomorphism between two reduced subalgebras eAe and fAf for some projections e and f of A. We write $\operatorname{PAut} A$ for the set of all partial $*$-automorphisms of A.

These X arise naturally in relation to a coaction of a discrete group on a von Neumann algebra. Our original motivation of this work was the analysis of such a coaction, rather than Morita equivalence. The actual study of coactions will be done in a subsequent paper [14]. But we briefly mention it to see how the results in the present paper are useful in such a study. Let A be a von Neumann algebra and G a discrete group. Our objective is to describe the von Neumann algebra B with a coaction β of G with fixed-point subalgebra $A = B^\beta := \{x \in B : \beta(x) = x \otimes 1\}$ as a sort of a "twisted crossed product" of A by G. Here a coaction of G on B, see [11, 22], is a unital normal $*$-monomorphism β of B into the von Neumann tensor product $B \mathbin{\overline{\otimes}} R(G)$ such that $(\beta \mathbin{\overline{\otimes}} \mathrm{id}_{R(G)}) \circ \beta = (\mathrm{id}_B \mathbin{\overline{\otimes}} \delta_G) \circ \beta$, $R(G)$ is the von Neumann algebra on $l^2(G)$ generated by the image of the right regular representation ρ of G, $\delta_G\colon R(G) \to R(G) \mathbin{\overline{\otimes}} R(G)$ is the normal $*$-monomorphism defined by $\delta_G(\rho(g)) = \rho(g) \otimes \rho(g)$, and id's denote the identity maps. Then B is the σ-weak closure of its $*$-subalgebra $\sum_{g \in G} X_g$, where $X_g := \{x \in B : \beta(x) = x \otimes \rho(g)\}$, which is a direct sum of the X_g's and is G-graded

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

in the sense that for all $g_1, g_2, g \in G$,

$$X_{g_1}X_{g_2} \subset X_{g_1g_2}, \quad X_g^* = X_{g^{-1}};$$

and each X_g is a σ-weakly closed linear subspace of B satisfying $(*)$ above, since β is normal, $X_e = A$ (e is the unit element of G), $X_gX_g^* = X_gX_{g^{-1}} \subset X_e$, etc. It follows from the foregoing that $X_g = As_gA$ for some $s_g \in N_B(A)$ and $\theta_g := \operatorname{Ad} s_g \in \operatorname{PAut} A$. Under a technical assumption on A and a special choice of the s_g, it turns out that B is recovered as a sort of the "twisted crossed product" $A \rtimes_{\theta,u} G$ from the pair (θ, u), a "twisted action" of G on A, where $\theta\colon G \to \operatorname{PAut} A$, $g \mapsto \theta_g$, $u\colon G \times G \to \operatorname{PI} A$, $(g_1, g_2) \mapsto u(g_1, g_2) := s_{g_1}s_{g_2}s_{g_1g_2}^*$, and $\operatorname{PI} A$ denotes the set of all partial isometries of A. This description is very natural, since the usual crossed product $B = A \rtimes_\theta G$ of A by G with respect to an action θ has a coaction of G with the decomposition as above given by $X_g = As_g = s_gA$, where the s_g are unitaries in B implementing the $*$-automorphisms θ_g.

In what follows and in [14], we work with monotone complete C^*-algebras, rather than von Neumann algebras in terms of which the results were stated so far. The readers who are interested only in von Neumann algebras may regard monotone complete C^*-algebras in the text as von Neumann algebras (though such a restriction does not simplify the arguments that follow). Indeed, suppose a monotone complete C^*-algebra under the consideration below happens to be a von Neumann algebra. Then, by [17], its $*$-subalgebra is monotone closed (*i.e.,* closed under the order-convergence, [10, 18]) if and only if the $*$-subalgebra is σ-weakly closed, and the same is true for a linear subspace X such that $XX^*X \subset X$, a triple subsystem (see the remark before Theorem 6.6). Since all linear subspaces we consider here are triple subsystems, we may replace, in such a situation, "monotone closed", "generated as a monotone complete C^*-algebra", by "σ-weakly closed", "generated as a von Neumann algebra", respectively.

There now follows the definition of a monotone complete C^*-algebra and related comments. We call a C^*-algebra *monotone complete* if every bounded increasing net in the self-adjoint part has a supremum with respect to the partial order. Hence, it is an AW^*-algebra in the sense of Kaplansky (see [1, 19]), and we can speak of its type as an AW^*-algebra. The difference between monotone completeness and AW^*-ness is not known, though concrete AW^*-algebras known so far are all monotone complete. Moreover, every von Neumann algebra is monotone complete, but *not* vice versa, and Tomiyama showed that every injective C^*-algebra is monotone complete [33, Theorem 7.1]. The examples of Dixmier [5] provide commutative non-W^* (not von Neumann), AW^*-algebras. Takenouchi and Dyer independently showed the existence of non-W^*, AW^*-factors (AW^*-algebras with trivial center). Moreover, monotone complete non-W^*, AW^*-factors outnumber von Neumann factors. Indeed, for an $\aleph_0$-dimensional Hilbert space H (the $*$-isomorphism classes of) von Neumann factors on H have cardinality $c = 2^{\aleph_0}$. However, simple, monotone complete, non-W^*, AW^*-factors which are completely isometrically embedded in $B(H)$ have cardinality 2^c ($c < 2^c!$). (See [13]; the completely isometric embeddings cannot be replaced by $*$-*homomorphisms*, since a $*$-homomorphism of a simple AW^*-factor into $B(H)$ is injective and by [34] an AW^*-factor which acts faithfully on H as a C^*-algebra is necessarily a von Neumann algebra.)

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Now let A and B be monotone complete C^*-algebras with A contained in B as a monotone closed C^*-subalgebra. Then, as stated above, the set of monotone closed linear subspaces X of B satisfying $(*)$ corresponds to a subset of $N_B(A)$, the normalizer of A in B, and $N_B(A)$ corresponds to a subset of PAut A, the partial $*$-automorphisms of A. The composition of these correspondences, up to some equivalences, induces an inverse semigroup homomophism when we introduce inverse semigroup structures in these sets. Moreover, if we fix A and take B to be large enough, then the homomorphism becomes surjective.

The key observation for establishing these assertions is that X as above, when A is fixed and B varies, has an intrinsic characterization as a "self-dual invertible A-module" or the A-bimodule associated with a "regular" partial $*$-automorphism of A (see Theorem 6.6, Proposition 9.1).

If we consider C^*-algebras $A \subset B$ instead of monotone complete C^*-algebras as above, then a similar reasoning proceeds, to some extent, and the notion of an invertible A-module (see Definition 5.1) gives, in turn, an intrinsic characterization of a subspace $X \subset B$ as above in the C^*-situation (see the last two sentences of Section 9). Here the adjective "invertible" is attached to mean that an invertible A-module for a C^*-algebra A or a self-dual invertible A-module for a monotone complete C^*-algebra A is embedded as an element of a certain inverse semigroup associated with A, called the Picard semigroup of A (see Theorems 5.2, 6.5, 6.17).

The paper is arranged as follows. Sections 2–4 are devoted to purely algebraic preliminaries for later use. In Sections 3–4, we consider AW^*-algebras; however, the proofs there use the properties (WSB), (ELCP), together with the existence of central covers of projections, rather than the full strength of AW^*-ness. In Section 5, we define the Picard semigroup of a C^*-algebra A as a direct generalization of the Picard group of A in [3]. In Sections 6–8, we consider a fixed monotone complete C^*-algebra A. In Section 6, we study the Picard semigroup of A, *i.e.,* the set of all the isomorphism classes of self-dual invertible A-modules or the set $\{\text{PAut}\, A\}$ of certain equivalence classes in PAut A; the main purpose is to describe it in terms of the composition in PAut A (see Theorem 6.17). In Section 7, we investigate the reduction, by a direct sum decomposition of A, of the study of PAut A and hence that of the coaction of a discrete group on a monotone complete C^*-algebra with fixed-point subalgebra A. We point out here that if A is σ-finite, then the rather involved arguments in Sections 3, 4, and 6 can be much simplified. Indeed, see the last but one paragraph of Section 9; one can prove Corollary 6.19 and Proposition 7.4(i) cited there independently of other results in Sections 6–7. The reader may notice the resemblance between our inverse semigroup $\{\text{PAut}\, A\}$ and the fundamental group of a von Neumann factor of type II_1 (see [21]). In Section 8, we confirm it by defining, for any finite A, a homomorphism from $\{\text{PAut}\, A\}$ into another inverse semigroup, whose image may be thought of as a generalization of the fundamental group. In Section 9, for any monotone complete A, we realize PAut A via $N_B(A)$ for large enough monotone complete B with $A \subset B$, and we also consider the C^*-version. Note that our main technical tool in the arguments of Sections 6–8 is the comparability theorem for projections in an AW^*-algebra (see Theorem 6.6) and that the reason we consider monotone complete C^*-algebras rather than AW^*-algebras there is the validity of the structure theorem of self-dual Hilbert modules over monotone complete

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

C^*-algebras (see Remark 6.1(ii)).

We refer to the literature related to the present work, with only very selective citation. The following result in Takesaki [31] may be viewed as giving a decomposition of a von Neumann algebra associated with a coaction of $\mathbb{Z}$ as stated above: A von Neumann algebra B with a certain periodic action of $\mathbb{R}$ (so essentially an action of the 1-dimensional torus $\mathbb{T}$) is a sort of a crossed product of the fixed-point subalgebra A by $\mathbb{Z} = \widehat{\mathbb{T}}$, *i.e.,* B is generated by a $\mathbb{Z}$-graded $*$-subalgebra $\sum_{n\in\mathbb{Z}} X_n$, where $X_n = Au^n$, $n \geq 0$, $X_n = (u^*)^{-n}A$, $n < 0$, and u is an isometry in $N_B(A)$. The Cuntz algebra [4] may be viewed as a C^*-version of B in Takesaki's situation in which A is a UHF C^*-algebra (see also [24]). The notion of "normalizer" in operator algebras has appeared in many papers, and the origin perhaps dates back to Dixmier [6]. We cite only Power [26]; the term "partial isometry normalizer" (see Definition 4.1) is his. However, its relation to submodules and partial $*$-automorphisms shown here seems to have been overlooked. The interplay between inverse semigroups and C^*-algebras is treated in the monograph [24], a series of papers by Exel [7–9], and others (see the bibliography of [24]), which seem to be related to, but, not to overlap with the present paper (see Sections 5, 9). At this point we explain the difference in the terminology and the viewpoint between these authors and ours, since our main interest here is in the monotone complete C^*-case, and we will not touch on it in the text below. The term "partial automorphism" in [7] is used in the C^*-context to mean a triple (θ, I, J) of closed two-sided ideals I, J of A and a $*$-isomorphism $\theta\colon I \to J$ for a fixed C^*-algebra A, and our "partial $*$-automorphism" is used only in the monotone complete C^*-context. Hence the former may be viewed as our invertible A-module for the C^*-algebra A, which is essentially the same as an I-J-imprimitivity bimodule for some closed two-sided ideals I, J of A in the sense of Rieffel (see Proposition 5.5).[1]

2 Algebraic Invertible Modules Over a $*$-Algebra

In this section we consider certain A-bimodules over a $*$-algebra A. Here A satisfies the following properties (ND is short for "Non-Degenerate"):

(ND) $\forall$ two-sided $*$-ideal I of A : $a \in I, aI = 0 \Longrightarrow a = 0,$

(ND′) $\forall$ two-sided $*$-ideal I of A : $a \in I, Ia = 0 \Longrightarrow a = 0;$

and a $*$-*algebra* is an associative algebra A over the complex number field $\mathbb{C}$, together with a map $A \to A$, $x \mapsto x^*$, called an *involution*, such that $(\lambda x + \mu y)^* = \overline{\lambda}x^* + \overline{\mu}y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for $\lambda, \mu \in \mathbb{C}$ and $x, y \in A$.

These properties are equivalent, since $a \in I$ and $Ia = 0$ imply $a^* \in I$ and $a^*I = 0$. Note further that for every two-sided $*$-ideal I of A,

$$\{a \in A : aI = 0\} = \{a \in A : Ia = 0\} =: I^\perp,$$

that this is also a two-sided $*$-ideal of A, and that

$$I \cap I^\perp = \{0\}. \tag{2.1}$$

[1] Part of the work described here was announced at a conference held at Hokkaido University, Sapporo, November 27–29, 1995.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Indeed, it suffices to show that $Ia = 0$ implies $aI = 0$. But, if $Ia = 0$, then $Ia^* \subset I$, $(Ia^*)I = I(Ia)^* = 0$, and so $aI = (Ia^*)^* = 0$ by (ND).

Note also that (ND) is true if the involution in A is *proper* [1, p. 10], *i.e.,* if $x^*x = 0$ implies $x = 0$ for all $x \in A$.

An *inner product A-module* is a left A-module X equipped with a non-degenerate sesquilinear map $\langle \cdot , \cdot \rangle : X \times X \to A$, called the *inner product* of X, *i.e.,* a map satisfying

$$\langle x, y\rangle = 0,\ \forall y \in X \implies x = 0; \tag{2.2}$$

$$\langle a_1 \cdot x_1 + a_2 \cdot x_2, y\rangle = a_1\langle x_1, y\rangle + a_2\langle x_2, y\rangle, \forall a_j \in A, \forall x_j, y \in X; \tag{2.3}$$

$$\langle x, y\rangle^* = \langle y, x\rangle,\ \forall x, y \in X. \tag{2.4}$$

A possibly degenerate sesquilinear map is called a *pre-inner product*.

For an inner product A-module X denote by $\mathrm{End}_A(X)$ the $*$-algebra of all module endomorphisms T of X with *adjoint* T^* (*i.e.,* $\langle xT^*, y\rangle = \langle x, yT\rangle$ for all $x, y \in X$), and by $F(X)$, its two-sided $*$-ideal of all finite-rank operators, *i.e.,* the linear span of rank-1 operators $\langle \cdot , x\rangle \cdot y : z \mapsto \langle z, x\rangle \cdot y$, $x, y \in X$. Note here that we apply a module map on the right of elements, since we are treating left modules, that the uniqueness of the adjoint, if it exists, is assured by (2.2), and that $\big(\langle \cdot , x\rangle \cdot y\big)^* = \langle \cdot , y\rangle \cdot x$.

Definition 2.1 An *algebraic invertible A-module* is a pair (X, θ) of an inner product A-module X and a $*$-homomorphism $\theta : A \to \mathrm{End}_A(X)$ such that

$$F(X) \subset \theta((\mathrm{Ker}\,\theta)^{\perp}). \tag{2.5}$$

Denote by $\mathrm{INV}'(A)$ the set of all algebraic invertible A-modules. Here the adjective "algebraic" is attached to distinguish the setting here and the C^*-algebraic setting later.

We regard X as an A-bimodule by setting $a \cdot x \cdot b = a \cdot x\theta(b)$, and abbreviate (X, θ) to X when there is no fear of confusion. Since $\theta|(\mathrm{Ker}\,\theta)^{\perp} : (\mathrm{Ker}\,\theta)^{\perp} \to \theta((\mathrm{Ker}\,\theta)^{\perp})$ is injective by (2.1), its inverse $\theta^{-1} : \theta((\mathrm{Ker}\,\theta)^{\perp}) \to (\mathrm{Ker}\,\theta)^{\perp}$ makes sense, and two-sided $*$-ideals $F_r(X), F_l(X)$ of A are defined as follows:

$$F_r(X) = \theta^{-1}(F(X)), \quad F_l(X) = \langle X, X\rangle, \tag{2.6}$$

where $\langle X, X\rangle$ is the linear span of $\{\langle x, y\rangle : x, y \in X\}$.

A *monomorphism* between algebraic invertible A-modules is an A-bimodule homomorphism which preserves the inner products, and the term *isomorphism* and *automorphism* have the obvious meaning. A monomorphism is injective by (2.2), and the monomorphic image of an algebraic invertible A-module is also an algebraic invertible A-module.

If (X, θ) is an algebraic invertible A-module and Y is a sub-A-bimodule of X, the restriction to which of the inner product of X is non-degenerate, then (Y, θ_1), with $\theta_1 : A \to \mathrm{End}_A(Y)$ defined by $\theta_1(a) := Y|\theta(a)$ (the restriction of $\theta(a)$ to Y), is also an

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

algebraic invertible A-module, which we call a *submodule* of (X,θ) and write (Y,θ) or Y. Indeed, it suffices to show that $Y|\langle\,\cdot\,,y_1\rangle\cdot y_2\in\theta_1((\operatorname{Ker}\theta_1)^{\perp})$ for $y_j\in Y$. But $a:=\theta^{-1}\big(\langle\,\cdot\,,y_1\rangle\cdot y_2\big)\in(\operatorname{Ker}\theta)^{\perp}$, and if $b\in\operatorname{Ker}\theta_1$, *i.e.*, $y\theta(b)=0$ for all $y\in Y$, then $x\theta(ab)=\langle x,y_1\rangle\cdot y_2\theta(b)=0$ for all $x\in X$, $\theta(ab)=0$, and by (2.1), $ab=0$. Hence $a\in(\operatorname{Ker}\theta_1)^{\perp}$ and $Y|\langle\,\cdot\,,y_1\rangle\cdot y_2=Y|\theta(a)=\theta_1(a)$.

We introduce two operations in the set $\mathrm{INV}'(A)$. First, for $(X,\theta)\in\mathrm{INV}'(A)$ define its *inverse* $(X,\theta)^{-1}$ as follows. Let $X^*:=\{x^*:x\in X\}$ be the linear space equipped with the scalar multiplication, A-bimodule operation and inner product given by

$$\lambda x^*=(\overline{\lambda}x)^*,\quad a\cdot x^*\cdot b=(b^*\cdot x\cdot a^*)^*,$$
$$\langle x^*,y^*\rangle=\theta^{-1}(\langle\,\cdot\,,x\rangle\cdot y)\in(\operatorname{Ker}\theta)^{\perp}\subset A. \tag{2.7}$$

Indeed, it follows from (ND) for $I=F_l(X)$ and (2.2) that $\langle\,\cdot\,,\,\cdot\,\rangle$ in X^* is nondegenerate. To see (2.3) it suffices to show that

$$\langle a\cdot x^*,y^*\rangle=a\langle x^*,y^*\rangle,\ \forall a\in A,\ \forall x,y\in X.$$

But

$$\langle a\cdot x^*,y^*\rangle=\theta^{-1}\big(\langle\,\cdot\,,x\theta(a^*)\rangle\cdot y\big)=\theta^{-1}\big(\langle\,\cdot\,\theta(a),x\rangle\cdot y\big)=\theta^{-1}\big(\theta(a)\big(\langle\,\cdot\,,x\rangle\cdot y\big)\big).$$

The substitution of ba, with $a\in A$ and $b\in(\operatorname{Ker}\theta)^{\perp}$, into a shows, in view of $ba\in(\operatorname{Ker}\theta)^{\perp}$ and so $\theta^{-1}(\theta(ba))=ba$, that

$$b\langle a\cdot x^*,y^*\rangle=\langle b\cdot(a\cdot x^*),y^*\rangle=\langle(ba)\cdot x^*,y^*\rangle=ba\langle x^*,y^*\rangle,$$

and the desired equality follows from (ND$'$). Then a $*$-homomorphism $\theta_{-1}\colon A\to\operatorname{End}_A(X^*)$ is defined by $x^*\theta_{-1}(a)=x^*\cdot a=(a^*\cdot x)^*$, and it follows that

$$\operatorname{Ker}\theta_{-1}=(F_l(X))^{\perp},$$

since $x^*\theta_{-1}(a)=0$, *i.e.*, $a^*\cdot x=0$ for all $x\in X$ if and only if

$$\langle y,x\rangle a=(a^*\langle x,y\rangle)^*=\langle y,a^*\cdot x\rangle=0,\ \forall x,y\in X,$$

and that

$$F(X^*)=\theta_{-1}(F_l(X))\subset\theta_{-1}((F_l(X))^{\perp\perp})=\theta_{-1}((\operatorname{Ker}\theta_{-1})^{\perp}),$$

since

$$\begin{aligned}z^*\theta_{-1}\big(\langle x,y\rangle\big)&=\big(\langle y,x\rangle\cdot z\big)^*=\big[y\theta\big(\theta^{-1}\big(\langle\,\cdot\,,x\rangle\cdot z\big)\big)\big]^*\\&=\big(y\theta\big(\langle x^*,z^*\rangle\big)\big)^*=\langle z^*,x^*\rangle\cdot y^*\end{aligned}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

and so

$$\theta_{-1}\big(\langle x, y\rangle\big) = \langle\,\cdot\,, x^*\rangle \cdot y^*, \ \forall x, y \in X. \tag{2.8}$$

Thus $X^{-1} = (X,\theta)^{-1} := (X^*, \theta_{-1})$ is an invertible A-module, and we call the map $(X,\theta) \to (X,\theta)^{-1}$, $x \mapsto x^*$, the *involution* of (X,θ).

It follows from (2) and (2.8) that

$$F_l(X^{-1}) = \langle X^*, X^*\rangle = \theta^{-1}(F(X)) = F_r(X), \tag{2.9}$$

$$F_r(X^{-1}) = (\theta_{-1})^{-1}(F(X^*)) = F_l(X). \tag{2.10}$$

So $F_l((X^{-1})^{-1}) = F_l(X)$, and we have

$$((X,\theta)^{-1})^{-1} = (X,\theta). \tag{2.11}$$

Indeed, for $x, y \in X$,

$$\langle (x^*)^*, (y^*)^*\rangle = (\theta_{-1})^{-1}\big(\langle\,\cdot\,, x^*\rangle \cdot y^*\big) =: a \in F_l((X^{-1})^{-1}) = F_l(X)$$

if and only if

$$\begin{aligned}(a^* \cdot z)^* = z^*\theta_{-1}(a) = \langle z^*, x^*\rangle \cdot y^* &= \theta^{-1}(\langle\,\cdot\,, z\rangle \cdot x) \cdot y^* \\ &= \big[y\theta\big(\theta^{-1}\big(\langle\,\cdot\,, x\rangle \cdot z\big)\big)\big]^* = (\langle y, x\rangle \cdot z)^*\end{aligned}$$

for all $z \in X$, and so $a = \langle x, y\rangle$ by (ND).

Remark 2.2

(i) If $\tau\colon (X,\theta) \to (Y,\psi)$ is a monomorphism between algebraic invertible A-modules, then a monomorphism $\tau^*\colon (X,\theta)^{-1} \to (Y,\psi)^{-1}$, called the *adjoint* of τ, is defined by $\tau^*(x^*) = \tau(x)^*$ for $x \in X$, so that $(\tau^*)^* = \tau$. Indeed, since the image of τ is an algebraic invertible A-module, we may assume τ to be surjective. Then for $x, y, z \in X$,

$$\begin{aligned}\tau(z)\psi\big(\langle x^*, y^*\rangle\big) &= \tau\big(z\theta\big(\langle x^*, y^*\rangle\big)\big) = \tau\big(\langle z, x\rangle \cdot y\big) \\ &= \langle z, x\rangle \cdot \tau(y) = \langle \tau(z), \tau(x)\rangle \cdot \tau(y)\end{aligned}$$

and so

$$\psi\big(\langle x^*, y^*\rangle\big) = \langle\,\cdot\,, \tau(x)\rangle \cdot \tau(y),$$

$$\langle x^*, y^*\rangle = \psi^{-1}\big(\langle\,\cdot\,, \tau(x)\rangle \cdot \tau(y)\big) = \langle \tau(x)^*, \tau(y)^*\rangle = \langle \tau^*(x^*), \tau^*(y^*)\rangle,$$

since $\langle x^*, y^*\rangle \in (\operatorname{Ker}\theta)^\perp = (\operatorname{Ker}\psi)^\perp$.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(ii) Let B be a $*$-algebra containing A as a $*$-subalgebra, regard it as an A-bimodule, and let $X \subset B$ be a sub-A-bimodule satisfying

(2.12) $$XX^* + X^*X \subset A;$$

(2.13) the A-valued pre-inner products of X and X^* defined by $\langle x, y\rangle = xy^*$ and $\langle x^*, y^*\rangle = x^*y,\ x, y \in X$, are non-degenerate,

where XX^*, denote the linear spans of $\{xy^* : x, y \in X\}$, Then, regarding the left multiplication by elements of A as a module operation and the right one as a *-representation of A, we obtain algebraic invertible A-modules X and X^*, so that $X^{-1} = X^*$ and the involution of X is the restricton to X of the involution of B. Indeed, (2.12) and (2.13) show that X and X^* are inner product A-modules and that $*$-homomorphisms $\theta\colon A \to \mathrm{End}_A(X)$ and $\theta_{-1}\colon A \to \mathrm{End}_A(X^*)$ are defined by $x\theta(a) = xa$ and $x^*\theta_{-1}(a) = x^*a$. If $a \in \operatorname{Ker}\theta$, then $(x^*y)a = x^*(ya) = 0$ for all $x, y \in X$. Hence $X^*X \subset (\operatorname{Ker}\theta)^\perp$. But

$$\langle\,\cdot\,, x\rangle \cdot y = \theta(x^*y), \quad F(X) = \theta(X^*X) \subset \theta((\operatorname{Ker}\theta)^\perp),$$

and, by symmetry, $F(X^*) \subset \theta_{-1}((\operatorname{Ker}\theta_{-1})^\perp)$.

(iii) In the situation (ii) above take $B = A$. Then each two-sided $*$-ideal of A is an algebraic invertible A-module, since the property (ND) means precisely that its pre-inner product, as above, is non-degenerate.

We define another operation $\odot_A$, called *product*, in $\mathrm{INV}'(A)$ as follows. For $(X_j, \theta_j) \in \mathrm{INV}'(A)$, $j = 1, 2$, denote by $X_1 \odot_{\theta_1} X_2$ the algebraic tensor product, $X_1 \odot_{\mathbb{C}} X_2$, with the module operation and pre-inner product

(2.14) $$a \cdot (x_1 \otimes x_2) = (a \cdot x_1) \otimes x_2,$$

(2.15) $$\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1\theta_1\big(\langle x_2, y_2\rangle\big), y_1\rangle,$$

divided by the submodule $\{u : \langle u, v\rangle = 0,\ \forall v\}$. For simplicity we use the same notation $x_1 \otimes x_2$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ to denote also the image in $X_1 \odot_{\theta_1} X_2$ of $x_1 \otimes x_2 \in X_1 \odot_{\mathbb{C}} X_2$ and the inner product in $X_1 \odot_{\theta_1} X_2$ induced from the pre-inner product, so that

(2.16) $$x_1 \otimes a \cdot x_2 = x_1\theta_1(a) \otimes x_2 = x_1 \cdot a \otimes x_2 \text{ in } X_1 \odot_{\theta_1} X_2.$$

Then $(X_1 \odot_{\theta_1} X_2, \widetilde{\theta_2})$ with $\widetilde{\theta_2}\colon A \to \mathrm{End}_A(X \odot_{\theta_1} X_2)$ defined by

$$(x_1 \otimes x_2)\widetilde{\theta_2}(a) = x_1 \otimes (x_2\theta_2(a))$$

is an algebraic invertible A-module, written $(X_1, \theta_1) \odot_A (X_2, \theta_2)$ or $X_1 \odot_A X_2$ and called the *product* of (X_j, θ_j), $j = 1, 2$. Indeed, for $x_j, y_j, z_j \in X_j$ it follows from (2.14)–(2.16) that

$$\begin{aligned}\langle z_1 \otimes z_2, x_2 \otimes x_2\rangle \cdot y_1 &= \langle z_1\theta_1\big(\langle z_2, x_2\rangle\big), x_1\rangle \cdot y_1\\ &= z_1\theta_1\big[\langle z_2, x_2\rangle\theta_1^{-1}\big(\langle\cdot, x_1\rangle \cdot y_1\big)\big],\end{aligned}$$

and

$$(2.17)\qquad \langle z_1\otimes z_2, x_1\otimes x_2\rangle\cdot(y_1\otimes y_2)$$
$$= z_1\theta_1\big[\langle z_2,x_2\rangle\theta_1^{-1}\big(\langle\cdot,x_1\rangle\cdot y_1\big)\big]\otimes y_2$$
$$= z_1\otimes\big[\langle z_2,x_2\rangle\theta_1^{-1}\big(\langle\cdot,x_1\rangle\cdot y_1\big)\big]\cdot y_2$$
$$= z_1\otimes z_2\theta_2\big(\theta_2^{-1}\big[\langle\cdot,x_2\rangle\cdot\big(\theta_1^{-1}\big(\langle\cdot,x_1\rangle\cdot y_1\big)\big)\cdot y_2\big]\big)$$
$$= (z_1\otimes z_2)\widetilde{\theta}_2\big(\theta_2^{-1}\big[\langle\cdot,x_2\rangle\cdot\big(\theta_1^{-1}\big(\langle\cdot,x_1\rangle\cdot y_1\big)\big)\cdot y_2\big]\big).$$

Hence it suffices to show that

$$(2.18)\qquad a:=\theta_2^{-1}\big[\langle\,\cdot\,,x_2\rangle\cdot\big(\theta_1^{-1}\big(\langle\,\cdot\,,x_1\rangle\cdot y_1\big)\big)\cdot y_2\big]\in(\operatorname{Ker}\widetilde{\theta}_2)^{\perp}.$$

We have $a\in\theta_2^{-1}(F(X_2))\subset(\operatorname{Ker}\theta_2)^{\perp}$, and $b\in\operatorname{Ker}\widetilde{\theta}_2$ if and only if

$$0=\langle(z_1\otimes z_2)\widetilde{\theta}_2(b), w_1\otimes w_2\rangle=\langle z_1\theta_1\big(\langle z_2\theta_2(b),w_2\rangle\big),w_1\rangle$$

for all z_j, w_j, *i.e.*, $\langle z_2\theta_2(b),w_2\rangle\in\operatorname{Ker}\theta_1$ for all z_2,w_2. Set $c:=\theta_1^{-1}\big(\langle\,\cdot\,,x_1\rangle\cdot y_1\big)\in(\operatorname{Ker}\theta_1)^{\perp}$ and so $a=\theta_2^{-1}\big[\langle\,\cdot\,,x_2\rangle\cdot(c\cdot y_2)\big]$. Then $c\cdot y_2\theta_2(b)=0$, since $\langle c\cdot y_2\theta_2(b),z_2\rangle=c\langle y_2\theta_2(b),z_2\rangle=0$ for all z_2, and so

$$\theta_2(ab)=\langle\,\cdot\,,x_2\rangle\cdot(c\cdot y_2\theta_2(b))=0,\quad ab\in\operatorname{Ker}\theta_2.$$

But, since $a\in(\operatorname{Ker}\theta_2)^{\perp}$ and so $ab\in(\operatorname{Ker}\theta_2)^{\perp}$, it follows from (2.1) that $ab=0$. Thus $a\in(\operatorname{Ker}\widetilde{\theta}_2)^{\perp}$.

In what follows, we regard each two-sided *-ideal of A as an algebraic invertible A-module as in Remark 2.2(iii).

Proposition 2.3 *Let $(X,\theta),(X_j,\theta_j)$ be in $\mathrm{INV}'(A)$.*

(i) *The maps $x\otimes y^*\mapsto\langle x,y\rangle$ and $x^*\otimes y\mapsto\theta^{-1}\big(\langle\,\cdot\,,x\rangle\cdot y\big)$ induce isomorphisms*

$$(X,\theta)\odot_A(X,\theta)^{-1}\cong F_l(X),\quad (X,\theta)^{-1}\odot_A(X,\theta)\cong F_r(X),$$

and for two-sided *-ideals I, J of A we have

$$I\odot_A(X,\theta)\cong(I\cdot X,\theta),\quad (X,\theta)\odot_A J\cong(X\theta(J),\theta),\quad I\odot_A J\cong IJ,$$

where $I\cdot X$, etc., denote the linear span of $\{a\cdot x : a\in I, x\in X\}$, etc.

(ii) *The map $(x_1\otimes x_2)^*\mapsto x_2^*\otimes x_1^*$, $x_j\in X_j$, induces an isomorphism*

$$((X_1,\theta_1)\odot_A(X_2,\theta_2))^{-1}\cong(X_2,\theta_2)^{-1}\odot_A(X_1,\theta_1)^{-1}.$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(iii) *We have*

$$F_r((X_1,\theta_1)\odot_A(X_2,\theta_2))\supset F_r(F_r(X_1,\theta_1)\cdot X_2,\theta_2),$$
$$F_l((X_1,\theta_1)\odot_A(X_2,\theta_2))\supset F_l(X_1\theta_1(F_l(X_2,\theta_2)),\theta_1),$$

and if further $F_r(X_1,\theta_1)=F_r(X_1,\theta_1)^2$ *and* $F_l(X_2,\theta_2)=F_l(X_2,\theta_2)^2$, *then these inclusions become equalities.*

(iv) *If* $\tau_j\colon(X_j,\theta_j)\to(Y_j,\psi_j)$, $j=1,2$, *are monomorphisms, then a monomorphism* $\tau_1\otimes\tau_2\colon(X_1,\theta_1)\odot_A(X_2,\theta_2)\to(Y_1,\psi_1)\odot_A(Y_2,\psi_2)$ *is defined by*

$$(\tau_1\otimes\tau_2)(x_1\otimes x_2)=y_1\otimes y_2.$$

(v) *The operation* $\odot_A$ *is associative in the sense that if* $\tau_j\colon(X_j,\theta_j)\to(Y_j,\psi_j)$, $j=1,2,3$, *are isomorphisms, then we have a natural isomorphism*

$$\begin{aligned}((X_1,\theta_1)\odot_A(X_2,\theta_2))\odot_A(X_3,\theta_3)&=((X_1\odot_{\theta_1}X_2)\odot_{\widetilde{\theta_2}}X_3,\widetilde{\theta_3})\\&\cong\big(X_1\odot_{\theta_1}(X_2\odot_{\theta_2}X_3)\,,\,\widetilde{(\widetilde{\theta_3})}\big)\\&=(X_1,\theta_1)\odot_A((X_2,\theta_2)\odot_A(X_3,\theta_3))\end{aligned}$$

and a similar one for (Y_j,ψ_j), *and with these identifications we have*

$$(\tau_1\otimes\tau_2)\otimes\tau_3=\tau_1\otimes(\tau_2\otimes\tau_3).$$

Proof Most of the proofs follow from direct computation, and we give only some of them.

(i) We have

$$(X,\theta)\odot_A(X,\theta)^{-1}=(X,\theta)\odot_A(X^*,\theta_{-1})=(X\odot_\theta X^*,\widetilde{\theta_{-1}}).$$

Then the map $\tau\colon X\odot_\theta X^*\to F_l(X)$ defined as above is an isomorphism, since for $x,x_j,y,y_j\in X$ and $a,b\in A$,

$$\begin{aligned}\langle x_1\otimes y_1^*,x_2\otimes y_2^*\rangle&=\langle x_1\theta\big(\langle y_1^*,y_2^*\rangle\big)\,,x_2\rangle\\&=\langle x_1\big(\langle\,\cdot\,,y_1\rangle\cdot y_2\big)\,,x_2\rangle\\&=\langle x_1,y_1\rangle\langle x_2,y_2\rangle^*\\&=\langle\tau(x_1\otimes y_1),\tau(x_2\otimes y_2)\rangle,\\ \tau(a\cdot(x\otimes y^*)\cdot b)&=\tau(a\cdot x\otimes(b^*\cdot y)^*)=\langle a\cdot x,b^*\cdot y\rangle\\&=a\langle x,y\rangle b=a\tau(x\otimes y^*)b.\end{aligned}$$

Note that $I\cdot X$ is a submodule of X and similarly for $X\theta(J)=(J\cdot X^*)^*$, since the pre-inner product in $I\cdot X$, *i.e.,* the restriction to $I\cdot X$ of that in X, is non-degenerate

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

by (ND). In $I \odot_A X$ we have for $x, x_j \in I$, $y, y_j \in X$ and $a, b \in A$,

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1 \cdot \langle y_1, y_2 \rangle, x_2 \rangle = x_1 \langle y_1, y_2 \rangle x_2^* \\ = \langle x_1 \cdot y_1, x_2 \cdot y_2 \rangle, \\ a \cdot (x \otimes y) \cdot b = ax \otimes y\theta(b).$$

Thus the map $x \otimes y \mapsto x \cdot y$, $x \in I$, $y \in X$, defines an isomorphism $I \odot_A (X, \theta) \cong (I \cdot X, \theta)$.

(iii) By (2.17) and (2.18) we have for $x_j, y_j \in X_j$, $\langle \cdot , x_1 \otimes x_2 \rangle \cdot (y_1 \otimes y_2) = \widetilde{\theta}_2(a)$,

$$a := \theta_2^{-1}\big(\langle \cdot, x_2 \rangle \cdot \big[\theta_1^{-1}\big(\langle \cdot , x_1 \rangle \cdot y_1\big) \cdot y_2\big]\big) \in (\operatorname{Ker} \widetilde{\theta}_2)^{\perp}, \tag{2.19}$$

and so

$$(\widetilde{\theta}_2)^{-1}\big(\langle \cdot , x_1 \otimes x_2 \rangle \cdot (y_1 \otimes y_2)\big) = \theta_2^{-1}\big(\langle \cdot , x_2 \rangle \cdot [\theta_1^{-1}\big(\langle \cdot, x_1 \rangle \cdot y_1) \cdot y_2]\big). \tag{2.20}$$

The right-hand side of the first inclusion in (iii) is a linear combination of elements of the form (2.19), where $x_2 \in F_r(X_1) \cdot X_2$, $x_1, y_1 \in X_1$, and $y_2 \in X_2$, and so (2.20) shows the inclusion. Suppose $F_r(X_1) = F_r(X_1)^2$. Then $\theta_1^{-1}\big(\langle \cdot , x_1 \rangle \cdot y_1\big)$ in (2.19) is a finite sum of the products bc, $b, c \in F_r(X_1)$, and since $\langle \cdot , x_2 \rangle \cdot ((bc) \cdot y_2) = \langle \cdot , b^* \cdot x_2 \rangle \cdot (c \cdot y_2)$, the inclusion becomes equality. Similarly for the second inclusion. ■

Remark 2.4 Let $A \subset B$ be as in Remark 2.2(ii) and take sub-A-bimodules X_j of B, $j = 1, 2$, satisfying (2.12) and (2.13), so that $X_j \in \mathrm{INV}'(A)$. Then X_1X_2 is also a sub-A-bimodule of B and satisfies (2.12). If, further, it satisfies (2.13) and so $X_1X_2 \in \mathrm{INV}'(A)$, then, as follows immediately from the definition, the map $x_1 \otimes x_2 \mapsto x_1x_2$, $x_j \in X_j$, induces an isomorphism $X_1 \odot_A X_2 \cong X_1X_2$ as algebraic invertible A-modules.

Proposition 2.5 *If $(X_j, \theta_j) \in \mathrm{INV}'(A)$, $j = 1, 2$, satisfy the following conditions*

$$F_r(X_j)F_r(X_k) = \{0\} = F_l(X_j)F_l(X_k), \quad j \neq k, \\ (\operatorname{Ker} \theta_j)^{\perp\perp} = \operatorname{Ker} \theta_j, \quad j = 1, 2,$$

then a direct sum $(X_1 \oplus X_2, \theta) = (X_1, \theta_1) \oplus (X_2, \theta_2)$ in $\mathrm{INV}'(A)$ is defined by

$$a \cdot (x_1 \oplus x_2) \cdot b = a \cdot (x_1 \oplus x_2) \cdot b = a \cdot x_1\theta_1(b) \oplus a \cdot x_2\theta_2(b), \\ \langle x_1 \oplus x_2, y_1 \oplus y_2 \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle.$$

Proof It suffices to check the condition (2.5) for $(X_1 \oplus X_2, \theta)$. Since $F_l(X_j)F_l(X_k) = \{0\}$, $j \neq k$,

$$\langle z_1 \oplus z_2, x_1 \oplus x_2 \rangle \cdot (y_1 \oplus y_2) = \langle z_1, x_1 \rangle \cdot y_1 \oplus \langle z_2, x_2 \rangle \cdot y_2 \\ = \langle z_1 \oplus z_2, x_1 \oplus 0 \rangle \cdot (y_1 \oplus 0) \\ + \langle z_1 \oplus z_2, 0 \oplus x_2 \rangle \cdot (0 \oplus y_2),$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

and so

$$F(X_1 \oplus X_2) = F(X_1 \oplus 0) + F(0 \oplus X_2).$$

For any $(Y, \psi) \in \mathrm{INV}'(A)$ we have $F_r(Y)^{\perp\perp} = [\psi^{-1}(F(Y))]^{\perp\perp} = (\mathrm{Ker}\,\psi)^{\perp}$. Indeed, $[\psi^{-1}(F(Y))]^{\perp\perp} \subset (\mathrm{Ker}\,\psi)^{\perp}$ by (2.5), and

$$[\psi^{-1}(F(Y))]^{\perp}(\mathrm{Ker}\,\psi)^{\perp} + (\mathrm{Ker}\,\psi)^{\perp}[\psi^{-1}(F(Y))]^{\perp} \subset [\psi^{-1}(F(Y))]^{\perp} \cap (\mathrm{Ker}\,\psi)^{\perp} = \{0\},$$

since $a \in [\psi^{-1}(F(Y))]^{\perp} \cap (\mathrm{Ker}\,\psi)^{\perp}$ implies $\psi(a)T = 0$ for all $T \in F(Y)$,

$$0 = \langle x\big(\psi(a)\big(\langle \cdot\,, y\rangle \cdot z\rangle\big)\big), w\rangle = \langle x\psi(a), y\rangle\langle z, w\rangle$$

for all $x, y, z, w \in Y$ and $a = 0$ by (2.1) and (2.2). Thus, $F_r(X_j)F_r(X_k) = \{0\}$, $j \neq k$, implies that $(\mathrm{Ker}\,\theta_j)^{\perp}(\mathrm{Ker}\,\theta_k)^{\perp} = F_r(X_j)^{\perp\perp}F_r(X_k)^{\perp\perp} = \{0\}$ and that

$$(\mathrm{Ker}\,\theta_j)^{\perp} \subset (\mathrm{Ker}\,\theta_k)^{\perp\perp} = \mathrm{Ker}\,\theta_k, \quad j \neq k,$$

by assumption. Hence

$$\theta(a) = \theta_1(a) \oplus 0, \quad a \in (\mathrm{Ker}\,\theta_1)^{\perp}, \quad \theta(a) = 0 \oplus \theta_2(a), \quad a \in (\mathrm{Ker}\,\theta_2)^{\perp},$$

and since $\mathrm{Ker}\,\theta = (\mathrm{Ker}\,\theta_1) \cap (\mathrm{Ker}\,\theta_2)$ and so $(\mathrm{Ker}\,\theta_1)^{\perp} + (\mathrm{Ker}\,\theta_2)^{\perp} \subset (\mathrm{Ker}\,\theta)^{\perp}$, it follows that

$$F(X_1 \oplus X_2) = F(X_1 \oplus 0) + F(0 \oplus X_2) \subset \theta((\mathrm{Ker}\,\theta_1)^{\perp} + (\mathrm{Ker}\,\theta_2)^{\perp}) \subset \theta((\mathrm{Ker}\,\theta)^{\perp}).$$

■

Remark 2.6 If A and $X \in \mathrm{INV}'(A)$ are unital (*i.e.,* A has a unit 1 and $1\cdot x = x = x\cdot 1$ for all $x \in X$) and if h, k are central projections of A, then $X \cdot h, k \cdot X \in \mathrm{INV}'(A)$ and $X \cong X\cdot h \oplus X\cdot(1-h) \cong k\cdot X \oplus (1-k)\cdot X$. (Note that the restriction to $X\cdot h$ of the inner product in X is non-degenerate, since $\langle x \cdot h, y \cdot h\rangle = \langle x \cdot h, y\rangle$ for $x, y \in X$.) Hence, for unital $X, Y \in \mathrm{INV}'(A)$, $X \cong Y$ if and only if $X \cdot h \cong Y \cdot h$, $X \cdot (1-h) \cong Y \cdot (1-h)$ or $k \cdot X \cong k \cdot Y$, $(1-k) \cdot X \cong (1-k) \cdot Y$.

3 Partial $*$-Automorphisms

Throughout this section A denotes a fixed AW^*-algebra. For the rest of the paper, the set of all projections (resp., partial isometries, central elements) of A is written as $\mathrm{Proj}\,A$ (resp., $\mathrm{PI}\,A$, $Z(A)$), so that $\mathrm{Proj}\,Z(A)$ denotes the set of central projections of A. For $e \in \mathrm{Proj}\,A$ denote by $C(e)$ the central cover of e, *i.e.,* the smallest central projection majorizing e.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Definition 3.1 A *reduced subalgebra* of A is a $*$-subalgebra of the form eAe for some $e \in \operatorname{Proj} A$. A *partial $*$-automorphism* of A is a $*$-isomorphism θ between reduced subalgebras eAe and fAf. Write then $r(\theta) := e$, $l(\theta) := f$, so θ is a $*$-isomorphism $r(\theta)Ar(\theta) \to l(\theta)Al(\theta)$, and call these the *right projection* and the *left projection* of θ, respectively. (The adjectives right and left come from the fact that when θ is realized by some partial isometry s of an AW^*-algebra containing A as an AW^*-subalgebra, see Proposition 4.2(iii), the right and left projections of θ are precisely the right and left projections of s.) Denote by $\operatorname{PAut} A$ the set of all partial $*$-automorphisms of A. Call $\theta_1, \theta_2 \in \operatorname{PAut} A$ *orthogonal* if $C(r(\theta_1))C(r(\theta_2)) = 0 = C(l(\theta_1))C(l(\theta_2))$ and so $(r(\theta_1) + r(\theta_2))A(r(\theta_1) + r(\theta_2)) = r(\theta_1)Ar(\theta_1) + r(\theta_2)Ar(\theta_2)$ and similarly for $l(\theta_j)$; define their *direct sum*, $\theta_1 \oplus \theta_2$, in $\operatorname{PAut} A$ by

$$r(\theta_1 \oplus \theta_2) = r(\theta_1) + r(\theta_2), \quad l(\theta_1 \oplus \theta_2) = l(\theta_1) + l(\theta_2),$$
$$(\theta_1 \oplus \theta_2)(a_1 + a_2) = \theta_1(a_1) + \theta_2(a_2), \quad a_j \in r(\theta_j)Ar(\theta_j);$$

and so call θ_j's *direct summands* of $\theta_1 \oplus \theta_2$.

For each $e \in \operatorname{Proj} A$ we have [1, p. 37, Proposition 4]

$$\operatorname{Proj} Z(eAe) = e \operatorname{Proj} Z(A)$$

and so the map $\operatorname{Proj} C(e)Z(A) \to e \operatorname{Proj} Z(A) = \operatorname{Proj} Z(eAe)$, $h \mapsto he$, is an isomorphism as lattices with inverse $f \mapsto C(f)$. Hence for $\theta \in \operatorname{PAut} A$ an isomorphism $\overline{\theta}\colon \operatorname{Proj} C(r(\theta))Z(A) \to \operatorname{Proj} C(l(\theta))Z(A)$ as lattices is defined by

$$\overline{\theta}(h) = C(\theta(hr(\theta))), \quad h \in \operatorname{Proj} C(r(\theta))Z(A), \tag{3.1}$$

so that

$$\theta(hr(\theta)) = \overline{\theta}(h)l(\theta), \quad h \in \operatorname{Proj} C(r(\theta))Z(A). \tag{3.2}$$

Indeed, $\overline{\theta}$ is a composition of the restriction

$$\theta| \operatorname{Proj} Z(r(\theta)Ar(\theta))\colon \operatorname{Proj} Z(r(\theta)Ar(\theta)) \to \operatorname{Proj} Z(l(\theta)Al(\theta))$$

and the isomorphisms $\operatorname{Proj} C(r(\theta))Z(A) \to \operatorname{Proj} Z(r(\theta)Ar(\theta))$, $h \mapsto hr(\theta)$, and $\operatorname{Proj} Z(l(\theta)Al(\theta)) \to \operatorname{Proj} C(l(\theta))Z(A)$, $f \mapsto C(f)$. The isomorphism $\overline{\theta}$ extends canonically to a $*$-isomorphism $C(r(\theta))Z(A) \to C(l(\theta))Z(A)$ as $*$-algebras, which we denote by the same letter $\overline{\theta}$.

For $\theta \in \operatorname{PAut} A$ and $h, k \in \operatorname{Proj} Z(A)$ define $k \cdot \theta \cdot h \in \operatorname{PAut} A$ by

$$r(k \cdot \theta \cdot h) = h\theta^{-1}(kl(\theta)), \quad l(k \cdot \theta \cdot h) = \theta(r(k \cdot \theta \cdot h)) = \theta(hr(\theta))k,$$
$$k \cdot \theta \cdot h = \theta | r(k \cdot \theta \cdot h)Ar(k \cdot \theta \cdot h),$$

and write $\theta \cdot h = 1 \cdot \theta \cdot h$, $k \cdot \theta = k \cdot \theta \cdot 1$. Then

$$\theta \cdot h = \theta \cdot (hC(r(\theta))) = \overline{\theta}(hC(r(\theta))) \cdot \theta, \quad k \cdot \theta = \theta \cdot (\overline{\theta})^{-1}(kC(l(\theta))) \tag{3.3}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

by (3.2), so $k \cdot \theta \cdot h$ is a direct summand of θ, and we have

$$\theta = \theta \cdot h \oplus \theta \cdot (1-h) = k \cdot \theta \oplus (1-k) \cdot \theta,$$
$$(k \cdot \theta \cdot h)^{-1} = h \cdot (\theta^{-1}) \cdot k \in \mathrm{PAut}\, A.$$

Hence the map $h \mapsto \theta \cdot h$ gives a bijection between $\mathrm{Proj}\, C(r(\theta))Z(A)$ and the set of all direct summands of θ.

Definition 3.2 We say that $\theta \in \mathrm{PAut}\, A$ is *positive* (resp., *negative, central*) if $r(\theta)$ (resp., $l(\theta)$, both $r(\theta), l(\theta)) \in \mathrm{Proj}\, Z(A)$, that it is *regular* if $\theta = \theta_1 \oplus \theta_2$ for some positive θ_1 and negative θ_2, and that it is *inner* if $\theta = \mathrm{Ad}\, u$ for some $u \in \mathrm{PI}\, A$ with $r(\theta) = u^*u$ and $l(\theta) = uu^*$, where $\mathrm{Ad}\, u\colon u^*uAu^*u \to uu^*Auu^*$, $x \mapsto uxu^*$. Denote by $(\mathrm{PAut}\, A)^+$ (resp., $(\mathrm{PAut}\, A)^0$, $\mathrm{RPAut}\, A$, $\mathrm{PInt}\, A$) the set of all positive (resp., central, regular, inner) elements of $\mathrm{PAut}\, A$.

Clearly, the map $\theta \mapsto \theta^{-1}$ interchanges positivity and negativity, and positivity, centrality, etc. are preserved by passage to the direct sums or the direct summands.

We associate with each $\theta \in \mathrm{RPAut}\, A$ an algebraic invertible A-module, written $\langle\theta\rangle$, as follows. (Note that A, being an AW^*-algebra, satisfies (ND). This algebraic invertible A-module turns out to be also an invertible A-module in the sense of Definition 5.1.)

If $\theta \in (\mathrm{PAut}\, A)^+$, then identify θ with the surjective $*$-homomorphism $A \to l(\theta)Al(\theta)$, $x \mapsto r(\theta)x \mapsto \theta(r(\theta)x) =: \theta(x)$, and write $\langle\theta\rangle = (Al(\theta), \theta) = Al(\theta)$, where $Al(\theta)$ is an inner product A-module defined by

$$a \cdot x \cdot b = ax\theta(b), \quad \langle x, y\rangle = xy^*, \quad a, b \in A,\ x, y \in Al(\theta).$$

Now $\mathrm{End}_A(Al(\theta))$ is naturally identified with $l(\theta)Al(\theta)$ (so (2.5) follows), and (2.2) holds, since $x \in Al(\theta)$ and $\langle x, y\rangle = 0$ for all $y \in Al(\theta)$ imply $x = xl(\theta) = \langle x, l(\theta)\rangle = 0$. Hence $\langle\theta\rangle \in \mathrm{INV}'(A)$. Further, as in Section 2, the inverse $\langle\theta\rangle^{-1} \in \mathrm{INV}'(A)$ is defined as $l(\theta)A$ with the operations

$$a \cdot x \cdot b = \theta(a)xb, \quad \langle x, y\rangle = \theta^{-1}(xy^*).$$

Indeed, $x \in l(\theta)A$ and $xAl(\theta) = 0$ imply $xAl(\theta)A = 0$, $xC(l(\theta))A = 0$ and $x = 0$. The kernel of the $*$-homomorphism $A \to \mathrm{End}_A(l(\theta)A)$, $a \mapsto [x \mapsto xa]$, is $(1 - C(l(\theta)))A$ and so $\mathrm{End}_A(l(\theta)A) \cong C(l(\theta))A$.

If $\theta = \theta_1 \oplus \theta_2 \in \mathrm{RPAut}\, A$ with $\theta_1, \theta_2^{-1} \in (\mathrm{PAut}\, A)^+$, then define $\langle\theta\rangle$ to be $\langle\theta_1\rangle \oplus \langle\theta_2^{-1}\rangle^{-1}$, *i.e.*, $Al(\theta_1) \oplus r(\theta_2)A$ with the module operation and inner product

$$a \cdot (x_1 \oplus x_2) \cdot b = ax_1\theta_1(b) \oplus \theta_2^{-1}(a)x_2b,$$
$$\langle x_1 \oplus x_2, y_1 \oplus y_2\rangle = x_1y_1^* + \theta_2(x_2y_2^*)$$

for $a, b \in A$, $x_1, y_1 \in Al(\theta_1)$ and $x_2, y_2 \in r(\theta_2)A$, which is an algebraic invertible A-module by Proposition 2.5.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

To be precise, the definition of $\langle\theta\rangle$ above depends on the decomposition $\theta = \theta_1 \oplus \theta_2$, and it is determined up to isomorphism. Indeed, observe first that if θ is central and so regarded as both positive and negative, then the map $x \mapsto \theta^{-1}(x)$ gives an isomorphism between $\langle\theta\rangle = Al(\theta)$ and $\langle\theta^{-1}\rangle^{-1} = r(\theta)A$. If $\theta = \theta_1 \oplus \theta_2 = \theta_1' \oplus \theta_2'$ for $\theta_1, \theta_2^{-1}, \theta_1', (\theta_2')^{-1} \in (\mathrm{PAut}\, A)^+$, then

$$\theta_1 = \theta_{11} \oplus \theta_{12}, \quad \theta_2 = \theta_{21} \oplus \theta_{22}, \quad \theta_1' = \theta_{11} \oplus \theta_{21}, \quad \theta_2' = \theta_{12} \oplus \theta_{22}$$

for θ_{11} positive, θ_{22} negative, and θ_{ij}, $i \neq j$, central, and so

$$\begin{aligned}\langle\theta_1\rangle \oplus \langle\theta_2^{-1}\rangle^{-1} &= \langle\theta_{11}\rangle \oplus \langle\theta_{12}\rangle \oplus \langle\theta_{21}^{-1}\rangle^{-1} \oplus \langle\theta_{22}^{-1}\rangle^{-1} \\ &\cong \langle\theta_{11}\rangle \oplus \langle\theta_{12}^{-1}\rangle^{-1} \oplus \langle\theta_{21}\rangle \oplus \langle\theta_{22}^{-1}\rangle^{-1} = \langle\theta_1'\rangle \oplus \langle(\theta_2')^{-1}\rangle^{-1}.\end{aligned}$$

For $\theta \in \mathrm{RPAut} A$ and $h, k \in \mathrm{Proj}\, Z(A)$ we have

$$\langle k \cdot \theta \cdot h\rangle = k \cdot \langle\theta\rangle \cdot h, \quad \langle k \cdot \theta \cdot h\rangle^{-1} = h \cdot \langle\theta^{-1}\rangle \cdot k. \tag{3.4}$$

Definition 3.3 Call an algebraic invertible A-module isomorphic to $\langle\theta\rangle$ for some $\theta \in \mathrm{RPAut}\, A$ (resp., $(\mathrm{PAut}\, A)^+$, $(\mathrm{PAut}\, A)^0$) *regular* (resp., *positive, central*), and denote by $\mathrm{RINV}(A)$ (resp., $\mathrm{INV}(A)^+$, $\mathrm{INV}(A)^0$) the set of all such modules. (The positivity and centrality for $\langle\theta\rangle$ are weaker than the corresponding notions for θ. See Proposition 7.7.)

The following result together with (3.4) tells us when $\langle\theta\rangle \cong \langle\psi\rangle$ for $\theta, \psi \in \mathrm{RPAut}\, A$.

Proposition 3.4 *Let $\theta \in (\mathrm{PAut}\, A)^+$.*

(i) *We have a monomorphism $\tau\colon \langle\theta\rangle \to \langle\psi\rangle$ for some $\psi \in (\mathrm{PAut}\, A)^+$ if and only if $r(\theta) \leq r(\psi)$ and $\psi \cdot r(\theta) = (\mathrm{Ad}\, u) \circ \theta$ for some $u \in \mathrm{PI}\, A$ with $u^*u = l(\theta)$. In this case $\tau(x) = xu^*$ for $x \in Al(\theta)$, $\tau\colon \langle\theta\rangle \to \langle\psi\rangle \cdot r(\theta) = \langle\psi \cdot r(\theta)\rangle$ is an isomorphism, and so τ is surjective if and only if $r(\theta) = r(\psi)$.*

(ii) *We have a monomorphism $\tau\colon \langle\theta\rangle \to \langle\psi\rangle^{-1}$ for some $\psi \in (\mathrm{PAut}\, A)^+$ if and only if $l(\theta) \leq r(\psi)$ and $\psi | l(\theta)Al(\theta) = (\mathrm{Ad}\, u) \circ \theta^{-1}$ for some $u \in \mathrm{PI}\, A$ with $u^*u = r(\theta)$. In this case, $\tau(x) = \psi(x)u$ for $x \in Al(\theta)$, $\tau\colon \langle\theta\rangle \to \langle\psi \cdot h\rangle^{-1} = h \cdot \langle\psi\rangle^{-1}$ is an isomorphism, where $h := \psi^{-1}(r(\theta)l(\psi)) \in \mathrm{Proj}\, Z(A)$, and τ is surjective if and only if $l(\psi) \leq r(\theta)$.*

Proof (i) If τ as above exists, then with $u := \tau(l(\theta))^* \in l(\psi)A$, we have $\tau(Al(\theta)) = Au^* = Auu^*$, $u \in \mathrm{PI}\, A$, and

$$u^*u = l(\theta), \quad \theta(a)u^* = u^*\psi(a), \ \forall a \in A, \tag{3.5}$$

since for $a, b \in A$,

$$\begin{gathered} a\theta(b)u^* = (a\theta(b)) \cdot \tau(l(\theta)) = \tau(a\theta(b)) = \tau(a \cdot l(\theta) \cdot b) = au^*\psi(b), \\ al(\theta)b^* = \langle al(\theta), bl(\theta)\rangle = \langle\tau(al(\theta)), \tau(bl(\theta))\rangle = \langle au^*, bu^*\rangle = au^*ub^*, \\ u^* = \tau(l(\theta) \cdot l(\theta)) = l(\theta)\tau(l(\theta)) = u^*uu^*. \end{gathered}$$

Conversely, any $u \in l(\psi)A$ and $\psi \in (\mathrm{PAut}\,A)^+$ satisfying (3.5) induce a monomorphism $\langle\theta\rangle \to \langle\psi\rangle$, $x \mapsto xu^*$. Hence it remains only to show that (3.5) implies that $r(\theta) \le r(\psi)$, $uu^* = \psi(r(\theta))$, and $u \in \mathrm{PI}A$. But

$$\theta(r(\theta)(1-r(\psi))) = \theta(r(\theta)(1-r(\psi)))u^*u = u^*\psi(r(\theta)(1-r(\psi)))u = 0$$

implies $r(\theta) \le r(\psi)$. Further $u^* = \theta(r(\theta))u^* = u^*\psi(r(\theta))$ implies

$$\begin{aligned} uu^* &= uu^*\psi(r(\theta)) = \psi(r(\theta))uu^* \in \psi(r(\theta))A\psi(r(\theta)) \\ &= \psi(r(\theta)A), \quad \psi^{-1}(uu^*) \in r(\theta)A; \\ \theta(r(\theta) - \psi^{-1}(uu^*)) &= u^*\psi(r(\theta) - \psi^{-1}(uu^*))u = u^*u - u^*u = 0 \end{aligned}$$

implies $uu^* = \psi(r(\theta))$; and $u^* = u^*\psi(r(\theta)) = u^*uu^*$.

(ii) If τ as above exists, then with $u := \tau(l(\theta)) \in l(\psi)A$, $\tau(al(\theta)) = \psi(a)u$ for $a \in A$, $\tau(Al(\theta)) = l(\psi)Au = l(\psi)Au^*u$, $u \in \mathrm{PI}\,A$, and

$$(3.6) \qquad l(\theta) \le r(\theta), \quad \psi(l(\theta)) = uu^*, \quad \psi(\theta(a))u = ua, \quad \forall a \in A,$$

since for $a, b \in A$,

$$\begin{aligned} al(\theta)b^* &= \langle al(\theta), bl(\theta)\rangle = \langle \tau(al(\theta)), \tau(bl(\theta))\rangle \\ &= \psi^{-1}(\psi(a)uu^*\psi(b^*)) = a\psi^{-1}(uu^*)b^*, \\ \psi(a\theta(b))u &= \tau(a \cdot l(\theta) \cdot b) = \psi(a)ub, \\ u &= \tau(l(\theta) \cdot l(\theta)) = \psi(l(\theta))\tau(l(\theta)) = uu^*u. \end{aligned}$$

Conversely, any $u \in l(\theta)A$ and $\psi \in (\mathrm{PAut}\,A)^+$ satisfying (3.6) induce a monomorphism $\langle\theta\rangle \to \langle\psi\rangle^{-1}$, $x \mapsto \psi(x)u$. Hence it suffices to show that (3.6) implies that $u^*u = r(\theta)$ and $u \in \mathrm{PI}\,A$. Indeed, then

$$\tau(Al(\theta)) = l(\psi)Au^*u = l(\psi)r(\theta)A = \psi(h)A = \langle\psi \cdot h\rangle^{-1}$$

with h as above. But, taking $a = r(\theta)$ and then $a = 1$ in (3.6) shows $ur(\theta) = u$, $u^*u \in r(\theta)A$. And taking $a = r(\theta) - u^*u$ in (3.6) shows $\psi(\theta(r(\theta) - u^*u)) = u(r(\theta) - u^*u) = 0$ and $r(\theta) = u^*u$, since $l(\theta) \le r(\psi)$, and $uu^*u = ur(\theta) = u$. ■

The following properties of A hold, since A is an AW^*-algebra (see [1]; WSB and ELCP are short for *Weak Schröder-Bernstein* and *Existence of the Largest Central Projection*, respectively):

$$(\mathrm{WSB}) \qquad \begin{cases} e \in \mathrm{Proj}\,A,\ f \in \mathrm{Proj}\,Z(A),\ u \in \mathrm{PI}\,A,\ u^*u \le e \le f = uu^* \\ \qquad \Longrightarrow \exists v \in \mathrm{PI}\,A: \quad e = v^*v, \quad vv^* = f; \end{cases}$$

$$(\mathrm{ELCP}) \qquad \begin{cases} \forall e \in \mathrm{Proj}A,\ \exists \text{ largest } h \in \mathrm{Proj}\,Z(A) \text{ with the property} \\ \qquad \exists u \in \mathrm{PI}\,A:\ he = u^*u,\ uu^* = h. \end{cases}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proposition 3.5

(i) *An* $X \in \mathrm{RINV}(A)$ *is both positive and negative in the sense of Definition* 3.3 *if and only if* X *is central.*

(ii) *Each* $X \in \mathrm{RINV}(A)$ *has a unique direct sum decomposition* $X = X^{++} \oplus X^{0} \oplus X^{--}$, *where* X^0 *is the largest central summand of* X, $X^+ := X^{++} \oplus X^0$ *is the largest positive summand of* X, *and* $X^- := X^0 \oplus X^{--}$ *is the largest negative summand of* X.

Proof (i) It suffices to show the necessity. Suppose $X \cong \langle\theta\rangle \cong \langle\psi\rangle^{-1}$ for some $\theta, \psi \in (\mathrm{PAut}\, A)^+$. Then, for $u \in \mathrm{PI}\, A$ as in Proposition 3.4(ii),

$$uu^* = (\mathrm{Ad}\, u) \circ \theta^{-1}(l(\theta)) = \psi(l(\theta)) \le l(\psi) \le r(\theta) = u^*u,$$

and by (WSB), $v^*v = l(\psi)$, $vv^* = r(\theta)$ for some $v \in \mathrm{PI}\, A$. Hence $(\mathrm{Ad}\, v) \circ \psi \in (\mathrm{PAut}\, A)^0$, $X \cong \langle((\mathrm{Ad}\, v) \circ \psi)^{-1}\rangle$ by Proposition 3.4(i), and X is central.

(ii) Observe first that if $\theta \in (\mathrm{PAut}\, A)^+$ and if we take, by (ELCP), the largest $h \in \mathrm{Proj}\, Z(A)$ such that $hl(\theta) = u^*u$ and $uu^* = h$ for some $u \in \mathrm{PI}\, A$, then $\langle h \cdot \theta\rangle$ is the largest central summand of $\langle\theta\rangle$. Indeed, $\langle h \cdot \theta\rangle \cong \langle(\mathrm{Ad}\, u) \circ \theta\rangle$ with $l((\mathrm{Ad}\, u) \circ \theta) = uu^*$, $r((\mathrm{Ad}\, u) \circ \theta) = \theta^{-1}(hl(\theta)) \in Z(A)$, and $\langle h \cdot \theta\rangle$ is central. Further, if $\langle\psi\rangle \cong \langle k \cdot \theta\rangle$ for some $\psi \in (\mathrm{PAut}\, A)^0$ and $k \in \mathrm{Proj}\, Z(A)$ with $k \le C(l(\theta))$, then, by Proposition 3.4(i), $\psi = (\mathrm{Ad}\, v) \circ (k \cdot \theta)$ for some $v \in \mathrm{PI}\, A$ with $v^*v = l(k \cdot \theta) = kl(\theta)$. But $vv^* = l(\psi) \in \mathrm{Proj}\, Z(A)$, so $vv^* = C(vv^*) = C(v^*v) = C(kl(\theta)) = kC(l(\theta)) = k$, and by the definition of h, $k \le h$, as desired.

For each $X \in \mathrm{RINV}(A)$, $X = X_1 \oplus X_2$ for X_1 positive and X_2 negative. Then application of the above argument to X_j shows the existence of the decomposition, and the uniqueness follows from (i). ■

Definition 3.6 For $X \in \mathrm{RINV}(A)$ we call $X^+, X^{++}, X^0, X^-, X^{--}$ as above the *positive, purely positive, central, negative, purely negative parts* of X, respectively.

The following property (GC) (Generalized Comparability) of A holds [1, Corollary 1, p. 80], and it implies the property (GC′):

(GC) $\forall e, f \in \mathrm{Proj}\, A,\ \exists h \in \mathrm{Proj}\, Z(A),\ \exists u, v \in \mathrm{PI}\, A:$

$$he = u^*u,\ uu^* \le hf,\ (1-h)e \ge v^*v,\ vv^* = (1-h)f.$$

(GC′) $\forall e, f \in \mathrm{Proj}\, A,\ \exists w \in \mathrm{PI}\, A:$

$$w^*w \le e,\ ww^* \le f,\ C(w^*w) = C(e)C(f).$$

Indeed, it suffices to take $w = u + v$ for u, v as in (GC), since

$$\begin{aligned} hC(e) &= C(he) = C(u^*u) = C(uu^*) \le hC(f),\\ (1-h)C(f) &= C(vv^*) = C(v^*v) \le (1-h)C(e),\\ C(w^*w) &= C(u^*u) + C(v^*v) = hC(e) + (1-h)C(f)\\ &= hC(e)C(f) + (1-h)C(e)C(f) = C(e)C(f). \end{aligned}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

If $\theta, \psi \in \mathrm{PAut}\, A$ and if $r(\theta)l(\psi) = l(\psi)r(\theta)$ and so this is a projection, then denote by $\theta \circ \psi \in \mathrm{PAut}\, A$ the composition

$$r(\theta \circ \psi)Ar(\theta \circ \psi) \to l(\theta \circ \psi)Al(\theta \circ \psi), \quad x \mapsto \theta(\psi(x)),$$

where $r(\theta \circ \psi) = \psi^{-1}(r(\theta)l(\psi))$ and $l(\theta \circ \psi) = \theta(r(\theta)l(\psi))$. If θ and ψ are both positive (resp., negative, central), then so is $\theta \circ \psi$.

Proposition 3.7 *Let $\theta, \psi \in \mathrm{PAut}\, A$.*

(i) *If θ and ψ are both positive (resp., negative), then an isomorphism $\langle\theta\rangle \odot_A \langle\psi\rangle \cong \langle\theta\circ\psi\rangle$ is defined by $x\otimes y \mapsto x\theta(y)$, $x \in Al(\theta)$, $y \in Al(\psi)$ (resp., $x\otimes y \mapsto \psi^{-1}(x)y$, $x \in r(\theta)A$, $y \in r(\psi)A$).*

(ii) *If θ is positive and ψ is negative, then we have a monomorphism $\tau\colon \langle\theta\rangle \odot_A \langle\psi\rangle \to \langle\omega_1\rangle \oplus \langle\omega_2\rangle = \langle\omega_1 \oplus \omega_2\rangle$ for some positive ω_1 and negative ω_2 if and only if there are an $h \in \mathrm{Proj}\, Z(A)$ and $u_1, u_2 \in \mathrm{PI}\, A$ such that*

$$(\mathrm{Ad}\, u_1) \circ \omega_1 | hr(\psi)Ar(\psi) = \theta \circ \psi | hr(\psi)Ar(\psi),$$

$$u_1u_1^* = (\theta \circ \psi)(hr(\theta \circ \psi)), \quad u_1^*u_1 = \omega_1(hr(\theta \circ \psi)),$$

$$\omega_2 \circ (\mathrm{Ad}\, u_2)|(1-h)r(\psi)Ar(\psi) = \theta \circ \psi|(1-h)r(\psi)Ar(\psi),$$

$$u_2u_2^* = (\omega_2^{-1} \circ \theta \circ \psi)((1-h)r(\theta \circ \psi)), \quad u_2^*u_2 = (1-h)r(\theta \circ \psi).$$

In this case, monomorphisms $\tau_1\colon \big(\langle\theta\rangle \odot_A \langle\psi\rangle\big) \cdot h \to \langle\omega_1\rangle$ and $\tau_2\colon \big(\langle\theta\rangle \odot_A \langle\psi\rangle\big) \cdot(1-h) \to \langle\omega_2\rangle$ are defined by

$$\tau_1(x \otimes y) = xu_1\omega_1(y), \quad x \in Al(\theta), y \in hr(\psi)A,$$

$$\tau_2(x \otimes y) = \omega_2^{-1}(x)u_2y, \quad x \in Al(\theta), y \in (1-h)r(\psi)A,$$

for h, u_1, u_2 as above, so that $\tau = \tau_1 \oplus \tau_2$.

(iii) *Let θ be negative and ψ positive. Take h, u, v as in* (GC) *for $e = l(\psi)$ and $f = r(\theta)$. Then isomorphisms $\tau_1\colon \langle\theta\rangle \odot_A h\cdot\langle\psi\rangle \to \langle\omega_1\rangle$ and $\tau_2\colon \langle\theta\rangle \odot_A (1-h)\cdot\langle\psi\rangle \to \langle\omega_2\rangle$ are defined by*

$$\tau_1(x \otimes y) = \theta(xyu^*), \quad x \in r(\theta)A, y \in Ahl(\psi),$$

$$\tau_2(x \otimes y) = \psi^{-1}(v^*xy), \quad x \in r(\theta)A, y \in A(1-h)l(\psi),$$

where $\omega_1 = \theta \circ (\mathrm{Ad}\, u) \circ \psi|\psi^{-1}(hl(\psi))A$ is positive and $\omega_2 = \theta \circ (\mathrm{Ad}\, v) \circ \psi|\psi^{-1}(v^*v)A\psi^{-1}(v^*v)$ is negative, so that we have $\langle\theta\rangle \odot_A \langle\psi\rangle \cong \langle\omega_1 \oplus \omega_2\rangle$.

Proof Part (i) follows, in view of the definitions of $\odot_A$, $\langle\theta\rangle$, and Proposition 3.4, from direct computation.

(ii) We show only the necessity, since the sufficiency follows immediately. If τ as above exists and if $h \in \mathrm{Proj}\, Z(A)$ is such that $\langle\omega_1\rangle = \big(\langle\omega_1\rangle \oplus \langle\omega_2\rangle\big) \cdot h$ and

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

$\langle\omega_2\rangle = \big(\langle\omega_1\rangle \oplus \langle\omega_2\rangle\big) \cdot (1-h)$, then τ is restricted to monomorphisms $\tau_1\colon \langle\theta\rangle \odot_A \langle\psi \cdot h\rangle = \big(\langle\theta\rangle \odot_A \langle\psi\rangle\big) \cdot h \to \langle\omega_1\rangle$ and $\tau_2\colon \langle\theta\rangle \odot_A \langle\psi \cdot (1-h)\rangle \to \langle\omega_2\rangle$. Set

$$u_1 := \tau(l(\theta) \otimes hr(\psi)) \in Al(\omega_1), \quad u_2 := \tau(l(\theta) \otimes (1-h)r(\psi)) \in r(\omega_2)A.$$

In view of

$$\theta(a) \otimes r(\psi)b = l(\theta) \cdot a \otimes r(\psi)b = l(\theta) \otimes a \cdot (r(\psi)b) = l(\theta) \otimes \psi^{-1}(a)b, \quad a, b \in A,$$

in $\langle\theta\rangle \odot_A \langle\psi\rangle$, we have

$$\begin{aligned} l(\theta) \otimes hr(\psi) &= \theta(r(\theta)) \otimes hr(\psi) = l(\theta) \otimes h\psi^{-1}(r(\theta)) \\ &= (l(\theta) \otimes hr(\psi)) \cdot \big(h\psi^{-1}(r(\theta)l(\psi))\big) = (l(\theta) \otimes hr(\psi)) \cdot (hr(\theta \circ \psi)), \\ (\theta \circ \psi)(hr(\psi)) \otimes hr(\psi) &= l(\theta) \otimes \psi^{-1}\big(\psi(hr(\psi))\big)\, h = l(\theta) \otimes hr(\psi). \end{aligned}$$

Hence

$$u_1 = \tau\big((l(\theta) \otimes hr(\psi)) \cdot (hr(\theta \circ \psi))\big) = u_1 \cdot (hr(\theta \circ \psi)) = u_1\omega_1(hr(\theta \circ \psi)),$$

and since

$$\begin{aligned} u_1u_1^* &= \langle \tau(l(\theta) \otimes hr(\psi)), \tau(l(\theta) \otimes hr(\psi))\rangle = \langle l(\theta) \otimes hr(\psi), l(\theta) \otimes hr(\psi)\rangle \\ &= \langle l(\theta) \cdot \langle hr(\psi), hr(\psi)\rangle, l(\theta)\rangle = (\theta \circ \psi)(hr(\psi)), \\ u_1u_1^*u_1 &= (u_1u_1^*)u_1 = \tau((\theta \circ \psi)(hr(\psi)) \otimes hr(\psi)) = \tau(l(\theta) \otimes hr(\psi)) = u_1. \end{aligned}$$

Thus $u_1 \in \mathrm{PI}\, A$ and $u_1^*u_1 \leq \omega_1(hr(\theta \circ \psi))$. Similarly $u_2 = u_2(1-h)r(\theta \circ \psi)$,

$$\begin{aligned} \omega_2(u_2u_2^*) &= \langle \tau(l(\theta) \otimes (1-h)r(\psi)), \tau(l(\theta) \otimes (1-h)r(\psi))\rangle \\ &= (\theta \circ \psi)((1-h)r(\theta \circ \psi)), \end{aligned}$$

and $u_2u_2^* = (\omega_2^{-1} \circ \theta \circ \psi)((1-h)r(\theta \circ \psi))$, since $u_2 \in r(\omega_2)A$. Since $a \cdot u_2 = \omega_2^{-1}(a)u_2$, $a \in A$, in $\langle\omega_2\rangle = r(\omega_2)A$,

$$\begin{aligned} u_2u_2^*u_2 &= (u_2u_2^*)u_2 \\ &= (\theta \circ \psi)((1-h)r(\theta \circ \psi)) \cdot \tau(l(\theta) \otimes (1-h)r(\psi)) \\ &= \tau((\theta \circ \psi)((1-h)r(\theta \circ \psi)) \otimes (1-h)r(\psi)) = u_2 \end{aligned}$$

as above. Hence $u_2 \in \mathrm{PI}\, A$ and $u_2^*u_2 \leq (1-h)r(\theta \circ \psi)$. For $a, b \in A$ we have

$$\begin{aligned} \tau(al(\theta) \otimes hr(\psi)b) &= \tau(a \cdot (l(\theta) \otimes hr(\psi)) \cdot b) = a \cdot u_1 \cdot b = au_1\omega_1(b), \\ \tau(al(\theta) \otimes (1-h)r(\psi)b) &= a \cdot u_2 \cdot b = \omega_2^{-1}(a)u_2b, \end{aligned}$$

and so

$$\begin{aligned}
u_1\omega_1(b)u_1^* &= \langle \tau(l(\theta)\otimes hr(\psi)b), \tau(l(\theta)\otimes hr(\psi))\rangle \\
&= (\theta\circ\psi)(hr(\psi)br(\psi)) = (\theta\circ\psi)(hr(\theta\circ\psi)br(\theta\circ\psi)), \\
\omega_2(u_2bu_2^*) &= \langle \tau(l(\theta)\otimes(1-h)r(\psi)b), \tau(l(\theta)\otimes(1-h)r(\psi))\rangle \\
&= (\theta\circ\psi)((1-h)r(\theta\circ\psi)br(\theta\circ\psi)).
\end{aligned}$$

Since $(\mathrm{Ad}\, u_1)\circ\omega_1|hr(\theta\circ\psi)Ar(\theta\circ\psi) = \theta\circ\psi|hr(\theta\circ\psi)Ar(\theta\circ\psi)$ is injective and $\omega_1 \in (\mathrm{PAut}\, A)^+$, we have $hr(\theta\circ\psi) \le r(\omega_1)$, and Ad u_1 on $\omega_1(hr(\theta\circ\psi)Ar(\theta\circ\psi)) = \omega_1(hr(\theta\circ\psi))A\omega_1(hr(\theta\circ\psi))$ is injective. Hence $(\mathrm{Ad}\, u_1)(\omega_1(hr(\theta\circ\psi)) - u_1^*u_1) = 0$ implies $u_1^*u_1 = \omega_1(hr(\theta\circ\psi))$. Similarly $u_2^*u_2 = (1-h)r(\theta\circ\psi)$.

(iii) Define τ_j, ω_j as above. Then $\omega_1 \in (\mathrm{PAut}\, A)^+$, since $u^*u = hl(\psi)$, $uu^* \le r(\theta)$ and so $r(\omega_1) = \psi^{-1}(hl(\psi)) \in \mathrm{Proj}\, Z(A)$. We have

$$\langle\theta\rangle \odot_A h\cdot\langle\psi\rangle = r(\theta)A \odot_A Ahl(\psi),$$

and for $a, b \in A$, $x, x_j \in r(\theta)A$ and $y, y_j \in Ahl(\theta) = Au^*u$ we have

$$\begin{aligned}
\tau_1(a\cdot(x\otimes y)\cdot b) &= \tau_1(\theta^{-1}(a)x\otimes y\psi(b)) = \theta(\theta^{-1}(a)xy\psi(b)u^*) \\
&= a\theta(xyu^*u\psi(b)u^*) = a\tau_1(x\otimes y)\omega_1(b) \\
&= a\cdot\tau_1(x\otimes y)\cdot b, \\
\langle\tau_1(x_1\otimes y_1), \tau_1(x_2\otimes y_2)\rangle &= \theta(x_1y_1u^*)\theta(uy_2^*x_2^*) = \theta(x_1y_1y_2^*x_2^*) \\
&= \langle x_1\otimes y_1, x_2\otimes y_2\rangle, \\
\tau_1(r(\theta)A\odot_A Ahl(\psi)) &= \theta(r(\theta)Ahl(\psi)u^*) = \theta(r(\theta)Auu^*) \\
&= \theta(r(\theta)Ar(\theta)uu^*) = Al(\theta)\theta(uu^*) \\
&= Al(\omega_1).
\end{aligned}$$

Hence τ_1 is an isomorphism, and similarly for τ_2. ■

4 Normalizers

Throughout this section, A denotes a fixed AW^*-algebra.

Definition 4.1 Let B be an AW^*-algebra containing A as an AW^*-subalgebra with the same unit $1_A = 1_B =: 1$, and regard B as an A-bimodule.

(i) Write $\mathrm{INV}'_B(A)$ for the set of all sub-A-bimodules X of B satisfying (2.12) so that $\mathrm{INV}'_B(A) \subset \mathrm{INV}'(A)$ ((2.13) is automatically satisfied here). Call an element of $\mathrm{INV}'_B(A)$ an *algebraic invertible A-module in B*, and write $\mathrm{RINV}_B(A)$ (resp., $\mathrm{INV}_B(A)^+$, $\mathrm{INV}_B(A)^0$) for the subset of all elements in $\mathrm{INV}'_B(A)$ which are also regular (resp., positive, central).

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(ii) Call the following sets respectively the *normalizer, partial isometry normalizer, regular normalizer, positive normalizer* and *central normalizer of A in B*:

$$N_B(A) = \{x \in B : xAx^* \subset A,\ x^*Ax \subset A\},$$
$$\mathrm{PI}\,N_B(A) = N_B(A) \cap \mathrm{PI}\,B,$$
$$RN_B(A) = \{s \in \mathrm{PI}\,N_B(A) : \exists h \in \mathrm{Proj}\,Z(A), h \le s^*s,\ s(1-h)s^* \in \mathrm{Proj}\,Z(A)\},$$
$$N_B(A)^+ = \{s \in \mathrm{PI}\,N_B(A) : s^*s \in Z(A)\},$$
$$N_B(A)^0 = \{s \in \mathrm{PI}\,N_B(A) : s^*s, ss^* \in Z(A)\},$$

so that $N_B(A)^0 \subset N_B(A)^+ \subset RN_B(A) \subset \mathrm{PI}\,N_B(A) \subset N_B(A)$.

(iii) For $s \in RN_B(A)$ call $s^+ = sh$, $s^0 = sk$, $s^- = s(1-h+k)$, $s^{++} = s(h-k)$, and $s^{--} = s(1-h)$), respectively, the *positive, central, negative, purely positive* and *purely negative parts* of s, where h is the largest central projection of A such that $hs^*s = u^*u, uu^* = h$ for some $u \in \mathrm{PI}\,A$, and k is the largest central projection of A such that $ks^*s = v^*v, vv^* = k$, $sks^* = w^*w$ and $ww^* \in Z(A)$ for some $v, w \in \mathrm{PI}\,A$ (see (ELCP), and note that $s^*s, sks^* \in \mathrm{Proj}\,A$).

Following the previous convention, we write AxA, etc. for the linear span of the set $\{axb : a, b \in A\}$, etc. But the set itself can be linear in some cases (see (vi) below).

Proposition 4.2 *Keep the notation A, B, etc. as above.*

(i) *For $x \in B$ we have $AxA \in \mathrm{INV}'_B(A)$ if and only if $x \in N_B(A)$.*

(ii) *If $s \in \mathrm{PI}\,N_B(A)$, then $s^*s, ss^* \in \mathrm{Proj}\,A$, and $\mathrm{Ad}\,s = \mathrm{Ad}\,s|s^*sAs^*s$: $s^*sAs^*s \to ss^*Ass^*$, where $(\mathrm{Ad}\,s)(x) = sxs^*$ for $x \in B$, is a ∗-isomorphism,* i.e., $\mathrm{Ad}\,s \in \mathrm{PAut}\,A$.

(iii) *For $s \in \mathrm{PI}\,B$ we have $s \in N_B(A)^+$ if and only if $ss^* \in A$, $sA \subset As$ and $s^*As \subset A$. In this case, $\theta := \mathrm{Ad}\,s \in (\mathrm{PAut}\,A)^+$ with $r(\theta) = s^*s$ and $l(\theta) = ss^*$. In particular, for $s \in \mathrm{PI}\,B$ we have $s \in N_B(A)^0$ if and only if $ss^*, s^*s \in A$ and $As = sA$.*

(iv) *If $s \in N_B(A)^+$ and $t \in \mathrm{PI}\,N_B(A)$, then $st, ts^* \in \mathrm{PI}\,N_B(A)$. The set $N_B(A)$ is a multiplicative semigroup closed under involution, and $N_B(A)^0 \subset N_B(A)^+$ are its subsemigroups. If $x_1, x_2 \in N_B(A)$ (resp., $\mathrm{PI}\,N_B(A), RN_B(A), N_B(A)^+, N_B(A)^0$) and $x_1 = x_1h = kx_1$, $x_2 = x_2(1-h) = (1-k)x_2$ for some $h, k \in \mathrm{Proj}\,Z(A)$, then $x_1 + x_2 \in N_B(A)$ (resp., $\mathrm{PI}\,N_B(A), RN_B(A), N_B(A)^+, N_B(A)^0$).*

(v) *Let $s, t \in N_B(A)^+$. We have $As \subset At$ if and only if $s^*s \le t^*t$, $st^* \in A$, and in this case, $st^* \in \mathrm{PI}\,A$, $As = A(st^*)t$. In particular, $As = At$ if and only if $s^*s = t^*t$, $st^* \in A$. We have $As \subset t^*A$ if and only if $ss^* \le t^*t$, $ts \in A$, and in this case, $ts \in \mathrm{PI}\,A$, $t^*As^*s = t^*s^*sA = sA$. In particular, $As = t^*A$ if and only if $ss^* \le t^*t$, $tt^* \le s^*s$ and $ts \in A$, and in this case, $As = t^*A = Ar = rA$ for some $r \in N_B(A)^0$.*

(vi) *If $X \in \mathrm{RINV}_B(A)$, then $X = AsA$ for some $s \in RN_B(A)$. In particular, if $X \in \mathrm{INV}_B(A)^+$ (resp., $\mathrm{INV}_B(A)^0$), then we may take s so that $s \in N_B(A)^+$ (resp., $N_B(A)^0$). Conversely, if $s \in RN_B(A)$, then $\theta := \mathrm{Ad}\,s \in \mathrm{RPAut}\,A$ with $r(\theta) = s^*s$ and $l(\theta) = ss^*$, $AsA \cong \langle\theta\rangle$,* i.e., *$AsA \in \mathrm{RINV}_B(A)$, and $AsA = \{asb : a, b \in A\}$. Moreover As^+A, As^0A, etc. are respectively the positive, central parts of AsA, etc.,* i.e., *$(AsA)^+ = As^+A$, $(AsA)^0 = As^0A$, etc. (see Proposition 3.5).*

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proof Part (i) is clear, since $1 \in A$.

(ii) If $s \in \operatorname{PI} B$ and $s^*s \in A$, then s^*sAs^*s and sAs^* are $*$-subalgebras of B, and $s^*sAs^*s \to sAs^*$, $x \mapsto sxs^*$, is a $*$-isomorphism. If, further, $s \in N_B(A)$, then $sAs^* = ss^*Ass^*$, since $sAs^* \subset A$ implies $sAs^* = ss^*(sAs^*)ss^* \subset ss^*Ass^*$ and $s^*As \subset A$ implies $ss^*Ass^* \subset sAs^*$, and the assertion follows.

(iii) Necessity: If $s \in N_B(A)^+$, then $sA = ss^*sA = sAs^*s \subset As$, etc.

Sufficiency: For s and θ as above set $s^*s = r$, $ss^* = l$. Then we show that $r \in Z(A)$ and $\theta \in (\operatorname{PAut} A)^+$ with $r(\theta) = r$ and $l(\theta) = l$. By assumption, $l \in A$, $\theta(A) = sAs^* \subset Ass^* = Al \subset A$, and $r = s^* \cdot 1 \cdot s \in s^*As \subset A$. For $a \in A$, $sa \in sA \subset As$ and $sa = a's$ for some $a' \in A$; so $\theta(a) = sas^* = a'ss^*$ and $sa = a's = a'ss^*s = \theta(a)s$. For $a, b \in A$ we have

$$\theta(a^*) = sa^*s^* = (sa^*s^*)^* = \theta(a)^*,$$

$$\theta(ab)s = s(ab) = (sa)b = \theta(a)sb = \theta(a)\theta(b)s,$$

$$\theta(ab) = \theta(ab)ss^* = \theta(a)\theta(b)ss^* = \theta(a)\theta(b).$$

We have $\theta(A) = lAl$, since $\theta(A) = \theta(A)^* \subset Al \cap lA = lAl$ and $s^*As \subset A$ implies $lAl = ss^*Ass^* \subset sAs^* = \theta(A)$. We have $r \in Z(A)$, since for all $a \in A$,

$$ar = as^*s = (sa^*)^*s = (\theta(a^*)s)^*s = s^*\theta(a)s,$$

$$ra = (a^*r)^* = (s^*\theta(a^*)s)^* = s^*\theta(a)s = ar;$$

$\operatorname{Ker}\theta = (1 - r)A$, since $sas^* = 0$ if and only if $s^*sa = s^*sas^*s = 0$, and the assertion follows.

(iv) If $x, y \in N_B(A)$, then $xy \in N_B(A)$. For s, t as above, $s^*s \in Z(A)$, $tt^*, t^*t \in A$, and

$$st(st)^*st = stt^*s^*st = ss^*stt^*t = st,$$

$$ts^*(ts^*)^*ts^* = ts^*st^*ts^* = tt^*ts^*ss^* = ts^*.$$

If $s, t \in N_B(A)^+$, then $(st)^*st \in Z(A)$ and $st \in N_B(A)^+$. Indeed, $(st)^*st = t^*s^*st$ is the inverse image of $(s^*s)(tt^*) \in Z(tt^*Att^*)$ under the $*$-isomorphism $\operatorname{Ad} t\colon t^*tA \to tt^*Att^*$ (see (iii)). Hence the second assertion follows. To see the last assertion, it suffices to note that for x_1, x_2 as above $x_1Ax_2 = x_1hA(1-h)x_2 = \{0\}$ and similarly $x_2Ax_1 = \{0\}$.

(v) If $As \subset At$, then $s \in At$, $s = st^*t$, $s^*s = s^*st^*t$ and $st^* \in Ast^* \subset Att^* \subset A$. Further $st^* \in \operatorname{PI} A$ by (iv), and $As = A(st^*)t$. Conversely, if $s^*s \leq t^*t$, $st^* \in A$, then $s = ss^*s = ss^*st^*t = st^*t \in At$ and $As \subset At$.

If $As \subset t^*A$, then $s \in t^*A$, $s = t^*ts$, $ss^* \leq t^*t$ and $ts \in tAs \subset tt^*A \subset A$. Further $st \in \operatorname{PI} A$ and $As = t^*s^*sA$ as above. Hence, if $As = t^*A$, then $tt^* = tt^*s^*s$, $tt^* \leq s^*s$. Conversely, if $ss^* \leq t^*t$, $ts \in A$, then, since $t^*t \in Z(A)$ and $tA \subset At$ by (iii), $As = At^*ts = t^*tAs \subset t^*Ats \subset t^*A$.

Finally, suppose $As = t^*A$. Then with $u := st \in \operatorname{PI} A$ we have $u^*u = t^*s^*st = t^*t \in Z(A)$, since $tt^* \leq s^*s$, and $uu^* = stt^*s^* \leq ss^* \leq t^*t$. Hence, by (WSB),

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

$v^*v = ss^*$ and $vv^* = t^*t$ for some $v \in \mathrm{PI}\, A$. Then $r := vs \in N_B(A)^0$, since $r^*r = s^*v^*vs = s^*ss^*s = s^*s \in Z(A)$ and $rr^* = vss^*v^* = vv^*vv^* = vv^* = t^*t \in Z(A)$, and $As = Ar$, since $v^*v = ss^*$.

(vi) If there is an isomorphism $\tau \colon \langle\theta\rangle = Al(\theta) \to X \in \mathrm{INV}'_B(A)$ for some $\theta \in (\mathrm{PAut}\, A)^+$, then with $s := \tau(l(\theta)) \in B$ we have $X = \tau(Al(\theta)) = As$, $s \in N_B(A)^+$, and $\theta = \mathrm{Ad}\, s$ as in the proof of Proposition 3.4(i). Hence, in view of (iv) and (v), the first assertion follows.

If $s \in \mathrm{PI}\, N_B(A)$, then $\mathrm{Ad}\, s \in \mathrm{PAut}\, A$ by (ii). If, further, $s \in RN_B(A)$ and $h \le s^*s$, $s(1-h)s^* \in Z(A)$ for some $h \in \mathrm{Proj}\, Z(A)$, then $sh, (1-h)s^* \in N_B(A)^+$, and by (iii), $\mathrm{Ad}(sh), \mathrm{Ad}((1-h)s^*) \in (\mathrm{PAut}\, A)^+$, and $AshA = Ash \to Ashs^*$, $x \mapsto xhs^*$, gives an isomorphism $AshA \cong \langle \mathrm{Ad}(sh)\rangle$. Similarly $As(1-h)A = s(1-h)A \cong s^*s(1-h)A = \langle \mathrm{Ad}((1-h)s^*)\rangle^{-1} = \langle \mathrm{Ad}(s(1-h))\rangle$, and $AsA = Ash + s(1-h)A \cong \langle \mathrm{Ad}\, s\rangle$. To see $AsA = \{asb : a, b \in A\}$, since $AsA = Ash + s(1-h)A$, note that for $a, b \in A$ we have

$$ash + s(1-h)b = (ashs^* + s(1-h)s^*)s(hs^*s + (1-h)b).$$

To see the last assertion it suffices to show that for $s \in RN_B(A)$ we have $AsA \in \mathrm{INV}_B(A)^+$ if and only if $s^*s = u^*u$ and $uu^* \in Z(A)$ for some $u \in \mathrm{PI}\, A$. Indeed, since $(s^0)^*$ is the positive part of $(s^+)^*$, the assertion for s^0, etc. follows from that for s^+. Further, if u as above exists, then $\mathrm{Ad}(su^*) \in (\mathrm{PAut}\, A)^+$ and $AsA = Asu^* \cong \langle \mathrm{Ad}(su^*)\rangle$, since

$$sA = su^*uA = su^* \cdot uu^* \cdot uA = su^*uAuu^* = sAus^*su^* \subset Asu^*.$$

Conversely, if $s \in RN_B(A)$, *i.e.*, $h \le s^*s$ and $s(1-h)s^* \in Z(A)$ for some $h \in \mathrm{Proj}\, Z(A)$, and $AsA \in \mathrm{INV}_B(A)^+$, then $Ash + s(1-h)A = AsA = At$ for some $t \in N_B(A)^+$. Hence $Ash = Ath$ and $s(1-h)A = At(1-h)$ with $sh, (1-h)s^*, th, t(1-h) \in N_B(A)^+$. Then, as in the proof of (v), $s^*s(1-h) = v^*v$ and $vv^* = t^*t(1-h)$ for some $v = v(1-h) \in \mathrm{PI}\, A$. Thus $u := h + v \in \mathrm{PI}\, A$ satisfies $u^*u = h + v^*v = s^*s$ and $uu^* = h + vv^* = h + t^*t(1-h) \in Z(A)$. ■

Proposition 4.3 *If A, B, etc. are as before, then*

(i) $A \cdot RN_B(A) \cdot A := \{asb : a, b \in A, s \in RN_B(A)\} \subset N_B(A)$*;*

(ii) $x \in A \cdot RN_B(A) \cdot A \cap \mathrm{PI}\, B$ *if and only if*

$$\exists s \in RN_B(A), \exists u, v \in \mathrm{PI}\, A,\ vv^* = s^*u^*us,\ u^*u = svv^*s^* : \quad x = usv.$$

Proof Part (i) is obvious, since $N_B(A)$ is closed under multiplication.

(ii) The sufficiency is clear, since usv as above gives a partial isomerty.

Necessity: Suppose $x = asb \in \mathrm{PI}\, B$, where $a, b \in A$ and $s \in RN_B(A)$. We may assume further $s \in N_B(A)^+$. Indeed, for some $h \in \mathrm{Proj}\, Z(A)$ we have $h \le s^*s$, $k := s(1-h)s^* \in \mathrm{Proj}\, Z(A)$ and so $sh = (1-k)s, s(1-h) = ks, s = sh + s(1-h)$, with $sh, (s(1-h))^* \in N_B(A)^+$. Hence, if we have the expressions for $a(sh)b$ and $a(s(1-h))b$ as above, then so do we for asb. Then $\theta := \mathrm{Ad}\, s \in (\mathrm{PAut}\, A)^+$ (see Proposition 4.2(iii)), and since $x = asb \in \mathrm{PI}\, B$, $x^*x \in A \cap \mathrm{Proj}\, B = \mathrm{Proj}\, A$. Let $b^* = v_1|b^*|$ be the polar decomposition of b^* with $v_1 \in \mathrm{PI}\, A$ and $v_1^*v_1|b^*| = |b^*|$, which exists, since A is an AW^*-algebra, and let $u := a\theta(|b^*|)$, $v := v_1^* x^* x$. Then $v \in \mathrm{PI}\, A$ with $vv^* \le s^* s = r(\theta)$ and $v^* v = x^* x$, since $x = asb = asbs^* s = xs^* s$ implies $vv^* \le s^* s$ and $x = asb = as|b^*|v_1^*$ implies $x = xv_1 v_1^*$. Hence $\theta(vv^*) \in \mathrm{Proj}\, A$, and

$$\begin{aligned}\theta(vv^*) &= svv^* s^* = sv_1^* x^* x v_1 s^* = sv_1^* b^* s^* a^* asbv_1 s^* \\ &= sv_1^* v_1 |b^*| s^* a^* as |b^*| v_1^* v_1 s^* = s|b^*|s^* a^* as|b^*|s^* = u^* u.\end{aligned}$$

Thus $u \in \mathrm{PI}\, A$, $s^* u^* us = s^* svv^* s^* s = vv^*$, and

$$x = asb = asbx^* x = ass^* s|b^*|v_1^* x^* x = as|b^*|s^* sv_1^* x^* x = usv. \qquad \blacksquare$$

The following property of A holds (SRU is short for "Square Root of a Unitary"):

$$(\mathrm{SRU}) \quad \begin{cases} \text{For every } h \in \mathrm{Proj}\, Z(A), \text{ every unitary } u \in hA \ (\textit{i.e.}, u^* u = uu^* = h), \\ \text{and every } *\text{-automorphism } \theta \text{ of } hA \text{ with } \theta(u) = u, \text{ there exists a unitary} \\ v \in h \text{ such that } v^2 = u,\ \theta(v) = v. \end{cases}$$

Indeed, since in the notation above $\{x \in hA : \theta(x) = x\}$ is an AW^*-subalgebra of hA and u generates its commutative AW^*-subalgebra, it suffices to show that a unitary u in a commutative AW^*-algebra A has a square root v in A. Regard A as $C(\Omega)$, the C^*-algebra of all continuous complex-valued functions on a stonean (extremely disconnected, compact Hausdorff) space Ω, and so u as a continuous function of Ω to $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$. The function $f : \mathbb{T} \to \mathbb{C}$, $f(e^{ir}) = e^{ir/2}$, $r \in \mathbb{R}$, $0 \le r < 2\pi$, being continuous except at 1, is a Baire function (a pointwise limit of a sequence of continuous functions in this case), and so is $v' := f \circ u$ on Ω, with $v'^2 = u$. Since Ω is stonean, $v' = v$ except on a meager (of the first category) subset of Ω for some $v \in A$ (see [5] or [32, p. 104, 1.7; p. 113, 1.24]), and this v is the desired unitary.

Proposition 4.4 *If $X \in \mathrm{RINV}_B(A)$ and $X = X^*$, then $X = AsA$ for some $s \in RN_B(A)$, with $s^0 = (s^0)^*$ and $s^{--} = (s^{++})^*$, so that $s = s^*$.*

Proof Suppose first $X = X^* \in \mathrm{INV}_B(A)^0$. Then, by Proposition 4.2(v), $X = Ar$ for some $r \in N_B(A)^0$, and $Ar = X = X^* = r^* A = Ar^*$. Hence, by Proposition 4.2(v), $r^* r = rr^* =: h \in \mathrm{Proj}\, Z(A)$, r^2 is a unitary in hA, and so $\theta := (\mathrm{Ad}\, r)|hA$ is a $*$-automorphism of hA with $\theta(r^2) = r^2$. Then, by (SRU), $v^2 = r^2$, $\theta(v) = v$ for some unitary $v \in hA$, and $s := v^* r$ satisfies the assertion. Indeed, $X = As$ by Proposition 4.2(v), and $v^* r = \theta(v)^* r = rv^* r^* r = rv^*$. Hence

$$s^* = r^* v = vr^* = hvr^* = v^* vvr^* = v^* r^2 r^* = v^* r = s,$$

$$s^* s = ss^* = s^2 = v^* rv^* r = (v^*)^2 r^2 = (r^*)^2 r^2 = h.$$

In general, we have $X = X^{++} \oplus X^0 \oplus X^{--}$ (see Proposition 3.5). Since this decomposition is unique and since $X = X^*$, we have $X^{--} = (X^{++})^*$ and $(X^0)^* = X^0$. By the first paragraph and Proposition 4.2(v), $X^0 = As_0$ and $X^{++} = As_1$ for some $s_0 \in N_B(A)^0$ and $s_1 \in N_B(A)^+$ with $s_0^* = s_0$ and with s_0^2, $s_1^* s_1$ and $s_1 s_1^*$ pairwise orthogonal. Then $s := s_1 + s_0 + s_1^*$ satisfies the assertion. $\blacksquare$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

5 The Picard Semigroup of a C^*-Algebra

Throughout this section A denotes a fixed C^*-algebra.

Definition 5.1 An *invertible A-module* is an algebraic invertible A-module (X, θ) in the sense of Definition 2.1 for which X is a left Hilbert A-module, and $\mathrm{INV}(A)$ denotes the set of all such modules (so $\mathrm{INV}(A) \subset \mathrm{INV}'(A)$). That is, the inner product in X satisfies, in addition to (2.2)–(2.4),

$$\langle x, x\rangle \geq 0 \text{ in } A,\ \forall x \in X, \tag{5.1}$$

and X is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. Note further that $\mathrm{End}_A(X)$, each of whose elements is necessarily bounded, is a C^*-algebra. Denote by $K(X)$ (resp., $K_r(X), K_l(X)$) the norm closure of $F(X) \subset \mathrm{End}_A(X)$ (resp., $F_r(X), F_l(X) \subset A$, see (2.6)). Then these are closed two-sided ideals of C^*-algebras with $K_r(X) = \theta^{-1}(K(X))$, and (2.5) is rephrased as

$$K(X) \subset \theta((\mathrm{Ker}\,\theta)^{\perp}). \tag{5.2}$$

A *submodule* of an invertible A-module means a *norm closed* sub-A-bimodule; an *ideal* of A means a *norm closed two-sided* one (hence a ∗-ideal) and so they belong to $\mathrm{INV}(A)$.

Note also that for $X \in \mathrm{INV}(A)$ and an ideal I of A, the sets $I \cdot X$ and $X \cdot I$, being norm closed [15, p. 268], are submodules of X and that $\mathrm{RINV}_B(A) \subset \mathrm{RINV}(A) \subset \mathrm{INV}(A)$ if A is an AW^*-subalgebra of another AW^*-algebra B containing the unit (see Section 3 and Definition 4.1).

Define two operations, inversion and product, in $\mathrm{INV}(A)$ as follows. The *inverse* $(X, \theta)^{-1} = (X^*, \theta) \in \mathrm{INV}'(A)$ of $(X, \theta) \in \mathrm{INV}(A)$ in the sense of Section 2 is also in $\mathrm{INV}(A)$, since (5.1) holds for X^* by (2), $\|x^*\|^2 = \|\langle x^*, x^*\rangle\| = \|\theta^{-1}(\langle\,\cdot\,, x\rangle \cdot x)\| = \|\langle\,\cdot\,, x\rangle \cdot x\| = \|x\|^2$ and so X^* is complete. Further (2.9), (2.10), etc. hold with $F(\,\cdot\,)$ replaced by $K(\,\cdot\,)$. For $(X_j, \theta_j) \in \mathrm{INV}(A)$, $j = 1, 2$, define the *product* $(X_1, \theta_1) \otimes_A (X_2, \theta_2) = (X_1 \otimes_{\theta_1} X_2, \widetilde{\theta_2}) \in \mathrm{INV}(A)$ as the norm completion of $(X_1, \theta_1) \odot_A (X_2, \theta_2)$ in Section 2. That is, $X_1 \otimes_{\theta_1} X_2$ is the Hilbert A-module completion of $X_1 \odot_{\theta_1} X_2$ [2, p. 130] and $\widetilde{\theta_2}\colon A \to \mathrm{End}_A(X_1 \otimes_{\theta_1} X_2)$ is the natural extension of $\widetilde{\theta_2}$ in Section 2.

All the remaining arguments in Propositions 2.3 and 2.5 and Remarks 2.4 and 2.6 hold true with the obvious modifications (with $\odot_A, F(\cdot)$, etc. replaced by $\otimes_A, K(\cdot)$, etc.). We omit writing down the statements corresponding to those in 2.3–2.6, and we freely use them below. Note that the inclusions in the statement corresponding to Proposition 2.3(iii) become equalities.

Denote by $[X]$ the isomorphism class of $X \in \mathrm{INV}(A)$ and by $[\mathrm{INV}(A)]$ the set of all such $[X]$. Note that for ideals I, J of A regarded as invertible A-modules we have $[I] = [J]$ if and only if $I = J$, and so the brackets can be omitted in this case, and that the set of all ideals of A is a commutative inverse semigroup with the inverse $I^{-1} = I^* = I$ and the product IJ (this is norm closed). Moreover $K_r(X)$ and $K_l(X)$ depend only on the isomorphism class $[X]$.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Here recall (see for example [20]) that an *inverse semigroup* is a semigroup S such that there corresponds to each $x \in S$ a unique element $x^{-1} \in S$, called the inverse of x, satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$, and that the subset $\{xx^{-1} : x \in S\}$, which coincides with the subset of all idempotents in S, forms a commutative inverse subsemigroup of S.

Theorem 5.2
(i) *The set* $[\mathrm{INV}(A)]$ *is an inverse semigroup with the inverse and product given by*

$$[X]^{-1} = [X^{-1}], \quad [X_1] \cdot [X_2] = [X_1 \otimes_A X_2];$$

its subsemigroup $\{[X] \cdot [X]^{-1} : [X] \in [\mathrm{INV}(A)]\}$ *is identified with the semigroup of all ideals of A.*

(ii) *For an ideal I of A the set* $[\mathrm{INV}(A)]_I := \{[X] \in [\mathrm{INV}(A)] : [X] \cdot [X]^{-1} = [X]^{-1} \cdot [X] = I\}$ *is a subgroup of* $[\mathrm{INV}(A)]$. *In particular, if A is simple, then* $[\mathrm{INV}(A)] \setminus \{0\}$ *is a group.*

Proof By the considerations above it suffices to show $[X] \cdot [X]^{-1} \cdot [X] = [X]$ for $X = (X, \theta) \in \mathrm{INV}(A)$ and the uniqueness of the inverse $[X]^{-1}$. We have

$$X = K_l(X) \cdot X = X \cdot K_r(X),$$

since for an approximate unit $\{u_i\}$ for $K_l(X)$ and $x \in X$, $\lim u_i \cdot x = x$ and since $(X \cdot K_r(X))^* = K_l(X^*) \cdot X^*$. Hence

$$[X] \cdot [X]^{-1} \cdot [X] = K_l(X) \cdot [X] = [K_l(X) \cdot X] = [X].$$

Suppose $[X] \cdot [Y] \cdot [X] = [X]$ and $[Y] \cdot [X] \cdot [Y] = [Y]$ for $Y \in \mathrm{INV}(A)$. Then the first equality multiplied on both sides by $[X]^{-1}$ turns to $[K_r(X) \cdot Y \cdot K_l(X)] = [X]^{-1}$. Moreover $K_r(X) = K_l(Y)$ and $K_l(X) = K_r(Y)$, and it follows that $[Y] = [X]^{-1}$. Indeed,

$$\begin{aligned} K_l(X) &= [X] \cdot [X]^{-1} = [X] \cdot ([X]^{-1} \cdot [X]) \cdot [X]^{-1} \\ &= [X] \cdot \big([X]^{-1} \cdot [Y]^{-1} \cdot [X]^{-1}[X] \cdot [Y] \cdot [X]\big)\, [X]^{-1} \\ &= K_l(X) \cdot [Y]^{-1} \cdot K_r(X) \cdot [Y] \cdot K_l(X) \\ &\subset K_l(X)[Y]^{-1} \cdot [Y] K_l(X) = K_l(X) K_r(Y) \subset K_l(X), \end{aligned}$$

so $K_l(X)K_r(Y) = K_l(X)$, and $K_l(X) \subset K_r(Y)$. Similarly $K_r(X) \subset K_l(Y)$, and the desired equalities follow. ∎

Definition 5.3 We call the inverse semigroup $[\mathrm{INV}(A)]$ the *Picard semigroup* of A. This term comes from the fact that the group $[\mathrm{INV}(A)]_A$ above for $I = A$ coincides with the Picard group $\mathrm{Pic}(A)$ of A in [3].

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

The following result describes all the submodules of an invertible A-module X in terms of $K_r(X)$ or $K_l(X)$. Denote by $M(B)$ and $Z(B)$ the multiplier C^*-algebra and the center of a C^*-algebra B, respectively, and write $M_j(X) = M(K_j(X))$, $j = r, l$. As is well known [2, p. 129], $\mathrm{End}_A(X) = M(K(X))$ for any Hilbert A-module X.

Proposition 5.4

(i) *For $(X, \theta) \in \mathrm{INV}(A)$ the map $I \mapsto (I \cdot X, \theta)$ is a bijection between the set of all ideals of $K_l(X)$ and the set of all submodules of (X, θ), with inverse given by $Y \mapsto K_l(Y)$, and similarly for the map $J \mapsto (X \cdot J, \theta)$ from the set of all ideals of $K_r(X)$.*

(ii) *For $(X, \theta) \in \mathrm{INV}(A)$ there is a $*$-isomorphism $\overline{\theta}\colon Z(M_r(X)) \to Z(M_l(X))$ such that $x\hat{\theta}(a) = \overline{\theta}(a) \cdot x$ for all $a \in Z(M_r(X))$ and $x \in X$, where $\hat{\theta}\colon M_r(X) = M(K_r(X)) \to M(K(X)) = \mathrm{End}_A(X)$ is the unique extension of the $*$-isomorphism $\theta|\,K_r(X)\colon K_r(X) \to K(X)$, and the original left module operation of $K_l(X) \subset A$ on X is canonically extended to that of $M_l(X)$.*

Proof (i) Note that X is a $K_l(X)$-$K_r(X)$-imprimitivity bimodule in the sense of Rieffel [28] if, on the left Hilbert A-module X, the module operation is restricted to $K_l(X)$, and the $K_l(X)$-valued and $K_r(X)$-valued inner products are defined respectively as the original one and $\langle x, y\rangle = \theta^{-1}\big(\langle\,\cdot\,, x\rangle \cdot y\big)$. Hence the assertion follows from [28] and Proposition 3.2.

(ii) For $a \in Z(M_r(X))$ the map $x^* \mapsto (x\hat{\theta}(a^*))^*$, $x \in X$, defines an element of $Z(\mathrm{End}_A(X^*))$, written $\rho(a)$. Indeed, $ab = ba$ for $b \in A$, since

$$M_r(X) = M(\theta^{-1}(K(X)))$$

and $\theta^{-1}(K(X))$ is an ideal of A;

$$\begin{aligned}(b \cdot x^*)\rho(a) &= (x\theta(b^*))^*\rho(a) = (x\theta(b^*)\hat{\theta}(a^*))^*\\ &= (x\hat{\theta}(a^*)\theta(b^*))^* = b \cdot (x^*\rho(a));\end{aligned}$$

and $\rho(a) \in \mathrm{End}_A(X^*)$. Further, since $\hat{\theta}(a) \in Z(\mathrm{End}_A(X))$, it follows that $\big(\langle\,\cdot\,, x^*\rangle \cdot y^*\big)\,\rho(a) = \rho(a)\big(\langle\,\cdot\,, x^*\rangle \cdot y^*\big)$ for all $x, y \in X$ and that $\rho(a) \in Z(\mathrm{End}_A(X^*))$. Denote by $\widehat{(\theta_{-1})}\colon M_l(X) = M(K_l(X)) = M(K_r(X^*)) \to M(K(X^*)) = \mathrm{End}_A(X^*)$ the unique extension of the $*$-isomorphism $\theta_{-1}|K_r(X^*)\colon K_r(X^*) \to K(X^*)$, and write $b \cdot x = (x^*\widehat{(\theta_{-1})}(b^*))^*$ for $b \in M_l(X)$ and $x \in X$. Then $\overline{\theta} := \widehat{(\theta_{-1})}^{-1} \circ \rho\colon Z(M_r(X)) \to Z(M_l(X))$ is a $*$-homomorphism such that

$$x\hat{\theta}(a) = (x^*\rho(a^*))^* = (x^*\widehat{(\theta_{-1})} \circ \overline{\theta}(a^*))^* = \overline{\theta}(a) \cdot x$$

for $a \in Z(M_r(X))$ and $x \in X$. By left-right symmetry, $\overline{\theta}$ has an inverse and it is a $*$-isomorphism. ∎

As noted above, an invertible A-module X is a $K_l(X)$-$K_r(X)$-imprimitivity bimodule, and the definition of the Picard group $\mathrm{Pic}(A)$ involves the notion of A-A-imprimitivity bimodules. The following shows, conversely, that any I-J-imprimitivity bimodule for some ideals I, J of A is indeed an invertible A-module.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proposition 5.5 *Let X be a left Hilbert A-module such that there is a $*$-isomorphism $\theta\colon I \to K(X)$ for some ideal I of A. Then a unique $*$-homomorphism $\theta_1\colon A \to \mathrm{End}_A(X)$ is defined so that $\theta_1|I = \theta$ and $\mathrm{Ker}\,\theta_1 = I^{\perp}$. Hence $K(X) \subset \theta_1((\mathrm{Ker}\,\theta_1)^{\perp})$, and (X, θ_1) is an invertible A-module.*

Proof If $\pi\colon A \to A/I^{\perp}$ is the quotient $*$-homomorphism, then $\pi|I$ is injective and $\pi(I)$ is an essential ideal of $A/I^{\perp}$. Hence we have canonically $\pi(I) \subset A/I^{\perp} \subset M(\pi(I))$. Moreover $\theta\colon I \to K(X)$ and $\pi|I\colon I \to \pi(I)$ extend uniquely to $*$-isomorphisms $\hat{\theta}\colon M(I) \to M(K(X)) = \mathrm{End}_A(X)$ and $\hat{\pi}\colon M(I) \to M(\pi(I))$. Then $\theta_1 := \hat{\theta} \circ \hat{\pi}^{-1} \circ \pi\colon A \to \mathrm{End}_A(X)$ is a $*$-homomorphism with $\theta_1|I = \theta$ and $\mathrm{Ker}\,\theta_1 = I^{\perp}$. To see the uniqueness of θ_1 let θ_2 be another such $*$-homomorphism. Then $\theta_1|(I+I^{\perp}) = \theta_2|(I+I^{\perp})$. Since $I + I^{\perp}$ is an essential ideal of A and so $I + I^{\perp} \subset A \subset M(I + I^{\perp})$, the surjective $*$-homomorphisms $\theta_j|(I + I^{\perp}) : I + I^{\perp} \to \theta(I) = K(X)$, $j = 1, 2$, extend to unique $*$-homomorphisms $(\theta_j|(I + I^{\perp}))\hat{}\colon M(I + I^{\perp}) \to M(K(X)) = \mathrm{End}_A(X)$ (see [25, p. 82, lines 3–4]). Hence $(\theta_1|(I + I^{\perp}))\hat{} = (\theta_2|(I + I^{\perp}))\hat{}$, and $\theta_1 = (\theta_1|(I + I^{\perp}))\hat{}|A = (\theta_2|(I + I^{\perp}))\hat{}|A = \theta_2$. ■

6 The Picard Semigroup of a Monotone Complete C^*-Algebra

Throughout this section, A denotes a fixed monotone complete C^*-algebra. The proofs of Propositions 6.3, 6.4 and Theorem 6.5 are omitted, since they parallel those of the assertions in Sections 2 and 5.

We recall here the following facts, which will be used repeatedly later.

Remark 6.1 (i) If B is an AW^*-algebra and C is its C^*-subalgebra, then the multiplier algebra $M(C)$ of C is identified with the subset $\{x \in pBp : xC + Cx \subset C\}$ of B, where p is the smallest projection of B with $pC = C$ (see [16, 27]). Henceforth, in such a situation, we write $M(C)$ for the subset. In particular, if C is an ideal of B, then $M(C) = hB$ for some $h \in \mathrm{Proj}\,Z(B)$.

(ii) A left Hilbert A-module X is self-dual (*i.e.,* each bounded module homomorphism of X into A is of the form $\langle \cdot, x\rangle : y \mapsto \langle y, x\rangle$ for some $x \in X$) if and only if there are a monotone complete C^*-algebra B and $e, f \in \mathrm{Proj}\,B$, with $C(f) = 1$, so that X, A and $\mathrm{End}_A(X)$ are identified respectively with fBe, fBf and eBe, where $a \cdot x = ax$, $\langle x, y\rangle = xy^* \in fBf = A$ for $a \in A = fBf$, $x, y \in X = fBe$, and $\langle \cdot, x\rangle \cdot y \in K(X)$ for $x, y \in X$ is identified with $x^*y \in eBe$. (See [12]; an obvious change in the presentation is needed here, since right, rather than left, A-modules are treated in [12], and note that additive completeness there is now known to be equivalent to monotone completeness, [29].) To be more explicit, we may take for some index set I, $B = A \overline{\otimes} B(l^2(I))$, the monotone complete C^*-algebra consisting of matrices $[a_{ij}], i, j \in I$, with entries from A, $f = [\delta_{i_0 i}\delta_{i_0 j}1] = 1 \otimes e_0$ for some fixed $i_0 \in I$, $e_0 := [\delta_{i_0 i}\delta_{i_0 j}] \in B(l^2(I))$, and $e = [\delta_{ij}e_i]$ for some $e_i \in \mathrm{Proj}\,A, i \in I$. Further, A and $fBf = \{[\delta_{i_0 i}\delta_{i_0 j}a] : a \in A\}$ are identified by the $*$-monomorphism $\pi\colon A \to fBf \subset B$ defined by $\pi(a) = [\delta_{i_0 i}\delta_{i_0 j}a] = a \otimes e_0$.

Definition 6.2 An invertible A-module (X, θ) is called *self-dual* if X is a self-dual left Hilbert A-module (so $\mathrm{End}_A(X)$ is a monotone complete C^*-algebra, [12, 1.1])

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

and if the $*$-homomorphism $\theta\colon A \to \mathrm{End}_A(X)$ is normal (*i.e.,* preserves the suprema of bounded increasing nets). Denote by $\mathrm{SDINV}(A)$ the set of all self-dual invertible A-modules.

Then $\operatorname{Ker}\theta = (1-h)A$ for some $h \in \operatorname{Proj} Z(A)$, and $\theta|\,hA$ is a $*$-isomorphism of hA onto $\mathrm{End}_A(X)$, since $\theta(hA) = \theta((\operatorname{Ker}\theta)^{\perp})$ is monotone closed in $\mathrm{End}_A(X)$ and so contains $M(K(X)) = \mathrm{End}_A(X)$ by Remark 6.1(i). Thus a self-dual invertible A-module is identified with a pair (X,θ) of a self-dual left Hilbert A-module X and a $*$-isomorphism θ of hA onto $\mathrm{End}_A(X)$ for some $h \in \operatorname{Proj} Z(A)$.

For $(X,\theta) \in \mathrm{SDINV}(A)$ define two central projections $z_j(X) = z_j(X,\theta)$, $j = r, l$, of A, so that $M_j(X) = M(K_j(X)) = z_j(X)A$ (see Remark 6.1(i)), $\operatorname{Ker}\theta = (1-z_r(X))A$ and $\theta|\,z_r(X)A : z_r(X)A \to \mathrm{End}_A(X)$ is a $*$-isomorphism. Write θ^{-1} for the inverse of $\theta|\,z_r(X)A$. Equivalently, $z_j(X)$ are the smallest central projections of A such that $z_l(X)\cdot x = x = x\cdot z_r(X)$ for all $x \in X$.

We introduce two operations in $\mathrm{SDINV}(A)$ as follows. First, note that the inverse $(X,\theta)^{-1} = (X^*,\theta_{-1})$ of $(X,\theta) \in \mathrm{SDINV}(A)$ is also in $\mathrm{SDINV}(A)$. Indeed, as in Remark 6.1(ii), take B, e, f so that $X = fBe, A = fBf$, etc. Then, $\theta|\,hA$, where $h := z_r(X)$, is a $*$-isomorphism of hA onto $\mathrm{End}_A(X) = eBe$; $k := z_l(X) = C(e)f$, a central projection of $fBf = A$ (note that $M(fBeBf) = C(e)fBf$); X^* is identified with $eBf = (fBe)^*$ equipped with the module operation $a\cdot x^*\cdot b = \theta(a)x^*b$ and the inner product $\langle x^*, y^*\rangle = \theta^{-1}(x^*y) \in hA$ for $a, b \in A$ and $x, y \in X$; $\mathrm{End}_A(X^*) = M(fBeBf) = kA$; and $\theta_{-1}\colon A \to \mathrm{End}_A(X^*)$ is identified with the map $a \mapsto ka$. Hence, via the $*$-isomorphism $\theta|\,hA$, X^* is identified with eBf, regarded canonically as a left Hilbert eBe-module, so $X^{-1} = (X,\theta)^{-1}$ is self-dual, and $z_r(X^{-1}) = k = z_l(X)$, $z_l(X^{-1}) = h = z_r(X)$.

Next, for $(X_j,\theta_j) \in \mathrm{SDINV}(A)$, $j = 1, 2$, define their *product*, which we denote $(X_1,\theta_1)\overline{\otimes}_A(X_2,\theta_2)$, as $(X_1\,\overline{\otimes}_{\theta_1} X_2, \widetilde{\theta}_2)$, where $X_1\,\overline{\otimes}_{\theta_1} X_2$ is the self-dual completion [12, 2.2] of the Hilbert A-module $X_1 \otimes_{\theta_1} X_2$, *i.e.,* a unique self-dual Hilbert A-module containing $X_1 \otimes_{\theta_1} X_2$ and generated by it, and $\widetilde{\theta}_2$ is the $*$-homomorphism of A into $\mathrm{End}_A(X_1\,\overline{\otimes}_{\theta_1} X_2)$, which we obtain by extending each $\widetilde{\theta}_2(a) \in \mathrm{End}_A(X_1 \otimes_{\theta_1} X_2)$ to the whole of $X_1\,\overline{\otimes}_{\theta_1} X_2$. The normality of $\widetilde{\theta}_2$ follows from the normality of the maps $a \mapsto \langle (x_1\otimes x_2)\widetilde{\theta}_2(a), x_1\otimes x_2\rangle = \langle x_1\theta_1\big(\langle x_2\theta_2(a), x_2\rangle\big), x_1\rangle$ for $x_j \in X_j$, and $\widetilde{\theta}_2$ is surjective, since $\widetilde{\theta}_2(A)$ is monotone closed in $\mathrm{End}_A(X_1\,\overline{\otimes}_{\theta_1} X_2)$, it contains $K(X_1 \otimes_{\theta_1} X_2)$, and $X_1 \otimes_{\theta_1} X_2$ generates $X_1\,\overline{\otimes}_{\theta_1} X_2$. Thus $(X_1\,\overline{\otimes}_{\theta_1} X_2, \widetilde{\theta}_2)$ is self-dual.

We see that a submodule (Y,θ) of $(X,\theta) \in \mathrm{SDINV}(A)$ is self-dual if and only if $Y = X\theta(h) = \overline{\theta}(h)\cdot X$ for some $h \in \operatorname{Proj}(z_r(X)Z(A))$, where $\overline{\theta}$ is the $*$-isomorphism $z_r(X)Z(A) = Z(M_r(X)) \to Z(M_l(X)) = z_l(X)Z(A)$ in Proposition 5.4(ii) and we note that $\hat{\theta}$ there is just $\theta|\,z_r(X)A$. Indeed, identify X with fBe, with B, e, f as above. If (Y,θ) is self-dual and so Y is a self-dual Hilbert A-module, then $Y = Xp = fBp$ for some $p \in \operatorname{Proj}(\mathrm{End}_A(X))$, *i.e.,* $p \in \operatorname{Proj} B$, $p \le e$ [12, 1.9], and that $p \in Z(eBe)$ follows, since $fBpBe = (fBp)(eBe) \subset fBp$, $fBpB(e-p) = 0$ and $C(f) = 1$. Hence $h = \theta^{-1}(p)$ is the desired central projection in $z_r(X)A$, and the reverse implication is clear.

Moreover the following version of Proposition 5.4 holds.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proposition 6.3

(i) *For* $(X,\theta) \in \mathrm{SDINV}(A)$ *the map* $h \mapsto (X \cdot h, \theta) = (X\theta(h), \theta)$ *is a bijection between* $\mathrm{Proj}\, z_r(X)Z(A)$ *and the set of all self-dual submodules of* (X,θ)*, and similarly for the map* $k \mapsto (k \cdot X, \theta)$ *from* $\mathrm{Proj}\, z_l(X)Z(A)$.

(ii) *The* $*$*-isomorphism* $\overline{\theta}\colon z_r(X)Z(A) \to z_l(X)Z(A)$ *in Proposition* 5.4 *relates the two bijections above so that* $x\theta(a) = \overline{\theta}(a) \cdot x$ *for* $a \in z_r(X)Z(A)$ *and* $x \in X$.

We regard an ideal of A of the form hA for $h \in \mathrm{Proj}\, Z(A)$ canonically as a self-dual invertible A-module. Then we can state the following version of Proposition 2.3.

Proposition 6.4 *Let* (X,θ), (X_j,θ_j)*, etc. be in* $\mathrm{SDINV}(A)$.

(i) *We have*

$$(X,\theta)\,\overline{\otimes}_A\,(X,\theta)^{-1} \cong z_l(X)A, \quad (X,\theta)^{-1}\,\overline{\otimes}_A\,(X,\theta) \cong z_r(X)A,$$

and for $h,k \in \mathrm{Proj}\, Z(A)$ *we have*

$$hA\,\overline{\otimes}_A\,(X,\theta) \cong (h\cdot X,\theta), \quad (X,\theta)\,\overline{\otimes}_A\,kA \cong (X\theta(k),\theta), \quad hA\,\overline{\otimes}_A\,kA \cong hkA.$$

(ii) *We have*

$$((X_1,\theta_1)\,\overline{\otimes}_A\,(X_2,\theta_2))^{-1} \cong (X_2,\theta_2)^{-1}\,\overline{\otimes}_A\,(X_1,\theta_1)^{-1}.$$

(iii) *We have, for* $h,k \in \mathrm{Proj}\, Z(A)$,

$$z_r(k\cdot X\cdot h) = (\overline{\theta})^{-1}(kz_l(X))h, \quad z_l(k\cdot X\cdot h) = k\overline{\theta}(hz_r(X)),$$

and

$$z_r((X_1,\theta_1)\,\overline{\otimes}_A\,(X_2,\theta_2)) = (\overline{\theta_2})^{-1}(z_r(X_1)z_l(X_2)),$$
$$z_l((X_1,\theta_1)\,\overline{\otimes}_A\,(X_2,\theta_2)) = \overline{\theta_1}(z_r(X_1)z_l(X_2)).$$

(iv) *If* $\tau_j\colon X_j \to Y_j$, $j = 1,2$*, are monomorphisms, then a monomorphism*

$$\tau_1\,\overline{\otimes}\,\tau_2\colon X_1\,\overline{\otimes}_A\,X_2 \to Y_1\,\overline{\otimes}_A\,Y_2$$

is defined by $(\tau_1\,\overline{\otimes}_A\,\tau_2)(x\otimes x_2) = \tau_1(x_1)\otimes\tau_2(x_2)$.

(v) *The operation* $\overline{\otimes}_A$ *is associative in the sense of Proposition* 2.3(v).

As in Theorem 5.2 we associate with the monotone complete C^*-algebra A the following inverse semigroup $[\mathrm{SDINV}(A)]$, which we call the *Picard semigroup* of A:

Theorem 6.5

(i) *The set* $[\mathrm{SDINV}(A)]$ *of all isomorphism classes* $[X,\theta]$ *of elements* (X,θ) *in* $\mathrm{SDINV}(A)$ *is an inverse semigroup with the inverse and product given by*

$$[X]^{-1} = [X^{-1}], \quad [X_1]\cdot[X_2] = [X_1\,\overline{\otimes}_A\,X_2];$$

and its subsemigroup $\{[X]\cdot[X]^{-1} : [X] \in [\mathrm{SDINV}(A)]\}$ *is identified with the multiplicative semigroup* $\mathrm{Proj}\, Z(A)$ *of all central projections of* A.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(ii) *For* $h \in \operatorname{Proj} Z(A)$ *the subset*

$$[\mathrm{SDINV}(A)]_h := \{[X] \in [\mathrm{SDINV}(A)] : [X] \cdot [X]^{-1} = [X]^{-1} \cdot [X] = hA\}$$

is a subgroup of $[\mathrm{SDINV}(A)]$. *In particular, if* A *is a monotone complete* AW^**-factor, then* $[\mathrm{SDINV}(A)] \setminus \{0\}$ *is a group.*

The rest of this section establishes an isomorphism between $[\mathrm{SDINV}(A)]$ and the set of certain equivalence classes in PAut A, introducing an inverse semigroup structure in the latter set. Note that as A is an AW^*-algebra, all the results in Section 3 are available here.

In view of Definition 6.2, the regular invertible A-module $\langle\theta\rangle$ associated with $\theta \in$ RPAut A (see Definition 3.3) belongs to $\mathrm{SDINV}(A)$. It follows immediately that in the notation of Section 6,

$$z_r\big(\langle\theta\rangle\big) = C(r(\theta)), \quad z_l\big(\langle\theta\rangle\big) = C(l(\theta)),$$

and that the $*$-isomorphism $C(r(\theta))Z(A) \to C(l(\theta))Z(A)$ defined in Proposition 5.4(ii) for $\langle\theta\rangle$ coincides with the $*$-isomorphism $\overline{\theta}$ defined in Section 3 for $\theta \in$ PAut A (see also the argument before Proposition 6.3; this is the reason for the notational coincidence).

To state the following theorem we need a fact from [12, 2.10(ii)], whose omitted proof is filled in here. Let B be a monotone complete C^*-algebra and X a monotone closed triple subsystem of B (*i.e.,* a linear subspace of B closed under order-convergence such that $XX^*X \subset X$). Denote by $K_r(X), K_l(X)$ the norm closed linear spans of X^*X, XX^*, respectively, which are C^*-subalgebras of B. (Note that the usage of the subscripts r, l here is consistent with that in [12].) Then their monotone closures, $M_r(X), M_l(X)$, in B satisfy $XM_r(X) \subset X, M_l(X)X \subset X$, and so $K_r(X)M_r(X) \subset K_r(X)$, $M_l(X)K_l(X) \subset K_l(X)$, since $XK_r(X) + K_l(X)X \subset X$, and X is monotone closed in B (see [10, 18]). Hence $M_j(X)$, $j = r, l$, are monotone closed (so AW^*) subalgebras of B, and they are multiplier algebras of $K_j(X)$ realized in B (see Remark 6.1(i)). Moreover X is regarded canonically as a self-dual left Hilbert $M_l(X)$-module with $\mathrm{End}_{M_l(X)}(X) = M_r(X)$ or a self-dual right Hilbert $M_r(X)$-module with $\mathrm{End}_{M_r(X)}(X) = M_l(X)$. Indeed,

$$L(X) := \begin{bmatrix} M_l(X) & X \\ X^* & M_r(X) \end{bmatrix}$$

is a monotone closed C^*-subalgebra of $B \otimes M_2$, the monotone complete C^*-algebra of 2×2 matrices over B, and hence a monotone complete C^*-algebra, and the argument in Remark 6.1(ii) applies.

We also remark that if the above B is a von Neumann algebra, then a triple subsystem X of B is monotone closed in B if and only if it is σ-weakly closed in B. Indeed, the sufficiency is clear, since the supremum of a bounded increasing net in the self-adjoint part of B is a σ-weak limit of the net. Conversely, if X is monotone closed in B, then so is the $*$-algebra $L(X)$ in $B \otimes M_2$, and by [17], $L(X)$ and hence $\left[\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] L(X) \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is σ-weakly closed in $B \otimes M_2$. Thus X is σ-weakly closed in B.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Theorem 6.6 *We have* SDINV(A) = RINV(A). *That is, for an invertible A-module (X,θ) the following conditions are equivalent:*

(i) (X,θ) *is self-dual;*

(ii) (X,θ) *is regular,* i.e., *isomorphic to* $\langle\psi\rangle$ *for some* $\psi \in \mathrm{RPAut}\, A$*;*

(iii) *There exists a monotone complete C^*-algebra B which contains X as a monotone closed triple subsystem, and $*$-isomorphisms $\theta'\colon hA \to M_r(X)$, $\psi\colon kA \to M_l(X)$ for some $h, k \in \mathrm{Proj}\, Z(A)$, so that the module operation and inner product in X are given by*

$$a\cdot x\cdot b = \psi(ka)x\theta'(hb), \quad \langle x, y\rangle = \psi^{-1}(xy^*)$$

for $a, b \in A$, $x, y \in X$, and the $*$-homomorphism θ is given by $a \mapsto ha \mapsto \theta'(ha)$.

Proof The implications (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i), and (iii) $\Rightarrow$ (i) follow from Remark 6.1 and the above arguments; it remains only to show (i) $\Rightarrow$ (ii).

Let (X,θ) be self-dual. As in Remark 6.1(ii), take B, e, f so that $X = fBe$, $A = fBf$, $\mathrm{End}_A(X) = eBe$ and $a\cdot x = ax$, $\langle x, y\rangle = xy^*$ for $a \in A$, $x, y \in X$. By (GC) (see Section 3) applied to $e, f \in \mathrm{Proj}\, B$ there exist $h \in \mathrm{Proj}\, Z(B)$ and $u, v \in \mathrm{PI}\, B$ such that

$$he = u^*u, \quad uu^* \le hf, \quad (1-h)e \ge v^*v, \quad vv^* = (1-h)f.$$

Then $X_1 := hX$ and $X_2 := (1-h)X$ are submodules of X with $(X,\theta) = (X_1,\theta) \oplus (X_2,\theta)$. With $e_1 := uu^* \in fBf = A$ we have

$$X_1u^* = hfBeu^* = fBeu^* = fBu^* = fBfuu^* = Ae_1,$$

and

$$\theta_1 := (\mathrm{Ad}\, u)\circ\theta : A \to eBe \to heBe = u^*uBu^*u \to uu^*Buu^* = uu^*fBfuu^* = e_1Ae_1$$

is a surjective normal $*$-homomorphism. Hence $\theta_1 \in (\mathrm{PAut}\, A)^+$ when regarded as the composition of the map $a \mapsto ka$ and the $*$-isomorphism $\theta_1|kA$ for some $k \in \mathrm{Proj}\, Z(A)$, and an isomorphism $\tau_1\colon (X_1,\theta) \to \langle\theta_1\rangle = (Ae_1,\theta_1)$ is defined by $\tau_1(x) = xu^*$. Moreover, with $e_2 := \theta^{-1}(v^*v) \in \theta^{-1}(eBe) = z_r(X)A$ we have

$$v^*X_2 \subset eBe, \quad \theta^{-1}(v^*X_2) = \theta^{-1}(v^*(1-h)fBe) = \theta^{-1}(v^*Be) = \theta^{-1}(v^*veBe) = e_2A,$$

and

$$\theta_2 := \theta^{-1}\circ(\mathrm{Ad}\, v^*) : A = fBf \to (1-h)fBf = vv^*Bvv^* \to v^*vBv^*v$$
$$= v^*veBev^*v \to e_2Ae_2$$

is a surjective normal $*$-homomorphism. Hence $\theta_2 \in (\mathrm{PAut}\, A)^+$, and an isomorphism $\tau_2\colon (X_2,\theta) \to \langle\theta_2\rangle^{-1} = (Ae_2,\theta_2)^{-1} = (e_2A,(\theta_2)_{-1})$ is defined by $\tau_2(x) = \theta^{-1}(v^*x)$. We have $\psi := \theta_1 \oplus \theta_2^{-1} \in \mathrm{RPAut}\, A$, since

$$\theta_1(l(\theta_2)) = \theta_1(e_2) = u\theta(\theta^{-1}(v^*v))u^* = 0$$

and $\theta_2(l(\theta_1)) = \theta_2(e_1) = 0$, and $(X,\theta) \cong \langle\theta_1\rangle \oplus \langle\theta_2\rangle^{-1} = \langle\psi\rangle$, as desired. ∎

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Corollary 6.7 *For a monotone complete C^*-algebra B and its linear subspace X the following are equivalent:*

(i) *There exists a monotone closed C^*-subalgebra C of B, satisfying $CX+XC \subset X$ and $XX^* + X^*X \subset C$, so that X, regarded canonically as a left pre-Hilbert C-module, is a self-dual Hilbert C-module;*

(ii) *There exist a monotone closed C^*-subalgebra C of B and $s \in RN_{pBp}(C)$ such that $X = CsC$, where p is the unit of C;*

(iii) *X is monotone closed in B, and there exists a monotone closed C^*-subalgebra C of B such that $CX + XC \subset X$ and $XX^* + X^*X \subset C$;*

(iv) *X is a monotone closed triple subsystem of B such that $X^*XX + XXX^* \subset X$.*

For X as in (iv)*, we may take $M_r(X) + M_l(X)$ as C in* (i) *or* (ii) *or* (iii)*, and it is the smallest in the sense that such a C contains $M_r(X) + M_l(X)$ as a monotone closed two-sided ideal.*

Proof (i) ⇒ (ii): Note first that under the stated condition we may assume the unit p of C to be the unit 1 of B. Indeed, for $x \in X$ we have $x(1-p) = x - xp \in X$, $(x(1-p))^*x(1-p) \in X^*X \subset C = pC$, and $x(1-p) = 0$, $x = xp$. Similarly $x = px$, and so it suffices to consider pBp instead of B.

If we define, for the self-dual Hilbert C-module X as above, a $*$-homomorphism $\theta\colon C \to \mathrm{End}_C(X)$ by $x\theta(a) = xa$, then $(X,\theta) \in \mathrm{SDINV}(C)$. Indeed, C, being monotone closed in B, is an AW^*-algebra, and by Remark 6.1(i), $M_r(X) = M(K_r(X)) = hC$ for some $h \in \mathrm{Proj}\, Z(C)$, since the norm closure, $K_r(X)$, of X^*X, is a two-sided ideal of C. Moreover, $\theta|\, hC$ is a $*$-isomorphism onto $\mathrm{End}_C(X)$, since its restriction to $K_r(X)$ is a $*$-isomorphism $K_r(X) \to K(X)$, $M_r(X) = M(K_r(X))$, $M(K(X)) = \mathrm{End}_C(X)$ and the multiplier algebras are unique; $\mathrm{Ker}\,\theta = (1-h)C$, since $\theta(a) = 0$ for $a \in C$ if and only if $x^*ya = 0$ in C for all $x, y \in X$ if and only if $ha = 0$; and the assertion follows. Then, by Theorem 6.6, (i) ⇒ (ii), $X \cong \langle\theta\rangle$ for some $\theta \in \mathrm{RPAut}\, C$. If $\langle\theta\rangle = \langle\theta_1\rangle \oplus \langle\theta_2\rangle^{-1} = Cl(\theta_1) \oplus l(\theta_2)C$ for $\theta_1, \theta_2 \in (\mathrm{PAut}\, C)^+$ and τ is an isomorphism of $\langle\theta\rangle$ onto X, then $s := \tau(l(\theta_1) \oplus l(\theta_2)) \in RN_B(C)$ and $X = CsC$.

(ii) ⇒ (iii): It suffices to show that CsC is monotone closed in B. As above we may assume that the units of C and B coincide. For some $h \in \mathrm{Proj}\, Z(C)$ we have $h \le s^*s$, $s(1-h)s^* \in \mathrm{Proj}\, Z(C)$, and $CsC = Csh + s(1-h)C$. Suppose that $x_i \to x(O)$ (x_i order-converges to x in B) for some $x_i \in CsC$ and $x \in B$. Then $x_ihs^* \to xhs^*(O)$, $x_ihs^* \in Cshs^*$, and $Cshs^*$ is monotone closed in B, since so is C and $shs^* \in \mathrm{Proj}\, C$. Hence $xhs^* \in Cshs^*$, and $xh = xhs^*s \in Cshs^*s = Csh$, since $x_ih \in Csh$ and so $x_ihs^*s = x_ih$. Similarly, $x(1-h) \in s(1-h)C$, since $s^*x_i(1-h) \in s^*s(1-h)C$, $s^*s(1-h)C$ is monotone closed in B and $ss^*x_i(1-h) = x_i(1-h)$. Thus $x = xh + x(1-h) \in CsC$, as desired.

(iii) ⇒ (iv): If C is as in (iii), then $XX^*X = (XX^*)X \subset CX \subset X$ and $X^*XX + XXX^* \subset CX + XC \subset X$.

(iv) ⇒ (i): For X as in (iv), using the the argument and notation before Theorem 6.6, the monotone closure, C, of $M_r(X) + M_l(X)$ in B is a monotone closed C^*-subalgebra of B containing $M_r(X), M_l(X)$ as monotone closed two-sided ideals, since $X^*X \cdot XX^* = (X^*XX)X^* \subset XX^*$, $X^*X \cdot XX^* = X^*(XXX^*) \subset X^*X$, etc. imply $M_r(X)M_l(X) + M_l(X)M_r(X) \subset M_r(X) \cap M_l(X)$, and since $M_r(X)$, $M_l(X)$ are

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

monotone closed C^*-subalgebras of B.

Hence $M_r(X) = hC$, $M_l(X) = kC$ for some $h, k \in \operatorname{Proj} Z(C)$, so $h \vee k =: p$ is the unit of C, $C = M_r(X) + M_l(X)$, and (i) follows from Theorem 6.6, (iii) $\Rightarrow$ (i).

If X and C are as in (i) (or (ii), (iii)), then C contains $M_r(X), M_l(X)$ as monotone closed two-sided ideals, and hence the last assertion follows. ■

We can sharpen Proposition 4.3 in the monotone complete C^*-situation as follows.

Corollary 6.8 *Let B be a monotone complete C^*-algebra containing A as a monotone closed C^*-subalgebra with the same unit. Then*

(i) $A \cdot RN_B(A) \cdot A = N_B(A)$;

(ii) *$x \in B$ is in* PI $N_B(A)$ *if and only if there exist $s \in RN_B(A)$, and $u, v \in$* PI A *such that $u^*u = svv^*s^*$, $vv^* = s^*u^*us$, and $x = usv$.*

(iii) *For $x \in N_B(A)$, AxA is monotone closed in B if and only if there exist $s, s' \in N_B(A)^+$, $a, a' \in A$, and finite $b_i, c_i, {b_j}', {c_j}' \in A$ such that*

$$ss' = 0 = s's, \quad x = as + s'^* a'^*, \quad a = ass^*, \quad a' = a's's'^*,$$

$$ss^* = \Big(\sum_i b_i a c_i\Big) ss^*, \quad \textit{and} \quad s's'^* = \Big(\sum_j {b_j}' a' {c_j}'\Big) s's'^*.$$

Proof To see (i) and (ii) it suffices, by Proposition 4.3, to show that $N_B(A) \subset A \cdot RN_B(A) \cdot A$. But, if $x \in N_B(A)$, then by Proposition 4.2(i) and Corollary 6.7, the monotone closure, $m\text{-cl}_B AxA$, of AxA in B equals AsA for some $s \in RN_B(A)$. Hence $x \in AsA$, as desired.

(iii) As above, AxA is monotone closed if and only if $AxA = AsA$ for some $s \in RN_B(A)$. Since $s = s_1 + s_2$ with $s_1 = sh = ks$, $s_2 = s(1-h) = (1-k)s$ for some $h, k \in \operatorname{Proj} Z(A)$ and $s_1, s_2^* \in N_B(A)^+$, it suffices to show that $x = xs^*s$ and $AxA = As$ for some $s \in N_B(A)^+$ if and only if the asserted condition holds (with $s' = a'_j = b'_j = 0$). But, if $AxA = As$, then $a := xs^* \in AxAs^* = Ass^* \subset A$, $a = ass^*$, $x = xs^*s = as$, and $ss^* \in Ass^* = AxAs^* = Aa(sAs^*) = Aass^*Ass^* = AaAss^*$. Hence $ss^* = (\sum_i b_i a c_i) ss^*$ for some $b_i, c_i \in A$. Conversely, if $x = as$, $a = ass^*$, and $ss^* = (\sum_i b_i a c_i) ss^*$, then $AxA = AasA \subset AaAs \subset As = A(\sum_i b_i a c_i)s = A(\sum_i b_i a s s^* c_i)s \subset Axs^*As \subset AxA$, and $AxA = As$. ■

By Theorem 6.6 each element of $\operatorname{SDINV}(A)$ is represented by an element of RPAut A. But, in order to describe the isomorphism from $[\operatorname{SDINV}(A)]$ onto certain equivalence classes of PAut A alluded to before, we need to consider not necessarily regular elements of PAut A and to relate them to regular ones.

Definition 6.9 Let $\theta, \psi \in$ PAut A. Write $\theta \simeq \theta_1$ if $\theta_1 = (\operatorname{Ad} v) \circ \theta \circ (\operatorname{Ad} u)$ for some $u, v \in$ PI A with $uu^* = r(\theta)$ and $v^*v = l(\theta)$; $\theta \leq \psi$ if $r(\theta) \leq r(\psi) \leq C(r(\theta))$ and $\psi | r(\theta) A r(\theta) = \theta$; $\theta \prec \psi$ if $\theta \simeq \theta_1 \leq \psi$ for some $\theta_1 \in$ PAut A, or equivalently,

$$(6.1) \qquad \begin{cases} \exists u, v \in \text{PI}\, A \text{ such that } \theta = (\operatorname{Ad} v) \circ \psi \circ (\operatorname{Ad} u), \\ u^*u = r(\theta), \quad uu^* \leq r(\psi) \leq C(uu^*), \quad \psi(uu^*) = v^*v; \end{cases}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

and $\theta \sim \psi$ if $\theta \geq \theta_1 \simeq \psi_1 \leq \psi$ for some $\theta_1, \psi_1 \in \operatorname{PAut} A$. Call θ_1 a *perturbation* of θ if $\theta \simeq \theta_1$, and ψ a *regularization* of θ if ψ is regular and $\theta \prec \psi$.

Remark 6.10 For later use we summarize here the following direct consequences of the definition:

(i) In the set $\operatorname{PAut} A$, $\simeq$ is an equivalence relation; and $\leq$ and $\prec$ are transitive, *i.e.*, $\theta \leq \psi \leq \omega$ (resp., $\theta \prec \psi \prec \omega$) implies $\theta \leq \omega$ (resp., $\theta \prec \omega$). The relations $\simeq$, $\leq$, $\prec$ and $\sim$ are compatible with the operations $\theta \mapsto \theta^{-1}$ and $\theta \mapsto k \cdot \theta \cdot h$:

$$\theta \simeq \psi \Leftrightarrow \theta^{-1} \simeq \psi^{-1}; \tag{6.2}$$

$$\forall h, k \in \operatorname{Proj} Z(A), \theta \simeq \psi \ \Rightarrow \ k \cdot \theta \cdot h \simeq k \cdot \psi \cdot h; \tag{6.3}$$

$$\left.\begin{array}{r} \theta \cdot h \simeq \psi \cdot h, \ \theta \cdot (1-h) \simeq \psi \cdot (1-h) \\ \text{or } k \cdot \theta \simeq k \cdot \psi, (1-k) \cdot \theta \simeq (1-k) \cdot \psi \end{array}\right\} \Longrightarrow \theta \simeq \psi; \tag{6.4}$$

and these are true with $\leq$, $\prec$, $\sim$ replaced by $\simeq$. Moreover

$$\theta \leq \psi \simeq \psi_1 \implies \theta \prec \psi_1. \tag{6.5}$$

Indeed, suppose that $\theta = \psi | r(\theta) A r(\theta)$, $r(\theta) \leq r(\psi) \leq C(r(\theta))$ and $\psi_1 = (\operatorname{Ad} v) \circ \psi \circ (\operatorname{Ad} u)$ for $u, v \in \operatorname{PI} A$ with $uu^* = r(\psi)$ and $v^* v = l(\psi)$. Set $u_1 = r(\theta) u$ and $v_1 = v l(\theta)$. Then $u_1 u_1^* = r(\theta)$, $v_1^* v_1 = l(\theta)$, and $\theta \simeq (\operatorname{Ad} v_1) \circ \theta \circ (\operatorname{Ad} u_1) = \psi_1 | u^* r(\theta) u A u^* r(\theta) u$, $C(u^* r(\theta) u) = C(r(\theta) u u^* r(\theta)) = C(r(\theta)) \geq u^* u = r(\psi_1)$, since $uu^* = r(\psi) \leq C(r(\theta))$.
We have
(6.6)

$$\theta \in \operatorname{PAut} A, u \in \operatorname{PI} A, u^* u \leq r(\theta), uu^* \leq r(\theta) \quad \Longrightarrow \quad \theta | u^* u A u^* u \simeq \theta | uu^* A uu^*.$$

Indeed, $u \in r(\theta) A r(\theta)$, and with $\psi := \theta | u^* u A u^* u$ we have $\theta | uu^* A uu^* = (\operatorname{Ad} \theta(u)) \circ \psi \circ (\operatorname{Ad} u^*) \simeq \psi$.

(ii) The relation $\sim$ is transitive and so it is an equivalence relation. Indeed, suppose that $\theta \sim \psi$, $\psi \sim \omega$, *i.e.*, $\theta \geq \theta_1 \simeq \psi_1 \leq \psi$, $\psi \geq \psi_2 \simeq \omega_1 \leq \omega$ for some θ_1, etc. We have $r(\psi_j) \leq r(\psi)$ and $C(r(\psi_j)) = C(r(\theta))$, $j = 1, 2$. By (GC$'$) (see Section 3), $r(\psi_1) \geq u^* u$, $uu^* \leq r(\psi_2)$ for some $u \in \operatorname{PI} A$ with $C(u^* u) = C(r(\psi_1)) = C(r(\psi_2))$, and

$$\psi | u^* u A u^* u \leq \psi_1 \leq \psi, \quad \psi | uu^* A uu^* \leq \psi_2 \leq \psi.$$

Hence, by (6.5) and (6.6),

$$\theta_1 \geq \theta_2 \simeq \psi | u^* u A u^* u \simeq \psi | uu^* A uu^* \simeq \omega_2 \leq \omega_1$$

for some θ_2, ω_2, and so $\theta \sim \omega$.

(iii) Clearly $\theta \sim \psi$ implies $C(r(\theta)) = C(r(\psi))$, $C(l(\theta)) = C(l(\psi))$ and $\overline{\theta} = \overline{\psi}$.

(iv) For inner partial $*$-automorphims $\operatorname{Ad} u$, $\operatorname{Ad} v$ of A for $u, v \in \operatorname{PI} A$ we have

$$\operatorname{Ad} u \sim \operatorname{Ad} v \iff C(u^* u) = C(v^* v).$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Indeed, suppose $C(u^*u) = C(v^*v)$. Then, by (GC′), $u^*u \geq w^*w$, $ww^* \leq v^*v$ for some $w \in \mathrm{PI}\, A$ with $C(w^*w) = C(u^*u) = C(v^*v)$, and

$$\mathrm{Ad}\, u \geq \mathrm{Ad}\, u|w^*wAw^*w =: \theta \simeq (\mathrm{Ad}\, vwu^*) \circ \theta \circ (\mathrm{Ad}\, w^*) = \mathrm{Ad}\, v|ww^*Aww^* \leq \mathrm{Ad}\, v.$$

The reverse implication is clear.

For $\theta \in \mathrm{RPAut}\, A$, denote by $[\theta]$ the isomorphism class in $[\mathrm{SDINV}(A)]$ of $\langle\theta\rangle \in \mathrm{SDINV}(A)$.

Theorem 6.11

(i) *Every* $\theta \in \mathrm{PAut}\, A$ *has a regularization* ψ, *which is unique in the sense that* $[\psi] = [\psi']$ *for another regularization* ψ'.

(ii) *For* $\theta_1, \theta_2 \in \mathrm{PAut}\, A$ *and their regularizations* ψ_1, ψ_2 *we have* $\theta_1 \sim \theta_2$ *if and only if* $[\psi_1] = [\psi_2]$, i.e., $\langle\psi_1\rangle \cong \langle\psi_2\rangle$.

The proof of Theorem 6.11 is contained in Lemmas 6.14(ii) and 6.15(ii).

Definition 6.12 Let B be a monotone complete C^*-algebra and $f \in \mathrm{Proj}\, B$ with $C(f) = 1$. Consider $\mathrm{PAut}\, fBf \subset \mathrm{PAut}\, B$ canonically. A *perturbed restriction of* $\omega \in \mathrm{PAut}\, B$ *to* fBf is an element $\theta \in \mathrm{PAut}\, fBf$ such that $\theta \prec \omega$ in $\mathrm{PAut}\, B$, (see (6.1)), $\theta = (\mathrm{Ad}\, v) \circ \omega \circ (\mathrm{Ad}\, u)$ for some $u, v \in \mathrm{PI}\, B$ such that

$$f \geq u^*u, \quad uu^* \leq r(\omega) \leq C(uu^*), \quad \omega(uu^*) = v^*v, \quad vv^* \leq f. \tag{6.7}$$

Lemma 6.13 *Let B, f be as above.*

(i) *For* $\theta_1, \theta_2 \in \mathrm{PAut}\, fBf$ *we have* $\theta_1 \simeq \theta_2$ *(resp.,* $\theta_1 \leq \theta_2$, $\theta_1 \sim \theta_2$, $\theta_1 \prec \theta_2$*) in* $\mathrm{PAut}\, fBf$ *if and only if so are* θ_1 *and* θ_2 *in* $\mathrm{PAut}\, B$.

(ii) *Each* $\omega \in \mathrm{PAut}\, B$ *has a perturbed restriction to* fBf. *If* $\theta_j \in \mathrm{PAut}\, fBf$ *and* $\theta_j \prec \omega$, $j = 1, 2$, *then* $\theta_1 \sim \theta_2$ *in* $\mathrm{PAut}\, fBf$. *If* $\theta_j \in \mathrm{PAut}\, fBf$, $\theta_j \prec \omega$, $j = 1, 2$, *and* $r(\theta_1) = w^*w$, $ww^* \leq r(\theta_2)$ *for some* $w \in \mathrm{PI}\, B$, *then* $\theta_1 \prec \theta_2$ *in* fBf.

(iii) *If* $\omega \in \mathrm{PAut}\, B$ *and* $C(r(\omega))f = u^*u$, $uu^* \leq r(\omega)$ *for some* $u \in \mathrm{PI}\, B$, *then there exists a perturbed restriction of* ω *to* fBf, *which is regular in* $\mathrm{PAut}\, fBf$.

Proof (i) If $\theta_1 \simeq \theta_2$ in $\mathrm{PAut}\, B$, *i.e.*, $\theta_2 = (\mathrm{Ad}\, v) \circ \theta_1 \circ (\mathrm{Ad}\, u)$ for $u, v \in \mathrm{PI}\, B$ with $uu^* = r(\theta_1)$, $v^*v = l(\theta_1) \in fBf$, then $u^*u = r(\theta_2)$, $vv^* = l(\theta_2) \in fBf$ and $u, v \in \mathrm{PI}\, fBf$. Hence $\theta_1 \simeq \theta_2$ in $\mathrm{PAut}\, fBf$, and the remaining assertions are clear.

(ii) Apply (GC′) twice to obtain $u_1, v \in \mathrm{PI}\, B$ so that

$$f \geq u_1^*u_1, \quad u_1u_1^* \leq r(\omega), \quad C(u_1u_1^*) = C(f)C(r(\omega)) = C(r(\omega)),$$
$$\omega(u_1u_1^*) \geq v^*v, \quad vv^* \leq f, \quad C(v^*v) = C(\omega(u_1u_1^*)).$$

Then $u := \omega^{-1}(v^*v)u_1$ and v satisfy (6.7), and $(\mathrm{Ad}\, v) \circ \omega \circ (\mathrm{Ad}\, u)$ is a perturbed restriction of ω to fBf. Indeed,

$$u^*u \leq u_1^*u_1 \leq f, \quad uu^* = \omega^{-1}(v^*v)u_1u_1^*\omega^{-1}(v^*v) = \omega^{-1}(v^*v).$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Further $C(v^*v) = C(\omega(u_1u_1^*))$ implies

$$C(uu^*) = C(\omega^{-1}(v^*v)) = C(u_1u_1^*) = C(r(\omega)),$$

since the $*$-isomorphism $\omega^{-1}\colon l(\omega)Bl(\omega) \to r(\omega)Ar(\omega)$ preserves central covers and since the central cover of $p \in \operatorname{Proj} l(\omega)Bl(\omega)$ in $l(\omega)Bl(\omega)$ equals $C(p)l(\omega)$, etc.

Suppose $\theta_j = (\operatorname{Ad} v_j) \circ \omega \circ (\operatorname{Ad} u_j)$ with u_j, v_j as in (6.7), $j = 1, 2$. Then $\theta_j \simeq \omega|\, u_ju_j^*Bu_ju_j^*$ and $C(u_1u_1^*) = C(r(\omega)) = C(u_2u_2^*)$. Hence, as in Remark 6.10(ii),

$$\omega|u_1u_1^*Bu_1u_1^* \geq \omega|w^*wBw^*w \simeq \omega|ww^*Bww^* \leq \omega|u_2u_2^*Bu_2u_2^*$$

for some $w \in \operatorname{PI} B$ with $u_1^*u_1 \geq w^*w$, $ww^* \leq u_2u_2^*$ and $C(w^*w) = C(r(\omega))$, and so $\theta_1 \sim \theta_2$ in PAut B, and hence in PAut fBf by (i).

If, further, $u_1^*u_1 = r(\theta_1) = w^*w$, $ww^* \leq r(\theta_2) = u_2^*u_2$ for $w \in \operatorname{PI} B$, then with $w_1 := u_2wu_1^*$ we have $u_1u_1^* = w_1^*w_1$, $w_1w_1^* \leq u_2u_2^*$, and we may take w_1 as the above w to conclude that $\omega|u_1u_1^*Bu_1u_1^* \prec \omega|u_2u_2^*Bu_2u_2^*$ and $\theta_1 \prec \theta_2$.

(iii) By (GC) applied to $\omega(uu^*)$ and f, take $k \in \operatorname{Proj} Z(B)$ and $v_1, v_2 \in \operatorname{PI} B$ so that

$$k\omega(uu^*) = v_1^*v_1, \quad v_1v_1^* \leq kf, \quad (1-k)\omega(uu^*) \geq v_2^*v_2, \quad v_2v_2^* = (1-k)f.$$

Set $k_1 := \omega^{-1}(kl(\omega)) \in \operatorname{Proj} Z(r(\omega)Br(\omega))$, so $k_1 = hr(\omega)$ with $h := C(k_1)$, and set $u_1 := k_1u = hr(\omega)u = hu$, $u_2 := \omega^{-1}(v_2^*v_2)u \in \operatorname{PI} B$. Then

$$u_ju_j^* \in \operatorname{PI} r(\omega)Br(\omega), \quad \omega(u_1u_1^*) = \omega(k_1uu^*) = k\omega(uu^*) = v_1^*v_1,$$
$$\omega(u_2u_2^*) = v_2^*v_2\omega(uu^*)v_2^*v_2 = v_2^*v_2, \quad u_1^*u_1 = hu^*u = hC(r(\omega))f = hf,$$

since $k_1 \leq r(\omega)$ implies $h = C(k_1) \leq C(r(\omega))$, and

$$u_2^*u_2 = u^*\omega^{-1}(v_2^*v_2)u \leq u^*(1-k_1)uu^*u = (1-h)u^*u = (C(r(\omega)) - h)f.$$

Hence $\psi_j := (\operatorname{Ad} v_j) \circ \omega \circ (\operatorname{Ad} u_j) \in \operatorname{PAut} fBf$, $j = 1, 2$, and

$$r(\psi_1) = u_1^*u_1 = hf, \quad l(\psi_1) = v_1v_1^* \leq kf,$$
$$r(\psi_2) = u_2^*u_2 \leq (1-h)f, \quad l(\psi_2) = v_2v_2^* = (1-k)f.$$

So ψ_1 and ψ_2 are orthogonal, ψ_1 (resp., ψ_2) is positive (resp., negative) in PAut fBf, and $\psi := \psi_1 \oplus \psi_2 \in \operatorname{PAut} fBf$ is regular. Then $\psi = (\operatorname{Ad} v) \circ \omega \circ (\operatorname{Ad} u')$ with $u' := u_1 + u_2$ and $v := v_1 + v_2$, and u' and v satisfy (6.7), *i.e.*, $\psi \prec \omega$. Indeed, $v_2v_2^* = (1-k)f$ implies $C(v_2^*v_2) = (1-k)C(f) = 1-k$, and so $C(u_2u_2^*) = C(\omega^{-1}(v_2^*v_2)) = (1-h)C(r(\omega)) = C(r(\omega)) - h$ as in (ii). Hence $C(u'^*u') = C(u_1{}^*u_1) + C(u_2{}^*u_2) = h + C(r(\omega)) - h = C(r(\omega))$. ■

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Lemma 6.14

(i) *If* $\theta, \psi \in \mathrm{PAut}\, A$ *and* $\theta \prec \psi$, *then we have* $\psi \prec \omega$ *in* $\mathrm{PAut}\, B$, *where, as in Remark* 6.1(ii), *for some index set* I, $B = A \mathbin{\overline{\otimes}} B(l^2(I))$, $f = 1 \otimes e_0 \in \mathrm{Proj}\, B$, *we identify* A *with* fBf, *and* $\omega = \theta \mathbin{\overline{\otimes}} \mathrm{id} \in \mathrm{PAut}\, B$.

(ii) *Each element of* $\mathrm{PAut}\, A$ *has a regularization in* $\mathrm{PAut}\, A$.

Proof (i) We have $\psi | u^*uAu^*u = (\mathrm{Ad}\, v) \circ \theta \circ (\mathrm{Ad}\, u)$ for some $u, v \in \mathrm{PI}\, A$ with $uu^* = r(\theta)$, $v^*v = l(\theta)$ and $u^*u \leq r(\psi) \leq C(u^*u) = C(r(\theta))$. By (GC) and a maximality argument there is a family $\{u_i\}_{i\in I}$ in $\mathrm{PI}\, A$ such that

$$r(\psi) = \sum_{i\in I} u_i^* u_i, \quad u_i u_i^* \leq r(\theta), \ \forall i \in I. \tag{6.8}$$

(Hence $\{u_i^* u_i\}_{i\in I}$ is orthogonal, *i.e.*, $u_i u_j^* = 0$ if $i \neq j$.) Then $u_i^* u_i = u_i^* r(\theta) u_i = u_i^* uu^* u_i$, and since ψ is normal, for all $a \in r(\psi) A r(\psi)$,

$$\begin{aligned} \psi(a) &= \sum_{i,j\in I} \psi(u_i^* uu^* u_i a u_j^* uu^* u_j) \\ &= \sum_{i,j\in I} \psi(u_i^* u)\psi(u^* u_i a u_j^* u)\psi(u^* u_j) \\ &= \sum_{i,j\in I} \psi(u_i^* u) v\theta(u_i a u_j^*) v^* \psi(u^* u_j) \\ &= \sum_{i,j\in I} v_i \theta(u_i a u_j^*) v_j^*, \end{aligned} \tag{6.9}$$

where $v_i := \psi(u_i^* u) v \in \mathrm{PI}\, A$, and

$$v_i^* v_j = v^* \psi(u^* u_i u_j^* u) v = \theta(u_i u_j^*).$$

Consider $B = A \mathbin{\overline{\otimes}} B(l^2(I))$, $f = 1 \otimes e_0 \in \mathrm{Proj}\, B$ as in Remark 6.1(ii), and set $\omega := \theta \mathbin{\overline{\otimes}} \mathrm{id} \in \mathrm{PAut}\, B$, $U := [\delta_{i_0 j} u_i]$, $V := [\delta_{i_0 i} v_j] \in B$, where $r(\omega) = r(\theta) \otimes 1$, $l(\omega) = l(\theta) \otimes 1$, and $(\theta \mathbin{\overline{\otimes}} \mathrm{id})([a_{ij}]) = [\theta(a_{ij})]$, $a_{ij} \in r(\theta) A r(\theta)$. Then

$$U^*U = [\delta_{i_0 i}\delta_{i_0 j} \sum_k u_k^* u_k] = r(\psi) \otimes e_0 \leq f,$$

$$VV^* = [\delta_{i_0 i}\delta_{i_0 j} \sum_k v_k v_k^*] = [\delta_{i_0 i}\delta_{i_0 j} \sum_k \psi(u_k^* u_k)] = l(\psi) \otimes e_0 \leq f,$$

and it follows from (6.6) and (6.7) that

$$UU^* = [u_i u_j^*] = [\delta_{ij} u_i u_i^*] \leq r(\theta) \otimes 1 = r(\omega), \quad \omega(UU^*) = V^*V,$$

$$(\mathrm{Ad}\, V) \circ \omega \circ (\mathrm{Ad}\, U) = \psi(\,\cdot\,) \otimes e_0 \text{ on } (r(\psi) \otimes e_0) B (r(\psi) \otimes e_0) = r(\psi) A r(\psi) \otimes e_0.$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Hence, identifying A with $A \otimes e_0 = fBf$ we obtain the conclusion.

(ii) For $\theta \in \operatorname{PAut} A$ take as above a family $\{u_i\}_{i\in I}$ in $\operatorname{PI} A$ such that

$$C(r(\theta)) = \sum_{i\in I} u_i^* u_i, \quad u_i u_i^* \le r(\theta),\ \forall i \in I;$$

set $B = A \overline{\otimes} B(l^2(I))$, $f = 1 \otimes e_0$, $U = [\delta_{i_o j} u_i] \in B$, and $\omega = \theta \overline{\otimes} \operatorname{id} \in \operatorname{PAut} B$, so that

$$C(r(\omega))f = C(r(\theta) \otimes e_0)(1 \otimes e_0) = C(r(\theta)) \otimes e_0 = U^* U,$$
$$UU^* = [\delta_{ij} u_i u_i^*] \le r(\theta) \otimes 1 = r(\omega);$$

and identify A with fBf and θ with $\omega|fBf = \omega|A$, so that $\theta \prec \omega$. By Lemma 6.13(iii) there is a regular $\psi \in \operatorname{PAut} A$ with $\psi \prec \omega$. Let $\psi = \psi_1 \oplus \psi_2$ with ψ_1 positive and ψ_2 negative in $\operatorname{PAut} A$. Then $r(\psi_1) = hf$ for some $h \in \operatorname{Proj} Z(A)$, and

$$\theta \cdot h \prec \omega \cdot h, \quad \psi_1 = \psi \cdot h \prec \omega \cdot h,$$
$$h = C(r(\psi_1)) = C(r(\omega \cdot h)) \ge r(\theta \cdot h), \quad r(\psi_1) = hf \ge r(\theta \cdot h).$$

Hence, by Lemma 6.13(ii), $\theta \cdot h \prec \psi_1$. In view of (6.2) it follows similarly that $\theta \cdot (1-h) \prec \psi_2$ and hence that $\theta \prec \psi$. ∎

Lemma 6.15

(i) *Suppose $\theta \prec \psi_k$, $k = 1, 2$, for $\theta, \psi_k \in \operatorname{PAut} A$.*

(1) *If ψ_1 and ψ_2 are both positive, then $\psi_2 = (\operatorname{Ad} u) \circ \psi_1$ for some $u \in \operatorname{PI} A$ with $u^* u = l(\psi_1)$.*

(2) *If ψ_1 is positive and ψ_2 is negative, then $\psi_2 | u^* u A u^* u = \psi_1 \circ (\operatorname{Ad} u)$ for some $u \in \operatorname{PI} A$ with $u^* u \le r(\psi_2)$ and $uu^* = r(\psi_1)$.*

(3) *If ψ_1 and ψ_2 are regular, then $\langle\psi_1\rangle \cong \langle\psi_2\rangle$.*

(ii) *For $\theta_j \in \operatorname{PAut} A$, $j = 1, 2$, and their regularizations ψ_j we have $\theta_1 \sim \theta_2$ if and only if $\langle\psi_1\rangle \cong \langle\psi_2\rangle$.*

Proof (i) There exists a monotone complete C^*-algebra B, $f \in \operatorname{Proj} B$ and $\omega \in \operatorname{PAut} B$ so that $C(f) = 1$, $fBf = A$ and $\psi_k \prec \omega$, $k = 1, 2$. Indeed, as in the proof of Lemma 6.14(i), take families $\{u_{ki}\}_{i\in I_k}$, $\{v_{ki}\}_{i\in I_k}$ in $\operatorname{PI} A$, $k = 1, 2$, satisfying (6.6) and (6.7) for θ and ψ_k. Set $I = I_1 \cup I_2$ (disjoint union), $u_{1i} = 0 = v_{1i}$ for $i \in I_2$ and $u_{2i} = 0 = v_{2i}$ for $i \in I_1$ and consider, as before, $B = A \overline{\otimes} B(l^2(I))$, $f = 1 \otimes e_0 \in \operatorname{Proj} B$, $\omega = \theta \overline{\otimes} \operatorname{id} \in \operatorname{PAut} B$, $U_k = [\delta_{i_0 j} u_{ki}]$, $V_k = [\delta_{i_0 i} v_{kj}] \in B$, $k = 1, 2$, where $i_0 \in I$ is a fixed element and e_0 is the minimal projection in $B(l^2(I))$ corresponding to it. Then, identifying A with fBf and ψ_k with $(\operatorname{Ad} V_k) \circ \omega \circ (\operatorname{Ad} U_k)$, we obtain the assertion.

(1) Since $\psi_k \prec \omega$, we have $C(r(\psi_k)) = C(r(\omega))$, and since the ψ_k are positive in $\operatorname{PAut} A = \operatorname{PAut} fBf$, we have

$$r(\psi_k) = C(r(\psi_k))f = C(r(\omega))f, \quad U_1^* U_1 = r(\psi_1) = r(\psi_2) = U_2^* U_2.$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Then $u := V_2\omega(U_2U_1^*)V_1^* \in fBf = A$, $u^*u = V_1V_1^* = l(\psi_1)$, and

$$(\mathrm{Ad}\, u)\circ\psi_1 = (\mathrm{Ad}\, uV_1)\circ\omega\circ(\mathrm{Ad}\, U_1) = (\mathrm{Ad}\, V_2)\circ\omega\circ(\mathrm{Ad}\, U_2) = \psi_2.$$

(2) As above, $U_1^*U_1 = r(\psi_1) = C(r(\omega))f$ and $V_2V_2^* = l(\psi_2) = C(l(\omega))f$. Then $u := U_1^*\omega^{-1}(V_1^*V_2)U_2 \in A$ satisfies

$$uu^* = U_1^*U_1 = r(\psi_1), \quad u^*u \le r(\psi_2), \quad \psi_1\circ(\mathrm{Ad}\, u) = \psi_2|u^*uAu^*u.$$

(3) If we consider the positive-negative decompositions of ψ_k, then the assertion follows from (1) and (2) by Proposition 3.4 and Remark 6.10(i).

(ii) If $\theta_1 \sim \theta_2$, then $\omega \prec \theta_1, \omega \prec \theta_2$ for some $\omega \in \mathrm{PAut}\, A$, and so $\langle\psi_1\rangle \cong \langle\psi_2\rangle$ by (i). Suppose, conversely, that the ψ_j are regular, $\theta_j \prec \psi_j$, $j = 1, 2$, and $\langle\psi_1\rangle \cong \langle\psi_2\rangle$. We may assume $\theta_j \le \psi_j$, and as above, it suffices to consider separately the two cases: (a) ψ_1 and ψ_2 are both positive; (b) ψ_1 is positive and ψ_2 is negative. (The case of the ψ_j being both negative reduces, by passing to inverses, to case (a).)

***Case* (a):** $\theta_1 \le \psi_1 \simeq \psi_2$ by Proposition 3.4(i), so $\theta_1 \prec \psi_2$ by (6.5). Since $\theta_2 \le \psi_2$ also, $\theta_1 \sim \theta_2$ by Lemma 6.13(ii).

***Case* (b):** By Proposition 3.4(ii), $\psi_2|u^*uAu^*u = \psi_1\circ(\mathrm{Ad}\, u)$ for some $u \in \mathrm{PI}\, A$ with $u^*u \le r(\psi_2) \le r(\psi_1) = uu^*$. Since $r(\theta_1) \le C(r(\theta_1)) = r(\psi_1) = uu^* \in Z(A)$, $u^*r(\theta_1)u \in \mathrm{Proj}\, A$ and

$$C(u^*r(\theta_1)u) = C(r(\theta_1)uu^*) = C(r(\theta_1))uu^* = r(\psi_1) = C(r(\psi_2)),$$

and since $\theta_1 \le \psi_1$ and $r(\theta_1)u(r(\theta_1)u)^* = r(\theta_1)$,

$$\begin{aligned}\theta_1 \simeq \theta_1\circ(\mathrm{Ad}\, r(\theta_1)u) &= \psi_1\circ(\mathrm{Ad}\, r(\theta_1)u)\\ &= \psi_1\circ(\mathrm{Ad}\, uu^*r(\theta_1)u) = \psi_2|u^*r(\theta_1)uAu^*r(\theta_1)u \prec \psi_2.\end{aligned}$$

Hence $\theta_1 \sim \theta_2$ as in Case (a). ■

For each $\theta \in \mathrm{PAut}\, A$ write $\{\theta\}$, $\langle\theta\rangle$, and $[\theta]$, respectively, for the equivalence class in $\mathrm{PAut}\, A$ of θ with respect to $\sim$, the regular, *i.e.,* self-dual, invertible A-module $\langle\psi\rangle$ associated with a regularization ψ of θ (which is unique up to isomorphism by Theorem 6.11) and the isomorphism class $[\psi]$ in $[\mathrm{SDINV}(A)]$ of such a ψ. Denote by $\{\mathrm{PAut}\, A\}$ the set of all $\{\theta\}$, $\theta \in \mathrm{PAut}\, A$. Then Theorem 6.11 yields a bijection $\{\theta\} \mapsto [\theta]$ between $\{\mathrm{PAut}\, A\}$ and $[\mathrm{SDINV}(A)]$.

Now we define a "composition", $\theta \bullet \psi$, of any $\theta, \psi \in \mathrm{PAut}\, A$ extending that in Section 3, so that we have

$$\langle\theta\rangle \,\overline{\otimes}_A\, \langle\psi\rangle \cong \langle\theta\bullet\psi\rangle. \tag{6.10}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Definition 6.16 Let $\theta, \psi \in \mathrm{PAut}\, A$. If the projections $r(\theta)$ and $l(\psi)$ are comparable in the sense that $hr(\theta) \leq hl(\psi)$, $(1-h)r(\theta) \geq (1-h)l(\psi)$ for some $h \in \mathrm{Proj}\, Z(A)$, then define the *composition*, $\theta \circ \psi \in \mathrm{PAut}\, A$, called *canonical*, as

$$\theta \circ \psi \mid \big(hr(\theta) + (1-h)l(\psi)\big) A \big(hr(\theta) + (1-h)l(\psi)\big) .$$

(As is readily checked, $hr(\theta) + (1-h)l(\psi)$, and hence $\theta \circ \psi$ also, does not depend on the choice of h.) For general θ, ψ take, by (GC′), any $u \in \mathrm{PI}\, A$ so that $u^*u \leq l(\psi)$, $uu^* \leq r(\theta)$ and $C(u^*u) = C(r(\theta))C(l(\psi))$, and define the *composition*, $\theta \bullet_u \psi$ or $\theta \bullet \psi$ for short, as $\theta \circ (\mathrm{Ad}\, u) \circ \psi | \psi^{-1}(u^*u) A \psi^{-1}(u^*u)$.

Clearly $(\mathrm{PAut}\, A)^+$ is a semigroup under the canonical composition, which contains the ∗-automorphism group $\mathrm{Aut}\, A$ as a subgroup. For $S \subset \mathrm{PAut}\, A$ write $\{S\} := \{\{\theta\} : \theta \in S\} \subset \{\mathrm{PAut}\, A\}$. Note that by identifying each $h \in \mathrm{Proj}\, Z(A)$ with id_{hA}, the identity map on hA, we have

$$\mathrm{Proj}\, Z(A) = \{\mathrm{Proj}\, Z(A)\} \subset \{\mathrm{PAut}\, A\},$$

and that the operation $\theta \mapsto k \cdot \theta \cdot h$ for $\theta \in \mathrm{PAut}\, A$ and $h, k \in \mathrm{Proj}\, Z(A)$ is viewed as the composition in $\mathrm{PAut}\, A$, *i.e.*,

$$k \cdot \theta \cdot h = (\mathrm{id}_{kA}) \circ \theta \circ (\mathrm{id}_{hA}).$$

Theorem 6.17

(i) *The set* $\{\mathrm{PAut}\, A\}$ *is an inverse semigroup with the inverse and product*

$$\{\theta\}^{-1} = \{\theta^{-1}\}, \quad \{\theta\} \cdot \{\psi\} = \{\theta \bullet \psi\},$$

the map $\{\theta\} \mapsto [\theta]$ *is an isomorphism of* $\{\mathrm{PAut}\, A\}$ *onto* $[\mathrm{SDINV}(A)]$, *and the multiplicative semigroup* $\mathrm{Proj}\, Z(A)$ *is its subsemigroup of all idempotents such that*

$$\{\mathrm{PInt}\, A\} = \mathrm{Proj}\, Z(A) \subset \{\mathrm{PAut}\, A\}.$$

(ii) *The subsets* $\{\mathrm{Aut}\, A\}$ *and* $\{(\mathrm{PAut}\, A)^+\}$ *are respectively a subgroup and a subsemigroup of* $\{\mathrm{PAut}\, A\}$*; the map* $\mathrm{Aut}\, A \to \{\mathrm{Aut}\, A\}$, $\theta \mapsto \{\theta\}$, *is a group homomorphism with the inner ∗-automorphism group* $\mathrm{Int}\, A$ *as the kernel. It induces a group monomorphism* $\mathrm{Out}\, A = \mathrm{Aut}\, A / \mathrm{Int}\, A \to \{\mathrm{PAut}\, A\}$.

In view of Theorem 6.5, Remark 6.10(iv), Theorem 6.11 and the inverse semigroup structure of $[\mathrm{SDINV}(A)]$, the proof of Theorem 6.17(i) is reduced to the proof of (6.10), which is given below, while (ii) follows from Proposition 3.4(i).

The following lemma shows that the isomorphism class of the right-hand side of (6.10) depends only on the equivalence classes $\{\theta\}$ and $\{\psi\}$.

Lemma 6.18 *Let* θ_j, ψ_j, *and* θ *be in* $\mathrm{PAut}\, A$.

(i) *If* $\psi_1 \sim \psi_2$ *and* $l(\psi_j) \leq r(\theta)$, $j = 1, 2$, *then* $\theta \circ \psi_1 \sim \theta \circ \psi_2$.

(ii) *If* $\psi_1 \sim \psi_2$, *then* $\theta \bullet \psi_1 \sim \theta \bullet \psi_2$.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(iii) *If* $\theta_1 \sim \theta_2$ *and* $\psi_1 \sim \psi_2$, *then* $\theta_1 \bullet \psi_1 \sim \theta_2 \bullet \psi_2$.

Proof (i) We have

$$\omega_j \leq \psi_j,\ j = 1, 2, \quad \omega_2 = (\operatorname{Ad} v) \circ \omega_1 \circ (\operatorname{Ad} u)$$

for some ω_j and $u, v \in \operatorname{PI} A$ with $uu^* = r(\omega_1)$ and $v^* v = l(\omega_1)$. Then $\theta \circ \omega_j \leq \theta \circ \psi_j$, since $r(\theta \circ \omega_j) = r(\omega_j)$ and $r(\theta \circ \psi_j) = r(\psi_j)$, and

$$\theta \circ \omega_2 = (\operatorname{Ad} \theta(v)) \circ (\theta \circ \omega_1) \circ (\operatorname{Ad} u) \sim \theta \circ \omega_1,$$

since $vv^* = l(\omega_2) \leq l(\psi_2) \leq r(\theta)$ and $v^* v = l(\omega_1) \leq l(\psi_1) \leq r(\theta)$. Hence $\theta \circ \psi_1 \sim \theta \circ \psi_2$.

(ii) We have $\theta \bullet \psi_j = \theta \circ (\operatorname{Ad} u_j) \circ \psi_j$ for $u_j \in \operatorname{PI} A$ with $u_j^* u_j \leq l(\psi_j)$, $u_j u_j^* \leq r(\theta)$ and $C(u_j^* u_j) = C(l(\psi_j))C(r(\theta))$. Since $\psi_1 \sim \psi_2$, $C(l(\psi_1)) = C(l(\psi_2))$ and so $C(u_1^* u_1) = C(u_2^* u_2) =: h$, say. Then

$$(\operatorname{Ad} u_1) \circ \psi_1 \prec h \cdot \psi_1 \sim h \cdot \psi_2 \succ (\operatorname{Ad} u_2) \circ \psi_2,$$

and $(\operatorname{Ad} u_1) \circ \psi_1 \sim (\operatorname{Ad} u_2) \circ \psi_2$. Hence $\theta \bullet \psi_1 \sim \theta \bullet \psi_2$ by (i).

(iii) By (ii) and the fact that $(\theta \bullet \psi)^{-1} = \psi^{-1} \bullet \theta^{-1}$, we have $\theta_1 \bullet \psi_1 \sim \theta_1 \bullet \psi_2$, $\theta_1 \bullet \psi_2 \sim \theta_2 \bullet \psi_2$, and so $\theta_1 \bullet \psi_1 \sim \theta_2 \bullet \psi_2$. ∎

Proof of equation (6.10) By Theorem 6.11 and Lemma 6.18 we may assume that θ and ψ are regular, and then, by taking their positive-negative decompositions we need only to consider the four cases separately: (a) θ and ψ are both positive; (b) θ is positive and ψ is negative; (c) θ is negative and ψ is positive; (d) θ and ψ are both negative.

In cases (a), (c) and (d), by Proposition 3.7(i), (iii), $\langle\theta\rangle \overline{\otimes}_A \langle\psi\rangle = \langle\theta\rangle \odot_A \langle\psi\rangle \cong \langle\omega\rangle$ for some $\omega \in \operatorname{PAut} A$ of the form $\theta \bullet \psi$.

In case (b), by Theorem 6.6, there is an isomorphism $\tau\colon \langle\theta\rangle \overline{\otimes}_A \langle\psi\rangle \to \langle\omega\rangle$ for some $\omega \in \operatorname{RPAut} A$. Then, by Proposition 3.7(ii), there are $h \in \operatorname{Proj} Z(A)$ and $u_1, u_2 \in \operatorname{PI} A$ such that $\omega_1 := \omega \cdot h$ is positive, $\omega_2 := \omega \cdot (1 - h)$ is negative, the conditions there hold, and monomorphisms τ_1, τ_2 are defined by the equalities there. Then, as noted in the proof of Proposition 3.7(ii), $hr(\theta \circ \psi) \leq r(\omega_1)$, and

$$\begin{aligned} \omega_1 |\, hr(\theta \circ \psi)Ar(\theta \circ \psi) &= (\operatorname{Ad} u_1^*) \circ (\theta \circ \psi)|\, hr(\theta \circ \psi)Ar(\theta \circ \psi) \\ &\simeq \theta \circ \psi|\, hr(\theta \circ \psi)Ar(\theta \circ \psi), \end{aligned}$$

since $\omega_1(hr(\theta \circ \psi)) = u_1^* u_1$ and $(\theta \circ \psi)(hr(\theta \circ \psi)) = u_1 u_1^*$. We have

$$u_1^* u_1 = \omega_1(hr(\theta \circ \psi)) \leq l(\omega_1) \leq C(u_1^* u_1),$$

since the image of τ_1 is $Au_1Al(\omega_1) = Au_1^* u_1 Al(\omega_1)$ and this generates $\langle\omega_1\rangle = Al(\omega_1)$ as a self-dual Hilbert A-module. Hence $\omega_1 | hr(\theta \circ \psi)Ar(\theta \circ \psi) \leq \omega_1$, and $(\theta \circ \psi) \cdot h \prec \omega_1$. Similarly $(\theta \circ \psi) \cdot (1 - h) \prec \omega_2$, and so $\omega \sim \theta \circ \psi$.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Corollary 6.19 *For each $e \in \operatorname{Proj} A$ the inclusion map $eAe \to C(e)A$ induces the equality $\{\operatorname{PAut} eAe\} = \{\operatorname{PAut} C(e)A\}$.*

Proof Immediate from Lemma 6.13(ii) and Theorem 6.17. ■

Corollary 6.20

(i) *We have $\{\operatorname{PAut} Z(A)\} = \operatorname{PAut} Z(A)$, and the map $\operatorname{PAut} A \to \operatorname{PAut} Z(A)$, $\theta \mapsto \overline{\theta}$, induces an inverse semigroup homomorphism $\{\operatorname{PAut} A\} \to \operatorname{PAut} Z(A)$, $\{\theta\} \mapsto \overline{\theta}$, whose restriction to $\operatorname{Proj} Z(A) = \{\operatorname{PInt} A\}$ is the identity map.*

(ii) *If A is of type I, then the map above is an isomorphism, and $\{\operatorname{PAut} A\} \cong \operatorname{PAut} Z(A)$.*

Proof (i) The first assertion is clear, and the second assertion follows from Remark 6.10(iii) and the definition of the product.

(ii) There is an abelian projection $e \in \operatorname{Proj} A$ with $C(e) = 1$, and so $eAe = Z(eAe) = eZ(A) \cong Z(A)$. Hence, by Corollary 6.19 and (i), it suffices to show that for each $\theta \in \operatorname{PAut} A$ its perturbed restriction, θ_1, to eAe is recovered from $\overline{\theta} \in \operatorname{PAut} Z(A)$. Let $\theta_1 = (\operatorname{Ad} v) \circ \theta \circ (\operatorname{Ad} u) : u^*uAu^*u \to vv^*Avv^*$ with u, v as in (6.7). Then, since $u^*u \leq e$ and $vv^* \leq e$, $u^*uAu^*u = u^*uZ(A)$, $vv^*Avv^* = vv^*Z(A)$, and these are canonically identified with $C(u^*u)Z(A) = C(r(\theta))Z(A) = r(\overline{\theta})Z(A)$, $C(vv^*)Z(A) = C(l(\theta))Z(A) = l(\overline{\theta})Z(A)$, respectively. If $hu^*u \in \operatorname{Proj} u^*uZ(A)$ with $h \in \operatorname{Proj} r(\overline{\theta})Z(A)$, then

$$\theta_1(hu^*u) = v\theta(huu^*)v^* = v\theta(hr(\theta))\theta(uu^*)v^* = v\overline{\theta}(h)l(\theta)v^* = \overline{\theta}(h)vv^*,$$

and the assertion follows. ■

7 A Decomposition of $\{\operatorname{PAut} A\}$

In this section we continue the study of the inverse semigroup $\{\operatorname{PAut} A\}$ for a monotone complete C^*-algebra A, assuming that A is totally of the same type (*i.e.,* of type I, II, III, etc.) as an AW^*-algebra. The relevance of such a restriction and the notions of "globally central", "global factor" defined below is as follows. Let B be a monotone complete C^*-algebra which has a coaction of a discrete group with fixed-point subalgebra A. Then B is generated by the normalizer $N_B(A)$ as a monotone complete C^*-algebra. Hence, irrespective of the coaction, each globally central projection of A is always a central projection of B (see Proposition 7.2), and A is a global factor if B is an AW^*-factor (see Proposition 7.9).

Definition 7.1 Call $h \in \operatorname{Proj} Z(A)$ *globally central* if $\theta(hr(\theta)) = hl(\theta)$ for all $\theta \in (\operatorname{PAut} A)^+$. For $e \in \operatorname{Proj} A$ set

$$GC(e) = \sup\{l(\theta) : \exists\theta \in \operatorname{PAut} A,\ r(\theta) \leq e\},$$

the supremum in $\operatorname{Proj} A$, and call this the *globally central cover* of e.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

We have $C(e) \le GC(e) \in \operatorname{Proj} Z(A)$, since the right-hand side above gives $C(e)$ when the membership of θ is restricted to inner partial $*$-automorphims of A and since $r((\operatorname{Ad} u)\circ\theta) = r(\theta)$, $l((\operatorname{Ad} u)\circ\theta) = ul(\theta)u^*$ for $\theta \in \operatorname{PAut} A$ and unitary u in A and so $GC(e)$ commutes with such a u.

Denote by h_ν the central projection of A corresponding to the type ν direct summand of A ($\nu = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{I}_n$, etc.).

Proposition 7.2

(i) *For $h \in \operatorname{Proj} Z(A)$ the following are equivalent:*

(1) *For each $(X,\theta) \in \operatorname{SDINV}(A)$ and each $x \in X$ we have $h \cdot x = x\theta(h) = x \cdot h$;*
(2) *h is globally central;*
(3) *$\theta(hr(\theta)) = hl(\theta)$ for all $\theta \in \operatorname{PAut} A$;*
(4) *$\overline{\theta}(hC(r(\theta))) = hC(l(\theta))$ for all $\theta \in \operatorname{PAut} A$.*

(ii) *For $e \in \operatorname{Proj} A$, $GC(e)$ is the smallest globally central projection of A majorizing e.*

(iii) *The projections $h_{\mathrm{I}}, h_{\mathrm{II}}$ and h_{III} are globally central, and we have the following direct sum decomposition as an inverse semigroup:*

$$\{\operatorname{PAut} A\} = \{\operatorname{PAut} h_{\mathrm{I}}A\} \oplus \{\operatorname{PAut} h_{\mathrm{II}}A\} \oplus \{\operatorname{PAut} h_{\mathrm{III}}A\}.$$

Proof (i) In view of Theorem 6.6 and the definition of $\langle\theta\rangle$, (1) $\Leftrightarrow$ (2) is clear, and (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) is clear by Definition 3.1 and Definition 7.1.

(2) $\Rightarrow$ (3): Each $\theta \in \operatorname{PAut} A$ has a regularization ψ, and (2) implies the validity of (3) for ψ and hence that of (4) for ψ. But, since $\overline{\theta} = \overline{\psi}$ (see Remark 6.10(iii)) and (3) $\Leftrightarrow$ (4) holds for fixed θ, (3) for θ follows.

(ii) Set $h = GC(e)$. For $\theta \in (\operatorname{PAut} A)^+$ and $\psi \in \operatorname{PAut} A$ with $r(\psi) \le e$ we have

$$\theta\circ\psi \in \operatorname{PAut} A, \quad r(\theta\circ\psi) \le r(\psi) \le e, \quad \theta(l(\psi)r(\theta)) = l(\theta\circ\psi).$$

This means that $\theta(l(\psi)r(\theta)) \le hl(\theta)$, so $\theta(hr(\theta)) \le hl(\theta)$ and hence that $\theta(hr(\theta)) = hl(\theta)$ by symmetry. Hence h is globally central. If k is globally central, $e \le k$, and $\psi \in \operatorname{PAut} A$, $r(\psi) \le e$, then $l(\psi) = \psi(r(\psi)) = \psi(kr(\psi)) = kl(\psi)$ by (i), and $l(\psi) \le k$. Thus $h \le k$.

(iii) It suffices to show the first assertion. We have $h_{\mathrm{I}} = C(e)$ for some abelian projection e of A. If $\theta \in \operatorname{PAut} A$, $r(\theta) \le e$, then $l(\theta)Al(\theta) = \theta(r(\theta)Ar(\theta)) \cong r(\theta)Ar(\theta)$ and $l(\theta)$ is also abelian. Hence $GC(e) \le h_{\mathrm{I}}$ and so $h_{\mathrm{I}} = GC(e)$. A similar argument, with e abelian replaced by e finite, shows that $h_{\mathrm{I}} + h_{\mathrm{II}}$ is globally central and so are $h_{\mathrm{II}} = (h_{\mathrm{I}} + h_{\mathrm{II}}) - h_{\mathrm{I}}$ and $h_{\mathrm{III}} = 1 - (h_{\mathrm{I}} + h_{\mathrm{II}})$. ■

Remark 7.3 Let $h_{\min}$ be the central projection of A which is the sum of all minimal central projections of A, *i.e.*, $h_{\min}A$ is the largest direct summand of A which is a C^*-sum of monotone complete AW^*-factors. Then $h_{\min}$ is clearly globally central. The central projections $h_{\mathrm{II}_1}, h_{\mathrm{II}_\infty}$ and h_{I_n} ($n < \infty$) in A (II_1 = finite continuous, II_∞ = semifinite, properly infinite, continuous, I_n = homogeneous of order n) need not be globally central. Indeed, if B is a type II_1 monotone complete AW^*-algebra and $A = B \oplus B \mathbin{\overline{\otimes}} B(l^2(I))$ (I infinite), then $h_{\mathrm{II}_1} = 1_B \oplus 0$, $h_{\mathrm{II}_\infty} = 0 \oplus 1_{B\overline{\otimes}B(l^2(I))}$, but $GC(h_{\mathrm{II}_1}) = GC(h_{\mathrm{II}_\infty}) = 1$, since $B \cong B \otimes e = (1\otimes e)(B \mathbin{\overline{\otimes}} B(l^2(I)))(1\otimes e)$ for any minimal projection e of $B(l^2(I))$. Similarly for h_{I_n}.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Now we consider the situations in which the equality $\{\mathrm{PAut}\, A\} = \{(\mathrm{PAut}\, A)^0\}$ holds.

Proposition 7.4

(i) *If A is properly infinite and if every orthogonal family of nonzero equivalent projections in A is countable, then* $\{\mathrm{PAut}\, A\} = \{(\mathrm{PAut}\, A)^0\}$.

(ii) *If A is of type* I_n $(n < \infty)$, i.e., $A = C(\Omega) \otimes M_n$, *where Ω is a stonean space and M_n is the C^*-algebra of all $n \times n$ complex matrices, then we have* $\mathrm{RPAut}\, A = (\mathrm{PAut}\, A)^0$, i.e., *each* $\theta \in (\mathrm{PAut}\, A)^+$ *is of the form* $\theta = (\mathrm{Ad}\, u) \circ (\theta_1 \otimes \mathrm{id}) \in (\mathrm{PAut}\, A)^0$, *where* $\theta_1 \in \mathrm{PAut}\, C(\Omega)$, id *is the identity map on M_n and* $u \in \mathrm{PI}\, A$ *with* $u^* u = u u^* = r(\theta_1) \otimes 1$.

Proof (i) It suffices to show under the assumptions that for every properly infinite $e \in \mathrm{Proj}\, A$ we have

$$e \sim C(e). \tag{7.1}$$

Here and henceforth $\sim$ and $\prec$, used in Definition 6.9, denote also the (Murray–von Neumann) equivalence and partial order for projections. Indeed, we have

$$\{\mathrm{PAut}\, A\} = \{(\mathrm{PAut}\, A)^0\}$$

if and only if for each $\theta \in (\mathrm{PAut}\, A)^+$ there is $\psi \in (\mathrm{PAut}\, A)^0$ with $\theta \sim \psi$, or equivalently, for each $\theta \in (\mathrm{PAut}\, A)^+$ there is $u \in \mathrm{PI}\, A$ such that $u^* u = l(\theta)$ and $(\mathrm{Ad}\, u) \circ \theta \in (\mathrm{PAut}\, A)^0$, *i.e.,* $u^* u = l(\theta), u u^* \in \mathrm{Proj}\, Z(A)$ (see Remark 6.10(ii), Proposition 3.4(i)). Moreover, if $0 \neq \theta \in (\mathrm{PAut}\, A)^+$, then $r(\theta)$ is properly infinite, since A is, and so is $l(\theta)$, since $l(\theta) A l(\theta) \cong r(\theta) A$. Hence (7.1) ensures the existence of such a u. The validity of (7.1) under the assumptions would be well known. But, for the sake of completeness, we give the following standard argument (see [30, proof of 2.2.14]). To see (7.1) it suffices to show that if $e \in \mathrm{Proj}\, A$ is properly infinite, then

$$he \sim h \leq C(e)$$

for some $0 \neq h \in \mathrm{Proj}\, Z(A)$, since $(1 - h)e$ (if $\neq 0$) is also properly infinite and a maximality argument implies that we may take $h = C(e)$. By [1, Theorem 1, p. 103], there is a maximal infinite orthogonal family $\{e_i\}_{i \in I}$ of projections in A such that $e \sim e_i \leq e$ for all $i \in I$. Enlarge the family to obtain a maximal orthogonal family $\{e_i\}_{i \in I_1}$, $I \subset I_1$, in $C(e)A$ such that $e_i \sim e$ for all $i \in I_1$. Then, by the second assumption on countability, $\mathrm{card}\, I = \mathrm{card}\, I_1 = \aleph_0$. (The argument that follows uses only $\mathrm{card}\, I = \mathrm{card}\, I_1 \geq \aleph_0$.) By (GC) take $h \in \mathrm{Proj}\, Z(A)$, $h \leq C(e)$, such that

$$\Big(C(e) - \sum_{i \in I_1} e_i\Big) h \prec eh,$$

$$\Big(C(e) - \sum_{i \in I_1} e_i\Big) (C(e) - h) \succ e(C(e) - h).$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

By the maximality of $\{e_i\}_{i\in I_1}$, $eh \neq 0$ and $h \neq 0$. For a fixed $i_0 \in I$, $e_{i_0} \sim e$, and $(C(e) - \sum_{i\in I_1} e_i)h \prec e_{i_0}h$. Adding this with $(\sum_{i\in I_1} e_i)h \sim (\sum_{i\in I_1\setminus\{i_0\}} e_i)h$ implies $h \prec (\sum_{i\in I_1} e_i)h$, $h \sim (\sum_{i\in I_1} e_i)h$. On the other hand, $\sum_{i\in I_1} e_i \sim \sum_{i\in I} e_i$ implies $h \sim (\sum_{i\in I} e_i)h \leq eh \leq h$, and so $he \sim h$, $0 \neq h \leq C(e)$, as desired.

(ii) If $\theta \in (\mathrm{PAut}\, A)^+$, then $r(\theta) = \chi_{\Omega_1} \otimes 1$, where $\chi_{\Omega_1} \in C(\Omega)$ is the characteristic function of some clopen (closed and open) subset Ω_1 of Ω, and $\theta\colon C(\Omega_1) \otimes M_n \to l(\theta)(C(\Omega) \otimes M_n)l(\theta)$ is a $*$-isomorphism. Regard each element of $A = C(\Omega) \otimes M_n$ as a continuous function of Ω into M_n. Then for each $t \in \Omega$,

$$\{l(\theta)(t)x(t)l(\theta)(t) : x \in C(\Omega) \otimes M_n\} = \{\theta(x)(t) : x \in C(\Omega) \otimes M_n\},$$

being the image of a $*$-homomorphism of $C(\Omega_1) \otimes M_n$ into M_n, is M_n or $\{0\}$. Hence $l(\theta)(t) = 1$ or 0, so $l(\theta) = \chi_{\Omega_2} \otimes 1 \in \mathrm{Proj}(C(\Omega) \otimes 1) = \mathrm{Proj}\, Z(A)$ for some clopen $\Omega_2 \subset \Omega$ so that $\theta(a \otimes 1) = \theta_1(a) \otimes 1$ for $a \in C(\Omega_1)$, and $\theta \circ ((\theta_1)^{-1} \otimes \mathrm{id})$ is a $*$-automorhism of $C(\Omega_2) \otimes M_n$ which fixes the center $C(\Omega_2) \otimes 1$ elementwise. Thus $\theta \circ ((\theta_1)^{-1} \otimes \mathrm{id}) = \mathrm{Ad}\, u$ for some unitary $u \in C(\Omega_2) \otimes M_n$ and $\theta = (\mathrm{Ad}\, u) \circ (\theta_1 \otimes \mathrm{id})$. ■

Remark 7.5

(i) Let A be of type I. Then the assumptions in Proposition 7.4(i) amount to saying that A is homogeneous of order $\aleph_0$ and it is not homogeneous of type $\aleph$ for any cardinal $\aleph > \aleph_0$, since the proper infiniteness of A implies that it has no nonzero homogeneous summand of finite order and since each nonzero projection of A majorizes a nonzero abelian projection. This fact shows also that the second assumption there is strictly weaker than the condition of A being σ-finite (or countably decomposable). Note here that a type I, non-W^*, AW^*-algebra can be both homogeneous of order κ and of order λ for different infinite cardinals κ and λ (see [23]).

(ii) The above proof of Proposition 7.4(i) uses the following fact, rather than the second assumption: every orthogonal family of "properly infinite" equivalent projections is countable. But, adding "properly infinite" to the statement does not improve Proposition 7.4(i). Indeed, the existence of an uncountable orthogonal family of equivalent, not necessarily properly infinite, nonzero projections $\{e_i\}_{i\in I}$ implies the existence of such a family of properly infinite projections, since I can be partitioned into an uncountable family of countably infinite subsets.

(iii) The assumptions in Proposition 7.4 are best possible (at least under the second assumption in Proposition 7.4(i)) in the following sense.

If A is of type II, but not of type II_∞, *i.e.,* $0 \neq h_{\mathrm{II}_1} \leq h_{\mathrm{II}} = 1$, then the equality $\{\mathrm{PAut}\, A\} = \{(\mathrm{PAut}\, A)^0\}$ does not necessarily hold. For example, if A is a type II_1 W^*-factor with nontrivial fundamental group in the sense of Murray–von Neumann [21], *i.e.,* there is a $*$-isomorphism $\theta\colon A \to eAe$ with $1 \neq e \in \mathrm{Proj}\, A$, then $\theta \in (\mathrm{PAut}\, A)^+$ is not equivalent to any element of $(\mathrm{PAut}\, A)^0$, *i.e.,* a $*$-automomorphism of A or 0.

If A is of type I, but not homogeneous, *i.e.,* $h_{\mathrm{I}_n} \neq 0 \neq h_{\mathrm{I}_m}$ for some cardinals $n < m$, then $\mathrm{RPAut}\, A = (\mathrm{PAut}\, A)^0$ does not necessarily hold. Indeed, $A = B(l^2(I)) \oplus B(l^2(J))$ is of type I and not homogeneous, where $\mathrm{card}\, I = n$ and $\mathrm{card}\, J = m$. But

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

$\theta \in (\mathrm{PAut}\, A)^+$ defined by $x \oplus 0 \mapsto 0 \oplus \varphi(x)$ is not equivalent to any element of $(\mathrm{PAut}\, A)^0$, where φ is the embedding $B(l^2(I)) \to B(l^2(J))$ as a reduced subalgebra.

Definition 7.6 If $\theta \in \mathrm{PAut}\, A$, then call the largest direct summand of θ which has a positive (resp., central, negative) regularization the *positive* (resp., *central, negative*) *part* of θ, and write it as θ^+ (resp., θ^0, θ^-), so that $\theta = \theta^{++} \oplus \theta^0 \oplus \theta^{--}$, where $\theta^+ = \theta^{++} \oplus \theta^0$ and $\theta^- = \theta^0 \oplus \theta^{--}$. Call θ^{++} (resp., θ^{--}) the *purely positive* (resp., *purely negative*) *part* of θ, and call θ itself *weakly positive, weakly central*, etc. if $\theta = \theta^+$, $\theta = \theta^0$, etc., *i.e.*, $\theta \sim \psi$ for some ψ positive, central, etc., or equivalently, if $\langle\theta\rangle$ (defined immediately before Definition 6.16) is positive, central, etc. in the sense of Definition 3.3.

Note that, in PAut A, weak positivity implies positivity if and only if A is abelian, since if A is not abelian and a non-central projection e exists, then $\mathrm{id}_{eAe} = \mathrm{Ad}\, e$ is weakly positive, but not positive (see Remark 6.10(iv)).

Proposition 7.7 *The algebra A is finite if and only if, in* RPAut *A, weak positivity, etc. imply positivity, etc.*

Proof *Sufficiency:* If A is infinite, *i.e.*, $1 = u^*u \neq uu^*$ for some $u \in \mathrm{PI}\, A$, then $\mathrm{Ad}\, u \sim \mathrm{id}_A$, id_A is central, (resp., negative), and so $\mathrm{Ad}\, u$ is weakly central, (resp., weakly negative), but not central, (resp., not negative), since $uu^* \notin Z(A)$. Also $\mathrm{Ad}\, u^* \sim \mathrm{id}_A$, and $\mathrm{Ad}\, u^*$ is weakly positive, but not positve.

Necessity: It suffices to show that if $\theta, \psi \in \mathrm{RPAut}\, A$ and $\theta \sim \psi$ with θ positive and ψ negative, then θ and ψ are both central. (Consider the positive-negative decompositions of θ and ψ.) But, by Theorem 6.11(ii) we have $\langle\theta\rangle \cong \langle\psi\rangle = \langle\psi^{-1}\rangle^{-1}$, and it follows from Proposition 3.4(ii) that $l(\theta) \leq l(\psi)$, $r(\theta) \leq r(\psi)$ and $\psi^{-1}|l(\theta)Al(\theta) = (\mathrm{Ad}\, u) \circ \theta^{-1}$ for some $u \in \mathrm{PI}\, A$ with $u^*u = r(\theta) \in Z(A)$. Then, since A is finite, $r(\psi) \geq \psi^{-1}(l(\theta)) = uu^* = u^*u = r(\theta)$, $r(\psi) = r(\theta) \in Z(A)$ and ψ is central. Similarly $\theta = \psi \circ (\mathrm{Ad}\, u)$ is central. ■

Definition 7.8 Call a monotone complete C^*-algebra a *global factor* if the unit is the only nonzero globally central projection.

By Proposition 7.2(iii) global factors are classified into the cases of types I, II, and III.

Proposition 7.9 *Let B be a monotone complete C^*-algebra containing A as a monotone closed C^*-subalgebra with the same unit. Suppose that B is generated as a monotone complete C^*-algebra by the normalizer $N_B(A)$ of A in B. Then each globally central projection of A belongs to the center of B. In particular, if B is an AW^*-factor, then A is a global factor.*

Proof By the assumption and Corollary 6.8, B is generated by A and $RN_B(A)$ as a monotone complete C^*-algebra. Hence it suffices to show that if h is a globally central projection of A and $s \in RN_B(A)$, then $sh = hs$. But, by Proposition 4.2(vi),

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Ad $s|s^*sAs^*s \in$ RPAut$A$, and so, by Proposition 7.2(i), $sh = ss^*sh = shs^*s = hss^*s = hs$, as desired. ∎

Proposition 7.10

(1) *If A is a global factor, then so is $Z(A)$. If, further, A is of type* I, *then the reverse implication holds.*
(2) *For $e \in \operatorname{Proj} A$ with $C(e) = 1$, A is a global factor if and only if so is eAe.*

Proof (i) For $h \in \operatorname{Proj} Z(A)$, h is globally central as an element of A if and only if $\overline{\theta}(hr(\overline{\theta})) = hl(\overline{\theta})$ for all $\theta \in \operatorname{PAut} A$ by Proposition 7.2(i). Hence h being globally central as an element of $Z(A)$, *i.e.*, $\psi(hr(\psi)) = hl(\psi)$ for all $\psi \in \operatorname{PAut} Z(A)$, implies that as an element of A, and the first assertion follows. If A is of type I, then $\{\overline{\theta} : \theta \in \operatorname{PAut} A\} = \operatorname{PAut} Z(A)$ by Corollary 6.20(ii), and the second assertion follows.

(ii) It suffices to show that for $h \in \operatorname{Proj} Z(A)$, h is globally central in A if and only if so is he in eAe. If he is globally central in eAe and $\theta \in \operatorname{PAut} A$, then, by Lemma 6.13(ii) we have $\psi \prec \theta$ for some $\psi \in \operatorname{PAut}(eAe)$. Hence $\overline{\theta}(hC(r(\theta))) = hC(l(\theta))$, since $\overline{\theta} = \overline{\psi}$ by Remark 6.10(iii), and it follows that h is globally central in A. Similarly for the reverse implication. ∎

If A is a global factor, then the central projection $h_{\min}$ is 1 or 0 (see Remark 7.3). Now we investigate the former case in more detail.

Proposition 7.11 *Let A be the C^*-sum, $\prod_{i\in I} A_i$, of a family of monotone complete AW^*-factors $\{A_i\}_{i\in I}$.*

(i) *If A is of type* I, *or equivalently, if each A_i is a type* I *W^*-factor, then A is a global factor.*
(ii) *If A is of type* II, *then A is a global factor if and only if there exist a type* II_1, *monotone complete AW^*-factor A_0 and families of index sets $\{J_i\}_{i\in I}$ and of projections $\{e_i\}_{i\in I}$, both parametrized by I, such that $e_i \in A_0 \mathbin{\overline{\otimes}} B(l^2(J_i))$, with $C(e_i) = 1$ in $A_0 \mathbin{\overline{\otimes}} B(l^2(J_i))$, and $A_i \cong e_i(A_0 \mathbin{\overline{\otimes}} B(l^2(J_i)))e_i$ for all $i \in I$. If, further, A is of type* II_1, *then we can take each J_i to be finite.*
(iii) *If A is of type* III *and if each A_i is σ-finite, then A is a global factor if and only if $A_i \cong A_j$ for all $i, j \in I$, or equivalently, $A \cong A_0 \mathbin{\overline{\otimes}} l^\infty(I)$ for some type* III, *monotone complete AW^*-factor A_0.*

Proof Observe first that A as above is a global factor if and only if for any $i_1, i_2 \in I$ there are nonzero projections $e_1 \in A_{i_1}$, $e_2 \in A_{i_2}$ such that $e_1A_{i_1}e_1 \cong e_2A_{i_2}e_2$. Indeed, A is a global factor if and only if $GC(h_i) = 1$ for all $i \in I$, where $h_i \in \operatorname{Proj} Z(A)$ with $h_iA = A_i$. The latter condition implies that for any $i_1, i_2 \in I$ there is $\theta \in \operatorname{PAut} A$ such that $r(\theta) \le h_{i_1}$ and $h_{i_2}l(\theta) \ne 0$ and hence that with $e_2 := h_{i_2}l(\theta)$ and $e_1 := \theta^{-1}(e_2)$ we have $e_1A_{i_1}e_1 \cong e_2A_{i_2}e_2$. The reverse implication follows similarly.

Part (i) follows from the above argument.

The sufficiency of the conditions in (ii) and (iii) follows from the fact that if $A_i \cong e_i(A_0 \mathbin{\overline{\otimes}} B(l^2(J_i)))e_i$ for some monotone complete AW^*-factor A_0 and some families of index sets $\{J_i\}_{i\in I}$ and projections $\{e_i\}_{i\in I}$ with $C(e_i) = 1$, then A is a global factor.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Indeed, by Proposition 7.10(ii), we may assume $e_i = 1$ for all i. Then we may apply the assertion in the first paragraph.

The necessity of (ii): For a fixed $i_0 \in I$ set $A_0 = A_{i_0}$. By the first paragraph, for each $i \in I$ there are nonzero projections $p_i \in A_0$, $q_i \in A_i$ such that $p_i A_0 p_i \cong q_i A_i q_i$. Since A_i is a monotone complete AW^*-factor and the central cover of q_i in A_i equals h_i, a standard argument shows that for some index set J_i and some projection $f_i \in q_i A_i q_i \mathbin{\overline{\otimes}} B(l^2(J_i))$ we have $A_i \cong f_i(q_i A_i q_i \mathbin{\overline{\otimes}} B(l^2(J_i)))f_i$, where J_i arises as the index set of an orthogonal family $\{r_j\}_{j\in J_i}$ of nonzero projections in A_i such that $\sum_j r_j = h_i$ and $r_j \prec q_i$ for all j. Then the assertion follows, since $q_i A_i q_i \mathbin{\overline{\otimes}} B(l^2(J_i)) \cong p_i A_0 p_i \mathbin{\overline{\otimes}} B(l^2(J_i)) = (p_i \otimes 1)(A_0 \mathbin{\overline{\otimes}} B(l^2(J_i)))(p_i \otimes 1)$. Suppose further that A is of type II_1 and hence that each A_i is a finite AW^*-factor of type II. Then A_i has a dimension function D_i [1, p. 153], which has values in the interval $[0, 1]$, and we may take the cardinality of J_i to be $\le 1/D_i(q_i) + 1$.

The necessity of (iii): If A satisfies the stated conditions, then, for q_i, J_i as above we have $p_i \sim h_{i_0}$, $q_i \sim h_i$ [30, 2.2.14], card $J_i \le \aleph_0$, etc., and the assertion follows. ■

8 The Fundamental Homomorphism on a Finite Monotone Complete C^*-Algebra

Throughout this section A denotes a *finite* monotone complete C^*-algebra and D denotes the unique dimension function for A [1, p. 153], *i.e.,* $D\colon \operatorname{Proj} A \to Z(A)^+$ is a unique completely additive function such that $e \sim f$ implies $D(e) = D(f)$ and $D(h) = h$ for $h \in \operatorname{Proj} Z(A)$, where $Z(A)^+ = \{a \in Z(A) : a \ge 0\}$.

Let Ω be the spectrum of the center $Z := Z(A)$ so that $Z = C(\Omega)$ and Ω is a stonean space, *i.e.,* the closure of any open subset or the interior of any closed subset of Ω is clopen, and use the notation Cl and Int to denote the closure and interior of subsets of Ω. Denote by Z^+_∞ the set of all equivalence classes of functions $a\colon \Omega \to \mathbb{R}^+ := \{r \in \mathbb{R} : r \ge 0\}$ such that $U_a := \{\omega \in \Omega : a(\omega) > 0\}$ is open and $a|U_a$ is continuous (hence a is lower semicontinuous on Ω), where the equivalence $\sim$ is defined by

$$a \sim b \iff \operatorname{Cl} U_a = \operatorname{Cl} U_b, \quad a = b \quad \text{on } U_a \cap U_b.$$

(That $\sim$ is an equivalence relation follows from the obvious fact that if O_1, O_2 are open subsets of Ω and $\operatorname{Cl} O_1 = \operatorname{Cl} O_2$, then $\operatorname{Cl}(O_1 \cap O_2) = \operatorname{Cl} O_1$.) For such a function a define a function a^{-1} and a projection $s(a) \in Z$ by

$$U_{a^{-1}} = U_a,$$

$$a^{-1}(\omega) = \begin{cases} a(\omega)^{-1} & \text{if } \omega \in U_a \\ 0 & \text{otherwise,} \end{cases}$$

$$s(a)(\omega) = \begin{cases} 1 & \text{if } \omega \in \operatorname{Cl} U_a, \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

For two such functions a, b, define the product ab by

$$U_{ab} = U_a \cap U_b,$$

$$(ab)(\omega) = \begin{cases} a(\omega)b(\omega) & \text{if } \omega \in U_{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

We have canonically $Z^+ \subset Z^+_\infty$, since each bounded lower semicontinuous function on Ω coincides with a unique element of Z except on a meager subset (see [5] or [32, p. 104, 1.7].

For simplicity, we use the same letter to denote both a function and its equivalence class. Then $aa^{-1} = s(a)$ for all $a \in Z^+_\infty$, and under the inversion and product above, Z^+_∞ is a commutative inverse semigroup with the subsemigroup of idempotents Proj Z. (Note that our later arguments depend only on the fact that the elements of the form ab^{-1}, $a, b \in Z^+$, have products and inverses in Z^+_∞.)

Note for later use that if $a, b, c, d \in Z^+_\infty$ and $s(b) = s(d)$, then

$$(8.1) \qquad ab^{-1} = cd^{-1} \iff ad = bc.$$

Definition 8.1 For $a \in Z^+$ define $a^0 \in \operatorname{Proj} Z$, $a^+, a^{++}, a^-, a^{--} \in Z^+$ by

$$a^0(\omega) = 1 \text{ on } \operatorname{Int}\{\omega \in \Omega : a(\omega) = 1\}, \; = 0 \text{ otherwise;}$$

$$a^{++}(\omega) = a(\omega) \text{ on } \operatorname{Cl}\{\omega \in \Omega : a(\omega) < 1\}, \; = 0 \text{ otherwise;}$$

$$a^{--}(\omega) = a(\omega) \text{ on } \operatorname{Cl}\{\omega \in \Omega : a(\omega) > 1\}, \; = 0 \text{ otherwise;}$$

$$a^+ = a^{++} + a^0, \quad a^- = a^{--} + a^0,$$

so that

$$a = a^{++} + a^0 + a^{--}, \quad s(a) = s(a^{++}) + a^0 + s(a^{--}).$$

(Note that the above three sets are pairwise disjoint clopen subsets with union Ω and that a^0, etc. are of the form ae for some $e \in \operatorname{Proj} Z$.)

Make $Z^+_\infty \times \operatorname{PAut} Z$ into an inverse semigroup with the operations:

$$(8.2) \qquad (a, \rho)^{-1} = (\rho^{-1}(a^{-1}), \rho^{-1}), \quad (a, \rho) \cdot (b, \sigma) = (a\rho(b), \rho \circ \sigma).$$

Here and henceforth the action of $\rho \in \operatorname{PAut} Z$ on elements of Z is naturally extended to that on elements of Z^+_∞, since ρ induces a homeomorphism between clopen subsets of Ω. Hence, for $\rho \in \operatorname{PAut} Z$ and $a \in Z^+_\infty$,

$$(8.3) \qquad \rho(a^{-1}) = \rho(a)^{-1}, \quad s(\rho(a)) = \rho(s(a)).$$

Lemma 8.2

(i) *For $e \in \operatorname{Proj} A$ we have $s(D(e)) = C(e)$, and $D(e)^0$ is the largest central projection of A such that $D(e)^0 \le e$.*

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

(ii) *Define a map* $d\colon \operatorname{PAut} A \to Z_\infty^+$ *by*

$$d(\theta) = D(l(\theta))[\overline{\theta}(D(r(\theta)))]^{-1}. \tag{8.4}$$

Then, for each $\theta \in \operatorname{PAut} A$,

$$D \circ \theta = d(\theta) \cdot (\overline{\theta} \circ D) \ \ on \ \operatorname{Proj} r(\theta)Ar(\theta), \tag{8.5}$$

$$\theta = \theta_1 \oplus \theta_2 \ \Rightarrow \ d(\theta) = d(\theta_1) + d(\theta_2), \quad d(\theta_1)d(\theta_2) = 0, \tag{8.6}$$

$$\theta \sim \psi \in \operatorname{PAut} A \ \Rightarrow \ d(\theta) = d(\psi). \tag{8.7}$$

(iii) *For* $\theta, \psi \in \operatorname{PAut} A$ *we have*

$$d(\theta \bullet \psi) = d(\theta)\overline{\theta}(d(\psi)). \tag{8.8}$$

Proof (i) We have $e = s(D(e))e$ and $C(e) \le s(D(e))$, since $D(e) \le s(D(e))$ and so $D[(1 - s(D(e)))e] = (1 - s(D(e)))D(e) = 0$. Further $D(e) \le D(C(e)) = C(e)$ and $s(D(e)) \le C(e)$. Since $0 \le D(e) \le 1$, $D[D(e)^0(1 - e)] = D(e)^0(1 - D(e)) = 0$ and so $D(e)^0(1 - e) = 0$, $D(e)^0 \le e$. Conversely, if $h \in \operatorname{Proj} Z$ and $h \le e$, then $h = D(h) \le D(e) \le 1$ and $h \le D(e)^0$.

(ii) Note first that by (i),

$$\begin{aligned} s(D(l(\theta))) &= C(l(\theta)) = l(\overline{\theta}), \\ s([\overline{\theta}(D(r(\theta)))]^{-1}) &= s[\overline{\theta}(D(r(\theta)))] = \overline{\theta}[s(D(r(\theta)))] \\ &= \overline{\theta}(C(r(\theta))) = \overline{\theta}(r(\overline{\theta})) = l(\overline{\theta}), \\ s(d(\theta)) &= l(\overline{\theta}). \end{aligned} \tag{8.9}$$

To see (8.5) define two maps $D', D''\colon \operatorname{Proj} r(\theta)Ar(\theta) \to Z(r(\theta)Ar(\theta)) = r(\theta)Z$ by

$$D'(e) = r(\theta)D(r(\theta))^{-1}D(e), \quad D''(e) = r(\theta)(\overline{\theta})^{-1}[D(l(\theta))^{-1}D(\theta(e))]$$

for $e \in \operatorname{Proj} r(\theta)Ar(\theta)$. (Note that the right-hand sides are bounded and so they give elements of $r(\theta)Z$.) Then both D' and D'' are dimension functions for $r(\theta)Ar(\theta)$. Indeed, for example, if $h \in \operatorname{Proj} Z(r(\theta)Ar(\theta))$, then $h = C(h)r(\theta)$, $\theta(h) = \overline{\theta}(C(h))l(\theta)$, and

$$\begin{aligned} D''(h) &= r(\theta)(\overline{\theta})^{-1}[D(l(\theta))^{-1}\overline{\theta}(C(h))D(l(\theta))] = r(\theta)(\overline{\theta})^{-1}[\overline{\theta}(C(h))C(l(\theta))] \\ &= r(\theta)(\overline{\theta})^{-1}[\overline{\theta}(C(h))l(\overline{\theta})] = r(\theta)C(h) = h \end{aligned}$$

by (i), etc. Hence, by the uniqueness of the dimension function, $D' = D''$, and (8.5) follows from (i), (8.1), (8.3) and (8.9).

The property (8.6) follows from (8.4) and (8.9).

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

To see (8.7) it suffices to consider the case $\theta \simeq \psi$ or $\theta \leq \psi$. Then $\overline{\theta} = \overline{\psi}$ by Remark 6.10(iii). In the former case, $\psi = (\mathrm{Ad}\, v)\circ\theta\circ(\mathrm{Ad}\, u)$ for some $u, v \in \mathrm{PI}\, A$ with $uu^* = r(\theta)$ and $v^*v = l(\psi)$, and so $D(r(\theta)) = D(uu^*) = D(u^*u) = D(r(\psi))$, etc. imply $d(\theta) = d(\psi)$. In the latter case, $r(\theta) \leq r(\psi) \leq C(r(\theta))$ and $\psi|r(\theta)Ar(\theta) = \theta$. Hence, by (8.5),

$$\begin{aligned} D(l(\theta)) &= D(\psi(r(\theta))) = d(\psi)\cdot\overline{\psi}(D(r(\theta))) \\ &= D(l(\psi))\cdot[\overline{\psi}(D(r(\psi)))]^{-1}\cdot\overline{\theta}(D(r(\theta))), \end{aligned}$$

and since $C(r(\psi)) = C(r(\theta))$ and so $s[\overline{\psi}(D(r(\psi)))] = s[\overline{\theta}(D(r(\theta)))]$ by (i) and (8.3),

$$d(\theta) = D(l(\theta))[\overline{\theta}(D(r(\theta)))]^{-1} = D(l(\psi))[\overline{\psi}(D(r(\psi)))]^{-1} = d(\psi).$$

(iii) If $u \in \mathrm{PI}\, A$ is as in Definition 6.16, then with $\theta_1 := \theta|uu^*Auu^*$ and $\psi_1 := (\mathrm{Ad}\, u)\circ\psi$ we have

$$\theta_1 \sim \theta, \quad \psi_1 \sim \psi, \quad \theta \bullet_u \psi = \theta_1 \circ \psi_1, \quad r(\theta_1) = uu^* = l(\psi_1).$$

Hence, by (ii), we may assume that $r(\theta) = l(\psi)$ and $\theta \bullet \psi = \theta \circ \psi$. But, then $l(\theta\circ\psi) = l(\theta)$, $r(\theta\circ\psi) = r(\psi)$, and

$$\begin{aligned} d(\theta\circ\psi) &= D(l(\theta))[\overline{\theta\circ\psi}(D(r(\psi)))]^{-1} \\ &= D(l(\theta))[\overline{\theta}(D(r(\theta)))]^{-1}\overline{\theta}(D(l(\psi)))\overline{\theta}\big([\overline{\psi}(D(r(\psi)))]^{-1}\big) \\ &= d(\theta)\overline{\theta}(d(\psi)), \end{aligned}$$

since

$$\begin{aligned} [\overline{\theta}(D(r(\theta)))]^{-1}\overline{\theta}(D(l(\psi))) &= \overline{\theta}[s(D(r(\theta)))] = \overline{\theta}(C(r(\theta))) \\ &= l(\overline{\theta}) = C(l(\theta)) \geq D(l(\theta)). \end{aligned}$$

■

We call the function F defined below the *fundamental homomorphism* on A.

Proposition 8.3

(i) *$\theta \in \mathrm{PAut}\, A$ is weakly positive, (resp., weakly central, weakly negative) if and only if $d(\theta) \leq C(l(\theta))$, (resp., $d(\theta) \in \mathrm{Proj}\, Z$, $d(\theta) \geq C(l(\theta))$),* i.e., *$d(\theta) = d(\theta)^+$, (resp., $d(\theta) = d(\theta)^0$, $d(\theta) = d(\theta)^-$). That is, the decomposition $\theta = \theta^{++} \oplus \theta^0 \oplus \theta^{--}$ in Definition 7.6 corresponds to the decomposition $d(\theta) = d(\theta)^{++} + d(\theta)^0 + d(\theta)^{--}$.*

(ii) *The map $F\colon \{\mathrm{PAut}\, A\} \to Z^+_\infty \times \mathrm{PAut}\, Z$ defined by $F(\{\theta\}) = (d(\theta), \overline{\theta})$ is a homomorphism between inverse semigroups.*

(iii) *If, in particular, A is a type II_1 W^*-factor and so $Z^+_\infty \times \mathrm{PAut}\, Z$ is identified with the multiplicative semigroup $\mathbb{R}^+$, then the map $\{\theta\} \mapsto d(\theta)$ from $\{\mathrm{PAut}\, A\} \setminus \{0\}$ to $\mathbb{R}^+_* = \mathbb{R}^+ \setminus \{0\}$ is a homomorphism between groups, whose image is the fundamental group $\mathcal{F}(A)$ of A and whose kernel is $\{\mathrm{Aut}\, A\} \cong \mathrm{Out}\, A$. Hence*

$$(\{\mathrm{PAut}\, A\} \setminus \{0\})\big/\{\mathrm{Aut}\, A\} \cong \mathcal{F}(A).$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proof (i) Each $\theta \in \operatorname{PAut} A$ has a regularization ψ, so that $\theta^{++} \sim \psi^{++}$, $\theta^0 \sim \psi^0$, etc. with θ^{++}, etc. positive, etc. Hence, in view of (8.7), it suffices to show, assuming θ to be regular, the validity of the assertion with weakly positive, etc. replaced by positive, etc.

If θ is positive, (resp., central, negative), then by (8.4) and (8.9), $d(\theta) = D(l(\theta)) \leq C(l(\theta))$, (resp., $d(\theta) = C(l(\theta))$, $d(\theta) = [\overline{\theta}(D(r(\theta)))]^{-1} \geq C(l(\theta))$).

To see the reverse implication, take $\theta \in \operatorname{RPAut} A$ with $\theta = \theta^{++} \oplus \theta^0 \oplus \theta^{--}$ as in Definition 7.6. Then θ^{++}, θ^0, etc. are positive, central, etc. by Proposition 7.7, and it follows from (8.6) and the foregoing that

$$d(\theta) = d(\theta^{++}) + d(\theta^0) + d(\theta^{--}), \quad d(\theta)^+ = d(\theta^{++}) + d(\theta^0),$$
$$d(\theta)^0 = d(\theta^0), \quad d(\theta)^- = d(\theta^0) + d(\theta^{--}).$$

Hence, if $d(\theta) = d(\theta)^+$, (resp., $d(\theta) = d(\theta)^0$, $d(\theta) = d(\theta)^-$), then $d(\theta^{--}) = 0$, *i.e.*, $\theta^{--} = 0$, (resp., $\theta^{++} = \theta^{--} = 0$, $\theta^{++} = 0$), and θ is positive, (resp., central, negative).

(ii) Immediate from Corollary 6.20(i), (8.2) and (8.8).

(iii) In this case, D is (identified with) the restriction to $\operatorname{Proj} A$ of the unique trace tr on A with $\operatorname{tr}(1) = 1$, and for $r \in \mathbb{R}^+_*$ we have $r \in \mathcal{F}(A)$ if and only if $r = \operatorname{tr}(l(\theta)) = \operatorname{tr}(l(\theta))/\operatorname{tr}(r(\theta))$ or $r = 1/\operatorname{tr}(l(\theta)) = \operatorname{tr}(l(\theta^{-1}))/\operatorname{tr}(r(\theta^{-1}))$ for some $\theta \in \operatorname{PAut} A$ with $r(\theta) = 1$, *i.e.*, $r = \operatorname{tr}(l(\theta))/\operatorname{tr}(r(\theta))$ for some $\theta \in \operatorname{PAut} A$ with $r(\theta) = 1$ or $l(\theta) = 1$. Further, by (8.4) and (8.8),

$$d(\theta) = \operatorname{tr}(l(\theta))/\operatorname{tr}(r(\theta)) \in \mathbb{R}^+_*, \quad d(\theta \bullet \psi) = d(\theta)d(\psi)$$

for $\theta, \psi \in \operatorname{PAut} A \setminus \{0\}$. If $\theta \in \operatorname{PAut} A \setminus \{0\}$ and ψ is its regularization, then $d(\theta) = d(\psi)$, $r(\psi) = 1$ or $l(\psi) = 1$, and $d(\psi) = \operatorname{tr}(l(\psi))$ or $d(\psi) = 1/\operatorname{tr}(r(\psi))$. Hence $\{d(\theta) : \theta \in \operatorname{PAut} A \setminus \{0\}\} = \mathcal{F}(A)$.

Finally, by (i), $d(\theta) = 1$ if and only if $\theta \sim \psi$ for some central $\psi \neq 0$, *i.e.*, $\theta \sim \psi \in \operatorname{Aut} A$. ■

9 Normalizers and Partial ∗-Automorphisms

In this closing section we summarize the consequences of the results in previous sections.

For the moment, let A denote a fixed monotone complete C^*-algebra and take another monotone complete C^*-algebra B containing A as a monotone closed C^*-subalgebra with the same unit. The C^*-version of the reasoning that follows holds with obvious modifications, and we shall touch on it later.

We will see how the operations defined in $[\operatorname{SDINV} A]$ and $\{\operatorname{PAut} A\}$ can be realized concretely as the operations in B.

In the notations $\operatorname{RINV}(A)$, $\operatorname{RINV}_B(A) \subset \operatorname{INV}'_B(A)$, $RN_B(A) \subset N_B(A)$, $\operatorname{RINV}_B(A)$, $\operatorname{SDINV}(A)$, etc. defined as before (see Sections 3, 4, 6) we have by Theorem 6.6 and Proposition 4.2(i),(vi), $\operatorname{RINV}(A) = \operatorname{SDINV}(A)$ and

$$\operatorname{RINV}_B(A) = \{AsA : s \in RN_B(A)\} \subset \{AxA : x \in N_B(A)\} \subset \operatorname{INV}'_B(A).$$

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

By Theorem 6.6 and Corollary 6.7 we may and shall identify $\mathrm{RINV}_B(A)$ with the set of all sub-A-bimodules X of B with $XX^* + X^*X \subset A$, which are also self-dual left Hilbert A-modules or monotone closed in B, and we obtain, by taking the monotone closure, a map reverse to the above inclusion:

$$\mathrm{INV}'_B(A) \to \mathrm{RINV}_B(A), \quad X \mapsto m\text{-}\mathrm{cl}_B X. \tag{9.1}$$

Moreover $\mathrm{RINV}_B(A)$ is an inverse semigroup with the inverse and product defined by

$$X^{-1} = X^*, \quad X \cdot Y = m\text{-}\mathrm{cl}_B(XY),$$

since for each $X \in \mathrm{RINV}_B(A)$ we have $X \cdot X^{-1} = hA$, $X^{-1} \cdot X = kA$ for some $h, k \in \mathrm{Proj}\, Z(A)$ (see Proposition 6.4(i)) and the argument in the proof of Theorem 5.2 applies. An inverse semigroup homomorphism is defined as follows:

$$\mathrm{RINV}_B(A) \to [\mathrm{SDINV}(A)], \quad X \mapsto [X], \tag{9.2}$$

so that we have a commutative diagram

$$\begin{array}{ccccc} N_B(A) & \longleftarrow & RN_B(A) & \longrightarrow & \{\mathrm{PAut}\, A\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{INV}'_B(A) & \longrightarrow & \mathrm{RINV}_B(A) & \longrightarrow & [\mathrm{SDINV}(A)], \end{array} \tag{9.3}$$

where the lower horizontal maps are as defined above, the upper left horizontal map is an inclusion map, the right vertical map is the isomorphism in Theorem 6.17, and the left vertical, middle vertical and upper right horizontal maps are defined respectively by $x \mapsto AxA$, $s \mapsto AsA$, and $s \mapsto \{\mathrm{Ad}\, s\}$ ($\mathrm{Ad}\, s = \mathrm{Ad}\, s|s^*sAs^*s$). The map (9.1) is also related to the passage from a partial $*$-automorphism of A to its regularization (see Definition 6.9) as follows. If $\theta \in \mathrm{PAut}\, A$ with $\theta = \mathrm{Ad}\, s$ for $s \in \mathrm{PI}\, N_B(A)$ and if $m\text{-}\mathrm{cl}_B(AsA) = AtA$ for some $t \in RN_B(A)$, then, as follows immediately, $\mathrm{Ad}\, t \in \mathrm{RPAut}\, A$ is a regularization of θ.

The following result shows that there is a large enough B for the map (9.2) or the map $\mathrm{RINV}_B(A) \to [\mathrm{SDINV}(A)] \cong \{\mathrm{PAut}\, A\}$ to be surjective, *i.e.*, for every element of $\mathrm{SDINV}(A)$ (resp., $\mathrm{PAut}\, A$) to be realized as some element of $\mathrm{RINV}_B(A)$ (resp., $\mathrm{Ad}\, s$ for some $s \in \mathrm{PI}\, N_B(A)$).

Proposition 9.1 *If $\{X_i\}_{i\in I}$ (resp., $\{\theta_i\}_{i\in I}$) is any subset of* $\mathrm{SDINV}(A)$ *(resp.,* $\mathrm{PAut}\, A$*), then there exists a monotone complete C^*-algebra B such that A is a monotone closed C^*-subalgebra of B containing the unit and each X_i is identified with some element of* $\mathrm{RINV}_B(A)$ *(resp., for each i there is an $s_i \in \mathrm{PI}\, N_B(A)$ with $\theta_i = \mathrm{Ad}\, s_i$) and such that B is generated as a monotone complete C^*-algebra by A and the X_i's (resp., s_i's).*

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Proof For $\{X_i\}_{i\in I}$ as above let F be the free group on card I generators $\{w_i\}_{i\in I}$ and make the algebraic direct sum $\mathcal{B} = \bigoplus_{g\in F} B_g$ into an F-graded $*$-algebra as follows. Set

$$B_e = A, \quad B_g = X_{i_1}^{\epsilon_1} \overline{\otimes}_A \cdots \overline{\otimes}_A X_{i_n}^{\epsilon_n},$$

where e is the unit element, $g = w_{i_1}^{\epsilon_1} \cdots w_{i_n}^{\epsilon_n}$, $i_1, \dots, i_n \in I$ (possibly duplicated), $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$, and $\overline{\otimes}_A$ is the product in SDINV(A). Write

$$x^* = x_n^* \otimes \cdots \otimes x_1^* \in B_{g^{-1}}, \quad x \cdot y = x \otimes y \in B_{gh},$$

where $x = x_1 \otimes \cdots \otimes x_n \in B_g$, $g = w_{i_1}^{\epsilon_1} \cdots w_{i_n}^{\epsilon_n}$, $x_k \in X_{i_k}^{\epsilon_k}$, and $y = y_1 \otimes \cdots \otimes y_m \in B_h$, $h = w_{j_1}^{\delta_1} \cdots w_{j_m}^{\delta_m}$, $y_k \in X_{j_k}^{\delta_k}$. Here, to make the second membership well defined we identify $A \overline{\otimes}_A B_g$ (or $B_g \overline{\otimes}_A A$) and $X \overline{\otimes}_A X^{-1}$, $X \in$ SDINV(A), with B_g and $z_r(X)A$ by the maps $a \otimes (x_1 \otimes \cdots \otimes x_n) \mapsto a \cdot x_1 \otimes \cdots \otimes x_n$ (or $(x_1 \otimes \cdots \otimes x_n) \otimes a \mapsto x_1 \otimes \cdots \otimes x_n \cdot a$) and $x \otimes y^* \mapsto \langle x, y \rangle$, $x, y \in X$ (see Proposition 6.4(i)), so that for x, y and g, h as above we have

$$\begin{aligned} x \otimes y &= (x_1 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes \cdots \otimes y_m) \\ &= x_1 \otimes \cdots \otimes x_{n-1} \otimes \langle x_n, y_1^* \rangle \cdot y_2 \otimes \cdots \otimes y_m = \cdots, \end{aligned}$$

etc. if $i_n = j_1$, $\epsilon_n = -\delta_1, \dots,$ etc.

Regard each element of $\mathcal{B}$ as a function $x \colon F \to \bigcup_{g\in F} B_g$ such that $x(g) \in B_g$ for all g and $x(g) = 0$ for all but finite g, identify A (resp., X_i) with the subset of $x \in \mathcal{B}$ such that $x(g) = 0$ for all $g \neq e$ (resp., $x(g) = 0$ for all $g \neq w_i$), and define the involution and product in $\mathcal{B}$ by

$$x^*(g) = x(g^{-1})^*, \quad (xy)(g) = \sum_{h\in F} x(h) \cdot y(h^{-1}g)$$

for $x, y \in \mathcal{B}$. Then $\mathcal{B}$ is both a $*$-algebra and a pre-Hilbert A-module with the module operation and inner product given by $a \cdot x = ax$ and $\langle x, y \rangle = (xy^*)(e)$, and each Hilbert A-module X_i is recovered as a subset of $\mathcal{B}$ via the product in $\mathcal{B}$ as in Remark 2.2(ii). If H is the self-dual completion of the pre-Hibert A-module $\mathcal{B}$, then a faithful $*$-representation π of $\mathcal{B}$ on H, *i.e.,* an injective $*$-homomorphism $\pi \colon \mathcal{B} \to \mathrm{End}_A(H)$, is defined by $y\pi(x) = yx$ for $x, y \in \mathcal{B}$, since $\langle y\pi(x), z \rangle = (yxz^*)(e) = \langle y, z\pi(x^*) \rangle$ and since each $\mathcal{B} \to \mathcal{B}$, $y \mapsto yx$, extends uniquely to a bounded linear map $H \to H$ (see the proof of [12, 3.1]). Denote by B the monotone closure of $\pi(\mathcal{B})$ in $\mathrm{End}_A(H)$. Then $\pi(A) \cong A$ is monotone closed in B, and so are $\pi(X_i)$, $i \in I$, by Corollary 6.7(i) $\Leftrightarrow$ (iii). Hence the assertion as for $\{X_i\}$ follows.

To see the assertion as for $\{\theta_i\}_{i\in I}$ take a regularization ψ_i of each θ_i (see Definition 6.9, Theorem 6.11) so that $\theta_i = (\mathrm{Ad}\, v_i) \circ \psi_i \circ (\mathrm{Ad}\, u_i)$ for some $u_i, v_i \in \mathrm{PI}\, A$ with $u_i^* u_i = r(\theta_i)$, $u_i u_i^* \leq r(\psi_i) \leq C(u_i u_i^*)$ and $\psi_i(u_i u_i^*) = v_i^* v_i$. If we apply the foregoing to $\{\langle \psi_i \rangle\}_{i\in I} \subset$ SDINV(A), then there exist a monotone complete C^*-algebra B and $\{s_i\}_{i\in I} \subset RN_B(A)$ such that $\langle \psi_i \rangle = As_iA$ and $\mathrm{Ad}\, s_i = \psi_i$, and hence we have $\theta_i = \mathrm{Ad}(v_i s_i u_i)$ with $v_i s_i u_i \in \mathrm{PI}\, N_B(A)$. ■

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

Corollary 9.2 *For a monotone complete C^*-algebra A, $h \in \operatorname{Proj} Z(A)$ is globally central in A if and only if h belongs to the center of each monotone complete C^*-algebra B which contains A as a monotone closed C^*-subalgebra with the same unit and is generated by the normalizer $N_B(A)$ as a monotone complete C^*-algebra.*

Proof The necessity has been proved in Proposition 7.9, and it remains to show the reverse implication. If h is not globally central, then $\theta(hr(\theta)) \neq hl(\theta)$ for some $\theta \in \operatorname{PAut} A$. By Proposition 9.1 there is a monotone complete C^*-algebra B which contains A as a monotone closed C^*-subalgebra with the same unit and is generated by A and a partial isometry $s \in N_B(A)$ such that $\operatorname{Ad} s \,|\, s^*sAs^*s = \theta$. Then $hs \neq sh$ and h is not in the center of B. ∎

Note that via the involution and product in B the sets $N_B(A)^0$ and $\mathrm{INV}_B(A)^0$ are inverse semigroups, *i.e.*, $s_1, s_2 \in N_B(A)^0 \Rightarrow s_1s_2 \in N_B(A)^0$, $\mathrm{INV}_B(A)^0 = \{As : s \in N_B(A)^0\}$, $As_1As_2 = As_1s_2 \in \mathrm{INV}_B(A)^0$ (see Proposition 4.2(iii)), etc. and the restrictions to $N_B(A)^0$ of the maps in (9.2), $N_B(A)^0 \to \mathrm{INV}_B(A)^0$ and $N_B(A)^0 \to \{\operatorname{PAut} A\}$ are inverse semigroup homomorphisms. Hence, if A is σ-finite and properly infinite and if B is as in Proposition 9.1 with $\{\theta_i\}_{i\in I} = \operatorname{PAut} A$, then $N_B(A)^0 \to \{\operatorname{PAut} A\}$ is a surjective homomorphism by Proposition 7.4(i), and so the inverse semigroup structure of $\{\operatorname{PAut} A\}$ is directly described by the product and involution in B. (But the map $N_B(A) \to \mathrm{INV}'_B(A)$, $x \mapsto AxA$, in (9.3) is not directly related to the product in B, although $N_B(A)$ and $\mathrm{INV}'_B(A)$ are $*$-semigroups under the operations $x \mapsto x^*$, $(x, y) \mapsto xy$, etc.) Even if A is only σ-finite, but not necessarily properly infinite, then this assertion is true to some extent. Indeed, $A \overline{\otimes} B(l^2)$, where l^2 is the $\aleph_0$-dimensional Hilbert space, is σ-finite and properly infinite, and by Corollary 6.19 and Proposition 7.4(i), $\{\operatorname{PAut} A\} \cong \{(\operatorname{PAut}(A \overline{\otimes} B(l^2)))^0\}$, since $A \cong (1 \otimes e)(A \overline{\otimes} B(l^2))(1 \otimes e)$ with $e \in B(l^2)$ a minimal projection.

From now on, let A be a C^*-algebra embedded in another C^*-algebra B as a C^*-subalgebra. Then the sets $N_B(A)$ and $\mathrm{INV}'_B(A) \subset \mathrm{INV}'(A)$ (where, see Section 2, $\mathrm{INV}'(A)$ is the set of all algebraic invertible A-modules and see Remark 2.2(ii) for the inclusion) make sense also for these $A \subset B$, *i.e.*,

$$N_B(A) = \{x \in B : xAx^* \subset A,\ x^*Ax \subset A\},$$
$$\mathrm{INV}'_B(A) = \{X \subset B : \text{sub-}A\text{-bimodules} : XX^* + X^*X \subset A\},$$

the set, $\mathrm{INV}_B(A)$, of all $X \in \mathrm{INV}'_B(A)$ which are also norm closed in B is an inverse semigroup with the product $X \cdot Y := n\text{-cl}_B(XY)$, the norm closure in B of the linear span XY, and the inversion $X^{-1} := X^*$; we have canonically $\mathrm{INV}_B(A) \subset \mathrm{INV}(A)$ (see Remark 2.2(ii), Remark 2.4, Definition 5.1); and we have the following canonical maps:

$$N_B(A) \to \mathrm{INV}'_B(A) \to \mathrm{INV}_B(A) \to [\mathrm{INV}(A)],$$

where the first map is as defined above, the second is defined by $X \mapsto n\text{-cl}_B X$, and the third, defined by $X \mapsto [X]$ with the identification $\mathrm{INV}_B(A) \subset \mathrm{INV}(A)$, is an inverse semigroup homomorphism. Moreover, a statement similar to the one in Proposition 9.1 holds with $\overline{\otimes}_A$ in the construction of B replaced by $\otimes_A$, and the last map

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

is surjective for some B. Finally, note that the notion of an invertible A-module gives an abstract characterization of an element of $\mathrm{INV}_B(A)$ for some B. Indeed, $\mathrm{INV}_B(A) \subset \mathrm{INV}(A)$ as seen above, and if $X \in \mathrm{INV}(A)$ and B is as constructed in Proposition 9.1 for $\{X\}$, then we have $X \in \mathrm{INV}_B(A)$.

Acknowledgement The author would like to thank the referee for the very careful reading of the manuscript.

References

[1] S. K. Berberian, *Baer ∗-Rings.* Grundlehren der Mathematischen Wissenschaften 195, Springer-Verlag, New York, 1972.
[2] B. Blackadar, *K-Theory for Operator Algebras.* Mathematical Sciences Research Institute Publications 5, Springer-Verlag, New York, 1986.
[3] L. G. Brown, P. Green, and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras.* Pacific J. Math. **71**(1977), no. 2, 349–363.
[4] J. Cuntz, *Simple C^*-algebras generated by isometries.* Comm. Math. Phys. **57**(1977), no. 2, 173–185.
[5] J. Dixmier, *Sur certains espaces considérés par M. H. Stone.* Summa Brasil. Math. **2**(1951), 151–182.
[6] ———, *Sous-anneaux abélian maximaux dans les factuers de type fini.* Ann. of Math. **59**(1954), 279-286.
[7] R. Exel, *Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence.* J. Funct. Anal. **122**(1994), no. 2, 361–401.
[8] ———, *Twisted partial actions: a classification of regular C^*-algebra bundles.* Proc. London Math. Soc.(3) **74**(1997), no. 2, 417–443.
[9] ———, *Amenability for Fell bundles.* J. Reine Angew. Math. **492**(1997), 41–73.
[10] M. Hamana, *Tensor products for monotone complete C^*-algebras. I.* Japan. J. Math. (N.S.) **8**(1982), no. 2, 259–283.
[11] ———, *Dynamical systems based on monotone complete C^*-algebras.* In: Current Topics in Operator Algebras. World Scientific. Publishing, River Edge, NJ, 1991, pp. 282–296.
[12] ———, *Modules over monotone complete C^*-algebras.* Intern. J. Math. **3**(1992), no. 2, 185–204.
[13] ———, *Infinite, σ-finite, non-W^*, AW^*-factors.* Internat. J. Math. **12**(2001), no. 1, 81–95.
[14] ———, *Coactions of discrete groups on monotone complete C^*-algebras*, in preparation.
[15] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. II.* Grundlehren der Mathematischen Wissenschaften 152, Springer-Verlag, New York, 1970.
[16] B. E. Johnson, *AW^*-algebras are QW^*-algebras.* Pacific J. Math. **23**(1967), 97–99.
[17] R. V. Kadison, *Operator algebras with a faithful weakly-closed representation.* Ann. of Math. **64**(1956), 175–181.
[18] R V. Kadison and G. K. Pedersen, *Equivalence in operator algebras.* Math. Scand. **27**(1970), 205–222.
[19] I. Kaplansky, *Projections in Banach algebras.* Ann. of Math. **53**(1951), 235–249.
[20] M. V. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries.* World Scientific Publishing, River Edge, NJ, 1998.
[21] F. J. Murray and J. von Neumann, *Rings of operators. IV.* Ann. of Math. **44**(1943), 716–808.
[22] Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras.* Lecture Notes in Mathematics 731, Springer-Verlag, Berlin, 1979.
[23] M. Ozawa, *Nonuniqueness of the cardinality attached to homogeneous AW^*-algebras.* Proc. Amer. Math. Soc. **93**(1985), no. 4, 681–684.
[24] A. L. T. Paterson, *Groupoids, Inverse Semigroups, and Their Operator Algebras.* Progress in Math. 170, Birkhäuser Boston, Boston, MA, 1999.
[25] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups.* London Mathematical Society Monographs 14, Academic Press, London, 1979.
[26] S. C. Power, *Limit Algebras: An Introduction to Subalgebras of C^*-Algebras.* Pitman Research Notes in Mathematics 278, Longman Scientific and Technical, Harlow, 1992.
[27] G. A. Reid, *A generalisation of W^*-algebras.* Pacific J. Math. **15**(1965), 1019–1026.
[28] M. A. Rieffel, *Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner.* In: Studies in Analysis, Adv. in Math. Suppl. Stud. 4, Academic Press, New York, 1979, pp. 43–82..
[29] K. Saitō and J. D. M. Wright, *All AW^*-factors are normal.* J. London Math. Soc.(2) **44**(1991), no. 1, 143–154.

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press

[30] S. Sakai, *C^*-algebras and W^*-algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete 60, Springer-Verlag, New York, 1971.
[31] M. Takesaki, *The structure of a von Neumann algebra with a homogeneous periodic state.* Acta Math. **131**(1973), 79-121.
[32] ———, *Theory of operator algebras. I.* Springer-Verlag, New York, 1979.
[33] J. Tomiyama, *Tensor products and projections of norm one in von Neumann algebras.* Lecture Notes, University of Copenhagen, 1970.
[34] J. D. M. Wright, *On some problems of Kaplansky in the theory of rings of operators.* Math. Z. **172**(1980), no. 2, 131–141.
[35] M. A. Youngson, *Completely contractive projections on C^*-algebras.* Quart. J. Math. Oxford **34**(1983), 507-511.
[36] H. H. Zettl, *A characterization of ternary rings of operators.* Adv. in Math. **48**(1983), no. 2, 117–143.

Department of Mathematics
Faculty of Science
University of Toyama
Toyama 930-8555
Japan
e-mail: hamana@sci.u-toyama.ac.jp

https://doi.org/10.4153/CJM-2006-042-0 Published online by Cambridge University Press