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GEOMETRIC CHARACTERIZATION OF INTERPOLATING VARIETIES FOR THE (FN)-SPACE A_p^0 OF ENTIRE FUNCTIONS

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ABSTRACT. A necessary and sufficient geometric characterization and a necessary and sufficient analytic characterization of interpolating varieties for the space of entire functions A_p^0 will be obtained in the paper, which as an application will also give a generalization of the well-known Pólya-Levinson density theorem.

1. **Introduction.** Let $p(z)$ be a weight (see Definition 2.1 below) and A_p^0 be the vector space of all the entire functions satisfying: $\sup_{z\in\mathbf{C}} |f(z)|e^{-\epsilon p(z)} < \infty$ for any $\epsilon > 0$. For instance, when $p(z) = |z|$, A_p^0 is the space of all entire functions of infraexponential type. Let $A_{p,n}$ denote the vector space of all entire functions on $\mathbf{C}$ satisfying: $\sup_{z\in\mathbf{C}} |f(z)|e^{-\frac{1}{n}p(z)} < \infty$. Then $A_p^0 = \bigcap_{n\in\mathbf{N}} A_{p,n}$ is the projective limit of the spaces $A_{p,n}$, where $\mathbf{N} := \{1, 2, \ldots\}$. Under its natural locally convex topology A_p^0 becomes a nuclear Fréchet (FN)-algebra. Algebras of this type appear naturally in complex and functional analysis. The present paper is concerned with the interpolation problem for A_p^0. That is, roughly speaking, find conditions for a given multiplicity variety $V = \{(z_k, m_k)\}$ such that for any doubly indexed complex sequence $\{a_{k,l}\}$ with convenient growth conditions there exists an entire function $f \in A_p^0$ satisfying $f_{k,l} := \frac{f^{(l)}(z_k)}{l!} = a_{k,l}$ for any $k \in \mathbf{N}$ and $0 \leq l \leq m_k - 1$. Interpolation problems are studied due to their applications to harmonic analysis. We are interested in finding necessary and sufficient conditions for a given multiplicity variety to be an interpolating variety for A_p^0, especially, getting purely geometric conditions that depend only on the distribution of points of a given variety. We note that in the case of the algebra A_p, *i.e.*, the algebra of all the entire functions satisfying: $\sup_{z\in\mathbf{C}} |f(z)|e^{-Bp(z)} < \infty$ for some $B > 0$, this problem has recently been completely solved by Berenstein and Li (see [BL1]). We also refer the reader to the papers [BL1], [BL2], [BL3], [BT], [S1], [S2], *etc.* for related results on interpolation theory in the space A_p.

It was shown in [V] that the very well-known density theorem of Pólya-Levinson ([L]) can be stated as follows. Let $V = \{\lambda_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers converging to infinity and satisfying

$$\operatorname{Re}\lambda_n > 0, \tag{1.1}$$

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$$|\operatorname{Im}\lambda_n| = o(|\lambda_n|), \qquad (1.2)$$
$$\frac{n}{\lambda_n} \longrightarrow 0$$

as $n \longrightarrow \infty$, and for some $c > 0$ and any $n, k \in \mathbf{N}$,

$$|\lambda_n - \lambda_k| \geq c|n-k|.$$

Then, V is an interpolating variety for the space A_p^0 with $p(z) = |z|$, *i.e.* the space of functions of infraexponential type (The conditions (1.1) and (1.2) are actually superfluous, see Theorem 5.1 and Remark 5.2). This theorem of Pólya-Levinson provides the motivation for this paper and leads naturally to ask when an arbitrarily given sequence $V = \{z_k\}$ in $\mathbf{C}$ or more general, a multiplicity variety $V = \{(z_k, m_k)\}$ is interpolating for the space A_p^0 with p being an arbitrary weight. We shall solve this problem in the present paper.

The paper is divided into four parts. In Section 2, we introduce the basic definitions and notations. In Section 3, we will give purely geometric characterization of interpolating varieties for A_p^0 which enables us to determine whether or not a given multiplicity variety is an interpolating variety by direct calculation. The geometric conditions will also yield necessary and sufficient analytic conditions as given in Section 4. Finally in Section 5, we obtain, as an application, a completely different proof of the above Pólya-Levinson density theorem with the conditions (1.1) and (1.2) removed.

Let us mention that our main result is the geometric characterization of interpolating varieties for A_p^0. The difficulty of this problem is that unlike the space A_p for which the geometric conditions follow form the analytic conditions (see [BL1]), the space A_p^0 is endowed with the rather complicated topology of the projective limits, which makes it hard to adopt the line of reasoning from [BL1] to the space A_p^0.

2. **Definitions and notation.** In this section, we recall and introduce some definitions and notation, which we need in the sequel.

DEFINITION 2.1. A subharmonic function $p: \mathbf{C} \longrightarrow [0, \infty)$ is called a *weight function* if it satisfies the following conditions:

$$\log(1 + |z|^2) = o\big(p(z)\big), \qquad (2.1)$$

$$p(z) = p(|z|) \qquad (2.2)$$

and

$$p(2z) = O\big(p(z)\big). \qquad (2.3)$$

Note that the subharmonicity of $p(z)$ and (2.2) imply that $p(e^r)$ is convex and $p(r)$ is increasing by Riesz's convexity theorem(see [BG, 4.4.27]). Also, (2.3) implies that there are constants $C_1 > 0, C_2 > 0$ such that

$$p(z + \zeta) \leq C_1 p(z) + C_2, \quad \text{whenever } |z - \zeta| \leq 1. \qquad (2.4)$$

DEFINITION 2.2. Let p be a weight. Then we set

$$A_p = \{f \in A(\mathbf{C}) : \exists A_f > 0, \sup_{z\in\mathbf{C}} |f(z)|e^{-A_f p(z)} < \infty\}$$

endowed with the inductive limit topology and

$$A_p^0 = \{f \in A(\mathbf{C}) : \forall\epsilon > 0, \sup_{z\in\mathbf{C}} |f(z)|e^{-\epsilon p(z)} < \infty\}.$$

endowed with the topology of the projective limit, where $A(\mathbf{C})$ is the set of all entire functions.

A basic example of such weight functions is $p(z) = |z|^\alpha$, $(\alpha > 0)$. As we said before, for $\alpha = 1, f \in A_p^0$ means that f is of *infraexponential* type.

Let $V = \{(z_k, m_k)\}_{k=1}^\infty$ be a multiplicity variety, that is, a sequence of points $\{z_k\}_{k=1}^\infty \subset \mathbf{C}$ with $|z_k| \longrightarrow \infty$, and a sequence of positive integers $\{m_k\}_{k=1}^\infty$ corresponding to the multiplicities of the points z_k. Associated to V, there is a unique closed ideal in $A(\mathbf{C})$,

$$I = I(V) = \{f \in A(\mathbf{C}) : f \text{ vanishes at } z_k \text{ with multiplicity } \geq m_k\}.$$

Two functions $g, h \in A(\mathbf{C})$ can be identified modulo I if and only if

$$\frac{g^{(l)}(z_k)}{l!} = \frac{h^{(l)}(z_k)}{l!} = a_{k,l}, \quad 0 \leq l < m_k,\ k = 1, 2, \ldots.$$

The quotient space $A(\mathbf{C})/I$ can be identified to the space of all sequences $\{a_{k,l}\}$ of complex numbers (*cf.* [BT]). We shall describe them as "analytic functions" on V, and denote that space by $A(V)$. The map

$$\rho: A(\mathbf{C}) \longrightarrow A(V), \quad \rho(g) = \left\{\frac{g^{(l)}(z_k)}{l!}\right\}$$

is called the *restriction map*.

DEFINITION 2.3. Let $V = \{(z_k, m_k)\}_{k=1}^\infty$ be a multiplicity variety. Then we define

$$A_p^0(V) := \left\{a = \{a_{k,l}\}_{k\in\mathbf{N}} : \forall\epsilon > 0, \sup_{k\in\mathbf{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| \exp\bigl(-\epsilon p(z_k)\bigr) < \infty\right\}.$$

It is easy to see that $\rho(A_p^0) \subset A_p^0(V)$, but in general, the space $A_p^0(V)$ is too large. We consider the following interpolation problem for A_p^0: Under what conditions does the map ρ map A_p^0 onto $A_p^0(V)$? That is, under what conditions is it true that for any doubly indexed sequence $\{a_{k,l}\} \in A_p^0(V)$ there exists an entire function $f \in A_p^0$ such that $f_{k,l} := \frac{f^{(l)}(z_k)}{l!} = a_{k,l}$ for any $k \in \mathbf{N}$ and $0 \leq l \leq m_k - 1$, *i.e.*, there is a $f \in A_p^0$ with prescribed first m_k Taylor coefficients at z_k for every $k \in \mathbf{N}$?

DEFINITION 2.4. If ρ maps A_p^0 onto $A_p^0(V)$, we will say that V is an *interpolating variety* for A_p^0.

We recall, by the way, that if ρ maps onto from A_p to the space

$$A_p(V) := \Big\{ a = \{a_{k,l}\}_{k\in\mathbf{N}} : \exists A_a > 0, \sup_{k\in\mathbf{N}} \sum_{l=0}^{m_k-1} |a_{k,l}| \exp\big(-A_a p(z_k)\big) < \infty \Big\},$$

then V is called an *interpolating variety* for A_p.

It turns out that the following valence function of V plays an essential role in interpolation theory.

DEFINITION 2.5. Let $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ be any multiplicity variety. The counting function of V, denoted by $n(r, V)$, is defined to be the number of points of V, counted with their multiplicities, in the disk $\{z : |z| \le r\}$. The valence function of V, denoted by $N(r, V)$, is defined to be:

$$\int_0^r \frac{n(t,V) - n(0,V)}{t}\,dt + n(0,V)\log r.$$

Similarly we can define the functions $n(r, z_0, V)$, $N(r, z_0, V)$ defined with respect to the disk $\{z : |z - z_0| \le r\}$.

3. **Geometric characterization of interpolating varieties for A_p^0.** In this section, we give purely geometric conditions necessary and sufficient for a given multiplicity variety to be interpolating for A_p^0.

THEOREM 3.1. *Let $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ be a multiplicity variety. Then V is an interpolating variety for A_p^0 if and only if*

$$N(r, V) = o\big(p(r)\big) \quad \text{as } r \longrightarrow \infty \tag{3.1}$$

and

$$N(|z_k|, z_k, V) = o\big(p(z_k)\big) \quad \text{as } k \longrightarrow \infty. \tag{3.2}$$

PROOF. Necessary conditions: Let

$$A_{p,n} = \{f \in A(\mathbf{C}) : \|f\|_{n,\infty} := \sup_{z\in\mathbf{C}} |f(z)| e^{-\frac{1}{n}p(z)} < \infty\}$$

for $n \in \mathbf{N}$. Then $A_p^0 = \bigcap_{n\in\mathbf{N}} A_{p,n}$ endowed with the natural projective limit topology is a nuclear Fréchet space. More precisely, let's consider the product space

$$\prod_{n\in\mathbf{N}} A_{p,n} = A_{p,1} \times A_{p,2} \times \cdots \times A_{p,n} \times \cdots$$

and the projections $g_{m,n}: A_{p,n} \longmapsto A_{p,m}$, whenever $m < n$. Let E be the subspace of $\prod_{n\in\mathbf{N}} A_{p,n}$ whose elements (f_n) satisfy the relation $f_m = g_{m,n}(f_n)$, whenever $m \le n$, that is, $E = \{(f, f, \ldots, f, \ldots) : f \in A_p^0\}$. We know that $E = \varprojlim g_{m,n}(A_{p,n})$ is the projective limit of the family $\{A_{p,n}, n \in \mathbf{N}\}$ with respect to the mappings $g_{m,n}$ (see *e.g.* [S] for basic

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properties of projective limit topology). The topology τ of E is the projective topology on E with respect to the family $\{A_{p,n}, \tau_n, \phi_n\}$, where τ_n denotes the topology of $A_{p,n}$, and ϕ_n the restriction to E of the projection of $\prod_{n\in\mathbf{N}} A_{p,n}$ onto $A_{p,n}$. Thus for any $(f) \in E$ and $f = \phi_n\big((f)\big)$, a τ neighborhood base of (f) is given by all the intersections $\bigcap_{n\in H} \phi_n^{-1}(U_n)$, where U_n is any neighborhood of f with respect to τ_n and H is any finite subset of N. Clearly, E is complete as a projective limit of complete locally convex spaces and is metrizable as a subset of a product of countable family of metrizable topological vector spaces (*cf.* [S]). In the same way we set

$$A_{p,n}(V) = \Big\{a = \{a_{k,l}\}_{\substack{k\in\mathbf{N}\\ 0\le l<m_k}} : \|a\|_{n,\infty} := \sup_{k\in\mathbf{N}}\Big(\sum_{l=0}^{m_k-1} |a_{k,l}|\Big)e^{-\frac{1}{n}p(z_k)} < \infty\Big\}.$$

Then $A_p^0(V) = \bigcap_{n\in\mathbf{N}} A_{p,n}(V)$. Let

$$E(V) = \{(a, a, \ldots, a, \ldots) : a \in A_p^0(V)\}.$$

The topology σ of $E(V)$ is the projective topology on $E(V)$ with respect to the family $\{A_{p,n}(V), \sigma_n, q_n\}$,where σ_n denotes the topology of $A_{p,n}(V)$, and q_n the restriction to $E(V)$ of the projection of $\prod_{n\in\mathbf{N}} A_{p,n}(V)$ onto $A_{p,n}(V)$. Thus for any $(a) \in E(V)$ and $a = q_n\big((a)\big)$, a σ neighborhood base of (a) is given by all the intersections $\bigcap_{n\in I} q_n^{-1}(V_n)$, where V_n is any neighborhood of a with respect to σ_n and I is any finite subset of $\mathbf{N}$. Also, $E(V)$ is complete and metrizable. Now consider the map $\varphi: E \longmapsto E(V)$ given by

$$\varphi\big((f)\big) = \Big((f_{k,l})_{\substack{k\in\mathbf{N}\\ 0\le l<m_k}}\Big)$$

with $f_{k,l} = \frac{f^{(l)}(z_k)}{l!} = a_{k,l}$. Here we have used the fact that $\varphi(E) \subset E(V)$. In fact by Cauchy's formula, for any $f \in A_p^0$, $\epsilon > 0$, there exists a $A_\epsilon > 0$ so that

$$|f_{k,l}| = \Big|\frac{1}{2\pi i}\int_{|z-z_k|=2} \frac{f(z)}{(z-z_k)^{l+1}}\,dz\Big| \le \frac{1}{2^l}A_\epsilon \exp\big(\epsilon \max_{|z-z_k|=2} p(z)\big)$$

and so that for any k:

$$\sum_{l=0}^{m_k-1} |f_{k,l}| \le \Big(\sum_{l=0}^{m_k-1} \frac{1}{2^l}\Big)A_\epsilon \exp\big(\epsilon \max_{|z-z_k|=2} p(z)\big) \le 2A_\epsilon \exp\big(\epsilon \max_{|z-z_k|=2} p(z)\big).$$

This implies that $(f_{k,l}) \in A_p^0(V)$ in view of (2.4). That is, $\varphi(E) \subset E(V)$. Obviously φ is linear and surjective since V is an interpolating variety for A_p^0. It is also obvious that φ is continuous. Thus, by Banach's homomorphism theorem ([H, p. 294]), φ is a strict morphism and maps every neighborhood of 0 in E onto a neighborhood of 0 in $E(V)$ ([H, p. 106]). For any fixed $n \in \mathbf{N}$, let

$$U_n = \text{ the unit ball of } A_{p,n} = \big\{f \in A(\mathbf{C}) : \|f\|_{n,\infty} = \sup_{z\in\mathbf{C}} |f(z)|e^{-\frac{1}{n}p(z)} \le 1\big\}.$$

Then

$$\phi_n^{-1}(U_n) = \{(f, f, \ldots, f, \ldots) : f \in A_p^0 \text{ and } \|f\|_{n,\infty} \le 1\}$$

is a neighborhood of 0 in E. By the above argument $\varphi\big(\phi_n^{-1}(U_n)\big)$ contains a neighborhood of 0 in $E(V)$ and so contains an open set of the form $\bigcap_{m\in I} q_m^{-1}(V_m)$, where V_m is a neighborhood of 0 with respect to σ_n and I is a finite subset of $\mathbf{N}$. It is readily seen that there exist a $t_n \in \mathbf{N}$ and $\delta_n > 0$ such that $\varphi\big(\phi_n^{-1}(U_n)\big)$ contains the set

$$V_n := \Big\{(a, a, \ldots, a, \ldots) : a = (a_{k,l}) \in A_p^0(V), \sup_{k\in\mathbf{N}}\Big(\sum_{l=0}^{m_k-1} |a_{k,l}|\Big)e^{-\frac{1}{t_n}p(z_k)} \le \delta_n\Big\}.$$

This implies that there exist entire functions $f_{n,k} \in A_p^0 (k \in \mathbf{N})$ with $\|f_{n,k}\|_{n,\infty} \le 1$ satisfying the following property:

$$(3.3) \qquad (f_{n,k})_{i,l} = 0, \quad i \in \mathbf{N},\ 0 \le l \le m_i - 1, \quad \text{except that} \quad (f_{n,k})_{k,m_k-1} = \delta_n e^{\frac{1}{t_n}p(z_k)}.$$

Also there exist entire functions $g_{n,k} \in A_p^0 (k \in \mathbf{N})$ with $\|g_{n,k}\|_{n,\infty} \le 1$ satisfying the following property:

$$(3.4) \qquad (g_{n,k})_{i,l} = 0, \quad i \in \mathbf{N},\ 0 \le l \le m_i - 1 \quad \text{except that} \quad (g_{n,k})_{k,0} = \delta_n e^{\frac{1}{t_n}p(z_k)}.$$

Define

$$(3.5) \qquad F_n(z) = \sum_{k=1}^{\infty}(z - z_k)\frac{g_{n,k}f_{n,k}}{e^{\frac{2}{t_n}p(z_k)}} := \sum_{k=1}^{\infty} F_k^*(z).$$

Recall that $\log(1 + |z|^2) = o\big(p(z)\big)$. For any $m \in \mathbf{N}$, there is a $D_m > 0$ such that for $z \in \mathbf{C}, |z| \le D_m e^{\frac{1}{m}p(z)}$. Thus, in view of the fact that $\|f_{n,k}\|_{n,\infty} \le 1, \|g_{n,k}\|_{n,\infty} \le 1$, we deduce that

$$(3.6) \qquad \begin{aligned} |F_k^*(z)| &\le (|z| + |z_k|)\frac{e^{\frac{2}{n}p(z)}}{e^{\frac{2}{t_n}p(z_k)}} \\ &\le (D_n e^{\frac{1}{n}p(z)} + D_{t_n} e^{\frac{1}{t_n}p(z_k)})\frac{e^{\frac{2}{n}p(z)}}{e^{\frac{2}{t_n}p(z_k)}} \\ &\le D_n' e^{\frac{3}{n}p(z)} e^{-\frac{1}{t_n}p(z_k)}. \end{aligned}$$

where D_n' is a constant depending on n. Let $A, B > 0$ be two integers such that $p(2z) \le Ap(z) + B$ (*cf.* (2.3)). Denote, for k fixed, $d_k = \inf_{j\ne k}\{|z_j - z_k|\}$, $\epsilon_k = \min\{1, d_k\}$, and $B_k = \{z : |z - z_k| < \epsilon_k\}$. For the integer $N := 2A(t_n + 1)$, similar to the proof of (3.3), one can obtain entire functions $f_{N,k} \in A_p^0 (k \in \mathbf{N})$ with $\|f_{N,k}\|_{N,\infty} \le 1$ satisfying the following property:

$$(3.7) \qquad (f_{N,k})_{i,l} = 0, \quad i \in \mathbf{N},\ 0 \le l \le m_i - 1 \quad \text{except that} \quad (f_{N,k})_{k,m_k-1} = \delta_N.$$

Set $G_{N,k} = \frac{f_{N,k}}{(z-z_k)^{m_k-1}}$. Then on $|z - z_k| = 1$ and so in $|z - z_k| \le 1$, by the maximum modulus theorem,

$$|G_{N,k}(z)| \le \max_{|z-z_k|=1} |f_{N,k}(z)| \le \max_{|z-z_k|=1} e^{\frac{1}{N}p(z)}.$$

We then have $|G_{N,k}(z)| \leq e^{\frac{B}{N}} e^{\frac{A}{N}p(z_k)}$ for $|z - z_k| \leq 1$ (we can assume that $|z_k| > 1$). Let

$$H_{N,k}(z) = \frac{G_{N,k}(z) - G_{N,k}(z_k)}{2e^{\frac{B}{N}} e^{\frac{A}{N}p(z_k)}}.$$

Then in $|z - z_k| \leq 1$, $|H_{N,k}(z)| \leq 1$ and $H_{N,k}(z_k) = 0$. By the Schwarz lemma ([BG]) and noting that $z_j, j \neq k$ is a zero of $G_{N,k}(z)$, we know that if $|z_j - z_k| \leq 1$, then in view of (3.7),

$$|z_j - z_k| \geq |H_{N,k}(z_j)| = \left| \frac{G_{N,k}(z_k)}{2e^{\frac{B}{N}} e^{\frac{A}{N}p(z_k)}} \right| = \frac{1}{2}\delta_N e^{\frac{-B}{N}} e^{\frac{-A}{N}p(z_k)}.$$

This implies that

$$\epsilon_k \geq \frac{1}{2}\delta_N e^{-\frac{B}{N}} e^{-\frac{A}{N}p(z_k)}.$$

Now, by assuming that $\delta_N < 1$, we deduce that for some integer $k_0 > 0$,

$$\begin{aligned}
\sum_{k=k_0}^{\infty} e^{-\frac{1}{t_n}p(z_k)} &= \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{\epsilon_k^2} \int_{B_k} e^{-\frac{1}{t_n}p(z_k)}\, dx\, dy \\
&\leq \left(\frac{2}{\delta_N}\right)^2 e^{\frac{2B}{N}} \sum_{k=k_0}^{\infty} \int_{B_k} e^{(\frac{2A}{N} - \frac{1}{t_n})p(z_k)}\, dx\, dy \\
&= \left(\frac{2}{\delta_N}\right)^2 e^{\frac{2B}{N}} \sum_{k=k_0}^{\infty} \int_{B_k} e^{-\frac{1}{(t_n+1)t_n}p(z_k)}\, dx\, dy \\
&\leq \frac{4}{\delta_N^2} e^{\frac{2B}{N}} e^{\frac{B}{A(t_n+1)t_n}} \int\int_{\mathbf{C}} e^{-\frac{1}{A(t_n+1)t_n}p(z)}\, dx\, dy := D_n'' \leq +\infty
\end{aligned}$$

by the property (2.1) of p. Combining this fact with (3.6), we see that the series (3.5) is uniformly convergent in compact sets and so $F_n(z)$ is an entire function. Moreover $|F_n(z)| \leq D_n' D_n'' e^{\frac{3}{n}p(z)}$, $(F_n)_{k,m_k} = \delta_n^2$, and $V \subset Z(F_n) := \{z : F_n(z) = 0\}$ by (3.4) and (3.5). Apply now Jensen's formula (see *e.g.* [BG]) to deduce that

$$\log|(F_n)_{k,m_k}| + N\big(|z_k|, |z_k|, Z(F_n)\big) = \frac{1}{2\pi} \int_0^{2\pi} \log|F_n(z_k + |z_k|e^{i\theta})|\, d\theta,$$

and so,

$$\begin{aligned}
N(|z_k|, |z_k|, V) &\leq N\big(|z_k|, |z_k|, Z(F_n)\big) \\
&\leq -2\log\delta_n + \log \max_{|z-z_k|=|z_k|} |F_n(z)| \\
&\leq -2\log\delta_n + \log D_n' + \log D_n'' + \frac{3}{n}p(2z_k) \\
&\leq -2\log\delta_n + \log D_n' + \log D_n'' + \frac{3A}{n}p(z_k) + \frac{3B}{n}.
\end{aligned}$$

This implies that $N(|z_k|, |z_k|, V) = o\big(p(z_k)\big)$ as $k \longrightarrow \infty$. Finally, noting that $V \subset Z(F_n)$, we conclude that, again by Jensen's formula,

$$\begin{aligned} N(r,V) &\leq N\Big(r,\frac{1}{F_n}\Big) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|F_n(re^{i\theta})|\,d\theta + \log\left|\frac{F_n^{(t)}(0)}{t!}\right|^{-1} \\ &\leq \log\max_{|z|=r}|F_n(z)| + \log\left|\frac{F_n^{(t)}(0)}{t!}\right|^{-1}, \end{aligned}$$

where $t := n(0, \frac{1}{F_n})$. Again by the fact that $|F_n(z)| \leq D'_n D''_n e^{\frac{3}{n}p(z)}$, it follows that $N(r,V) = o\big(p(z)\big)$, as $r \longrightarrow \infty$. This completes the proof of the necessity.

Sufficient conditions: Recall that $p(e^r)$ is a convex function of r and $p(r)$ is also increasing (see Definition 2.1). By [BMT, 1.7 and 1.8], for any continuous and increasing function $\omega(r)$, if $\omega(r)$ satisfies (2.1) and (2.3) and $\omega(e^r)$ is convex, then for any function $g\colon [0,\infty) \longrightarrow [0,\infty)$ satisfying $g(r) = o\big(\omega(r)\big)$ as $r \longrightarrow \infty$, there exists an increasing function $q(r)\colon [0,\infty) \longrightarrow [0,\infty)$ such that $q(r)$ also satisfies (2.1) and (2.3) and $q(e^r)$ is convex, and moreover $g(r) = o\big(q(r)\big)$, $q(r) = o\big(\omega(r)\big)$ as $r \longrightarrow \infty$. Therefore by this result, we deduce from (3.1) and (3.2) that:

$$N(r,V) = o\big(q(r)\big), \tag{3.8}$$

and

$$N(|z_k|, z_k, V) = o\big(q(|z_k|)\big), \tag{3.9}$$

where $q(r)\colon [0,\infty) \longrightarrow [0,\infty)$ is a function which satisfies (2.1) and (2.3). Moreover, $q(e^r)$ is convex and $q(r) = o\big(p(r)\big)$ as $r \longrightarrow \infty$. By the fact that $f \circ u$ is subharmonic if f is convex increasing and u is subharmonic [BG, 4.4.18], we deduce that $q(|z|) = q(e^{\ln|z|})$ is subharmonic. This shows that $q(r)$ is also a weight. Using Theorem 4.1 in [BL1], (3.8) and (3.9) imply that there are two functions $F \in A_q$ and $G \in A_q$ such that

$$V = Z(F,G) := \{z \in \mathbf{C} : F(z) = G(z) = 0\} \tag{3.10}$$

and

$$|G_{k,m_k}| \geq \epsilon_0 \exp\big(-Cq(z_k)\big). \tag{3.11}$$

for some ϵ_0, $C > 0$ and V is interpolating for A_q. Now the sufficiency can be finished by using the fact that V is also interpolating for $A_{q'}$, where $q' \geq q$ is any bigger weight or by the following direct argument. In fact, by [BT, Corollary 2], (3.10) and (3.11) imply that for some ϵ, $c > 0$, each $z_k \in V$ is contained in a bounded component S_k of

$$S_q(F,G,;\epsilon,c) := \big\{z \in \mathbf{C} : |F(z)| + |G(z)| \leq \epsilon \exp\big(-cq(z)\big)\big\},$$

whose diameter is at most one and no two distinct points of V lie in the same component of $S_q(F,G,;\epsilon,c)$. For any $a=\{a_{k,l}\}_{\substack{k\in\mathbf{N}\\ 0\le l\le m_k-1}}\in A_p^0(V)$, we may then define an analytic function $\tilde{\lambda}\colon S_q(F,G;\epsilon,c)\longmapsto\mathbf{C}$ by:

$$\tilde{\lambda}(z)=\begin{cases}\sum_{l=0}^{m_k-1}a_{k,l}(z-z_k)^l, & \text{if } z\in S_k;\\ 0, & \text{if } z\in S_q(F,G;\epsilon,c)\setminus\bigcup_{k\in\mathbf{N}}S_k.\end{cases}$$

For each $z\in S_k$, $m\in\mathbf{N}$, since $|z-z_k|\le 1$ we deduce that

$$|\tilde{\lambda}(z)|\le\sum_{l=0}^{m_k-1}|a_{k,l}|\,|z-z_k|^l\le\sum_{l=0}^{m_k-1}|a_{k,l}|\le A_m e^{\frac{1}{m}}p(z_k)\le A_m e^{\frac{M}{m}p(z)}$$

by the definition of $A_p^0(V)$ and the property (2.3), where M and A_m are two positive constants and A_m depends only on m. From this, we can easily verify that for $\forall n\in\mathbf{N}$:

$$\sup\{|\tilde{\lambda}(z)|e^{-\frac{1}{n}p(z)}:z\in S_q(F,G;\epsilon,c)\}<\infty.$$

Therefore $\tilde{\lambda}$ satisfies all the hypothesis of Proposition 2 in [MT]. Consequently there exist $\lambda\in A_p^0$, ϵ_1, c_1, with $0<\epsilon_1<\epsilon$, $c_1>c$ and u,v analytic in $S_q(F,G,;\epsilon,c)$, such that for all $z\in S_q(F,G,;\epsilon_1,c_1)$,

$$\lambda(z)=\tilde{\lambda}(z)+uF+vG.$$

and thus for $k\in\mathbf{N}$, $0\le l<m_k-1$, $\lambda_{k,l}=\tilde{\lambda}_{k,l}=a_{k,l}$.

The proof of Theorem 3.1 is thus complete. ∎

REMARK 3.2. From the proof of the sufficiency of Theorem 3.1, we see that any interpolating variety for A_p^0 is also interpolating for A_q, where q is some weight satisfying that $q(r)=o\big(p(r)\big)$, and thus for A_p.

REMARK 3.3. Thanks to Dr. A. Russakovskii, we have become aware of [GR] where they studied functions of fixed type. Their result,though related, does not apply either A_p or A_p^0.

4. **Analytic characterization of interpolating varieties for A_p^0.**

As a corollary of the last section, we give in this section the following analytic conditions necessary and sufficient for V to be interpolating for A_p^0.

THEOREM 4.1. *Let $V=\{(z_k,m_k)\}$ be a multiplicity variety. Then V is an interpolating variety for A_p^0 if and only if there exists an entire function $f\in A_p^0$ such that $V\subset Z(f)$ and as $k\longrightarrow\infty$,*

$$|f_{k,m_k}|^{-1}\le\exp\big(o\{p(z_k)\}\big).\tag{4.1}$$

PROOF. Necessary conditions: If V is an interpolating variety for A_p^0, then by Theorem 3.1, (3.1), (3.2) hold. By the proof of the sufficiency of Theorem 3.1, we know that there is a $f\in A_q\subset A_p^0$ so that $V\subset Z(f)$ and (4.1) holds (see (3.10) and (3.11)).

Sufficient conditions: By the fact that $V \subset Z(f)$, we have that

$$\begin{aligned}
N(r,V) &\le N\Big(r,\frac{1}{f}\Big)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta + \log\left|\frac{t!}{f^{(t)}(0)}\right|^{-1}\\
&\le \log\max_{|z|=r}|f(z)| + \log\left|\frac{t!}{f^{(t)}(0)}\right|^{-1} = o\big(p(r)\big)
\end{aligned}$$

since $f \in A_p^0$, where $t = n(0,\frac{1}{f})$. By (4.1), we have that

$$\begin{aligned}
N(|z_k|,z_k,V) &\le N\Big(|z_k|,z_k,\frac{1}{f}\Big)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|f(z_k+|z_k|e^{i\theta})|\,d\theta + \log\frac{1}{|f_{k,m_k}|}\\
&\le \log\max_{|z|=2|z_k|}|f(z)| + o\big(p(z_k)\big)\\
&= o\big(p(2z_k)\big) + o\big(p(z_k)\big) = o\big(p(z_k)\big).
\end{aligned}$$

Therefore (3.1), (3.2) are satisfied. It follows then that V is an interpolating variety for A_p^0 by Theorem 3.1. ■

As we said in Remark 3.2, any interpolating variety for A_p^0 is also interpolating for A_p. But the converse is not true. The interesting variety $\mathcal{Z}^2 := \{m+in\}_{m,n\in\mathbf{N}}$, the lattice in $\mathbf{C}$, gives such an example. Note that $\mathcal{Z}^2$ is the zero set of the Weierstrass function σ, which is the analogue of the function $\sin \pi z$ for the lattice. It is easy to check that $\mathcal{Z}^2$ satisfies that $N(r,\mathcal{Z}^2) \le Ar^2$ and for $z_k \in \mathcal{Z}^2$, $N(|z_k|,z_k,\mathcal{Z}^2) \le A(|z_k|^2)$, where $A > 0$ is a constant. Thus by Theorem 4.1 in [BL1] (*cf.* the proof of the sufficiency of Theorem 3.1), $\mathcal{Z}^2$ is an interpolating variety for A_ϱ, where $\varrho(z) := |z|^2$. However, $\mathcal{Z}^2$ does not satisfy (3.1). Thus $\mathcal{Z}^2$ is too "dense" to be interpolating for A_ϱ^0. It becomes natural to ask when a subvariety $V = \{z_k\}$ with $z_k \longrightarrow \infty$ of $\mathcal{Z}^2$ is interpolating for A_ϱ^0. The answer is provided by the following

PROPOSITION 4.2. *Let $V = \{z_k\}$ be a subvariety of $\mathcal{Z}^2$. Then V is an interpolating variety for A_ϱ^0 if and only if as $r \longrightarrow \infty$,*

$$n(r,V) = o\{\varrho(r)\}. \tag{4.2}$$

To prove the theorem, we need the following crucial lemma which is implicit in the proof of Theorem 4.1 in our paper [BL1].

LEMMA 4.3. *Let V be a multiplicity variety satisfying*

$$N(r,V) = O\{q(r)\}$$

for a weight $q(z)$. Then there exists another variety V_* such that

$$V \cap V_* = \emptyset, \tag{4.3}$$

$$\tilde{\tilde{V}} := V \cup V_* = Z(G) := \{z : G(z) = 0\} \tag{4.4}$$

for an entire function $G \in A_q$,

$$n(r, \tilde{\tilde{V}}) = O\big(n(2r, V)\big), \tag{4.5}$$

and

$$N(|z_k|, z_k, V_*) \leq \mathrm{A}n(B|z_k|, V) \tag{4.6}$$

for some constants $A, B > 0$.

PROOF OF PROPOSITION 4.2. The necessary condition follows from Theorem 3.1. We next prove the sufficiency. First, it is easy to check that (4.2) is equivalent to

$$N(r, V) = o(r^2). \tag{4.7}$$

Thus, by [BMT,1.7 and 1.8], we deduce that there exists a weight $q(r)$ with $q(r) = o(r^2)$ such that $N(r, V) = o\{q(r)\}$ as $r \longrightarrow \infty$ (*cf.* the proof of (3.8)). By Lemma 4.3, there exist a variety V_* and an entire function $G \in A_q$ satisfying (4.3)–(4.6). There is no loss of generality to assume that $G(0) = 1$. For any $0 < \epsilon < 1$ and $r > 0$, set

$$R(z) = \frac{\big(-(r + r\epsilon)\big)^n}{a_1 a_2 \cdots a_n} \prod_{k=1}^{n} \frac{(r + r\epsilon)(z - a_k)}{(r + r\epsilon)^2 - \bar{a_k} z},$$

where $a_1, a_2, \ldots, a_n$ are the zeros of $G(z)$ in $|z| < r + r\epsilon$, and

$$S(z) = G(z)/R(z).$$

Recall that if a function $f(z)$ is regular in $|z| \leq R$ without zeros and $f(0) = 1$ then

$$\log |f(z)| \geq -\frac{2r}{R - r} \log M(R,f) \quad (|z| \leq r < R),$$

where $M(R,f) := \max_{|z|=R}\{|f(z)|\}$(see [Le, p. 19]). Applying this result to $S(z)$ in $|z| \leq r + r\epsilon$, we have that in $|z| \leq r + \frac{1}{2}r\epsilon$,

$$\begin{aligned} \log |S(z)| &\geq \frac{-2(r + \frac{1}{2}r\epsilon)}{r + r\epsilon - (r + \frac{1}{2}r\epsilon)} \left\{ \log M(r + r\epsilon, G) - \log \frac{(r + r\epsilon)^n}{|a_1 a_2 \cdots a_n|} \right\} \\ &\geq -\frac{8}{\epsilon} \log M(r + r\epsilon, G). \end{aligned}$$

Since $G \in A_q \subset A_\varrho^0$, there is a constant $C > 0$ such that

$$\log M(r + r\epsilon, G) \leq C\epsilon^2 \varrho(r).$$

We then have that

$$\log S(z) \geq -8C\epsilon r^2.$$

Next using the well known Cartan's theorem [Le], we have

$$\prod_{k=1}^{n} |z - a_k| > \left(\frac{\frac{1}{8} r\epsilon}{e}\right)^n$$

for any z outside a union of circles with the sum of radii $\leq \frac{1}{4} r\epsilon$. Therefore for such z in $\{|z| \leq r + \frac{1}{2} r\epsilon\}$, we have

$$|R(z)| \geq \frac{(r + r\epsilon)^n}{(r + r\epsilon)^n} \frac{(r + r\epsilon)^n}{2^n (r + r\epsilon)^{2n}} \left(\frac{r\epsilon}{8e}\right)^n = \left(\frac{r\epsilon}{16e(r + r\epsilon)}\right)^n.$$

Also by (4.5) and (4.2),

$$n = n(r + r\epsilon, \tilde{\tilde{V}}) = O\{n(2r, V)\} = o(r^2).$$

We thus deduce that for large r,

$$\log |R(z)| \geq \epsilon r^2 \log \frac{\epsilon}{16e(1 + \epsilon)}$$

and so

$$\log |G(z)| \geq \left(-8C\epsilon + \epsilon \log \frac{\epsilon}{16e(1 + \epsilon)}\right) r^2 := -g(\epsilon) r^2 \tag{4.8}$$

outside a union of circles with the sum of radii $\leq \frac{1}{4} r\epsilon$. Here $g(\epsilon) \geq 0$ is defined obviously from the above equation. Therefore for any large k, (4.8) holds for $r = |z_k|$ and $|z| \leq |z_k| + \frac{1}{2}|z_k|\epsilon$ outside a family of excluded circles with the sum of radii $\leq \frac{1}{4}|z_k|\epsilon$. Hence, one can get a $\rho_k (0 < \rho_k < \epsilon|z_k|)$ such that on $|z - z_k| = \rho_k$, (4.8) holds for $r = |z_k|$. Applying Jensen's formula to G and in view of (4.3) and (4.4), we have

$$\begin{aligned}
\log |G'(z_k)| &= \frac{1}{2\pi} \int_0^{2\pi} \log |G(z_k + \rho_k e^{i\theta})\, d\theta - N(\rho_k, z_k, \tilde{\tilde{V}}) \\
&\geq -g(\epsilon)|z_k|^2 - N(\rho_k, z_k, \tilde{\tilde{V}}).
\end{aligned}$$

But

$$\begin{aligned}
N(\rho_k, z_k, \tilde{\tilde{V}}) &\leq N(\rho_k, z_k, V) + N(\rho_k, z_k, V_*); \\
N(\rho_k, z_k, V) &= \int_0^{\rho_k} \frac{n(t, z_k, V) - n(0, z_k, V)}{t}\, dt + n(0, z_k, V) \log \rho_k \\
&\leq \int_1^{\rho_k} \frac{t^2}{t}\, dt + \log \rho_k \\
&= \frac{1}{2}{\rho_k}^2 - \frac{1}{2} + \log \rho_k \leq \frac{1}{2}\epsilon^2 |z_k|^2 - \frac{1}{2} + \log |z_k|;
\end{aligned}$$

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and by (4.6) and (4.2),

$$N(\rho_k, z_k, V_*) \le \mathrm{An}(B|z_k|, V) = o\{|z_k|^2\}.$$

We thus obtain that

$$\log|G'(z_k)| \ge -g(\epsilon)|z_k|^2 - \frac{1}{2}\epsilon^2|z_k|^2 + \frac{1}{2} - \log|z_k| - o\{|z_k|^2\}.$$

Since $g(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$, we finally have that

$$\log|G'(z_k)| \ge -o\{|z_k|^2\}.$$

This together with (4.7) yields that $\mathcal{Z}^2$ is an interpolating variety for A_ϱ^0 by Theorem 4.1. ■

If we replace $\mathcal{Z}^2$ by an arbitrary interpolating variety V for A_p, the analogue of Proposition 4.2 might not hold. The obvious obstruction is on the separation between distinct points in V. To be interpolating for A_p, there must be constants $\delta > 0, c > 0$ such that for distinct points $z, w \in V$, $|z-w| \ge \delta \exp\bigl(-cp(z)\bigr)$, while the separation condition for A_p^0 is that for any $\epsilon > 0$, there is a ϵ_1 such that $|z-w| \ge \epsilon_1 \exp\bigl(-\epsilon p(z)\bigr)$. Therefore the latter needs a stronger restriction than the former. It is not clear to us at this moment whether or not Proposition 4.2 with the extra hypothesis on the separation still remains true for any variety which is interpolating for A_p.

5. **An application of interpolation theory for A_p^0.** As an application of Theorem 3.1 and Theorem 4.1, we have the following

THEOREM 5.1. *Let $V = \{\lambda_n\}_{n=1}^\infty$ be a sequence of non-zero complex numbers converging to infinity and satisfying that for some $\alpha > 0$:*

$$\frac{n^{\frac{1}{\alpha}}}{\lambda_n} \longrightarrow 0 \tag{5.1}$$

as $n \longrightarrow \infty$, and for some $c > 0$ and any $n, k \in \mathbf{N}$,

$$|\lambda_n - \lambda_k|^\alpha \ge c|n-k|. \tag{5.2}$$

Then there exists an entire function $F(z)$ vanishing at $z = \lambda_n$ and satisfying that for all $\epsilon > 0$

$$|F(re^{i\theta})| = O(e^{\epsilon r^\alpha}) \tag{5.3}$$

as $r \longrightarrow \infty$ and

$$\frac{1}{|F'(\lambda_n)|} = O(e^{\epsilon|\lambda_n|^\alpha}) \tag{5.4}$$

as $n \longrightarrow \infty$.

We see that from (5.3) and (5.4) or from the following proof, $V = \{\lambda_n\}_{n=1}^\infty$ is also an interpolating variety for A_p^0 with $p(z) = |z|^\alpha$.

REMARK 5.2. Letting $\alpha = 1$ in Theorem 5.1, one obtains the classical Pólya-Levinson density theorem([L]), which is very useful to study problems such as analytic continuation and domains of holomorphy for Dirichlet series $\sum_{n=1}^{n=\infty} a_n e^{-\lambda_n z}$ with Fabry gap $\frac{n}{\lambda_n} \longrightarrow 0$. One actually only needs the fact that V is an interpolating variety(*cf.* [BKS]). When $\frac{n}{\lambda_n} \longrightarrow D > 0$ instead of (5.1), the detailed proof was given in [L]. The essential point used in [L] is that in the case $D > 0$, the zeroes of the proposed function by him $F_1(z) = \prod_{n=1}^{n=\infty}(1 - \frac{z^2}{\lambda_n^2})$ are along an asymptotic direction, *i.e.*, satisfy conditions (1.1) and (1.2). However, the conditions (1.1) and (1.2) are consequences of the limit $\frac{n}{\lambda_n} \longrightarrow D$ when $D \neq 0$ but not when $D = 0$. It was pointed out by one of us [V] in his thesis that the same proof for the case $D = 0$ in [L] breaks down without these two conditions. By modifying Levinson's idea, he also gave another completely different, but rather lengthy, proof for the special case $\alpha = 1$ of Theorem 5.1, constructing explicitly an entire function f satisfying (5.3), and (5.4) for every zero of f.

PROOF OF THEOREM 5.1. For any $\epsilon > 0$, by (5.1), $n < \epsilon\lambda_n^\alpha$ for large n. Thus for any $t \geq 0$ large, $n(t, V) = \#\{n : |\lambda_n| \leq t\} \leq \epsilon t^\alpha$. Thus

$$N(r, V) = \int_0^t \frac{n(t, V) - n(0, V)}{t}\, dt + n(0, V)\log r \leq \frac{1}{\alpha}\epsilon r^\alpha + \log r.$$

This implies that

$$N(r, V) = o(r^\alpha). \tag{5.5}$$

Also by (5.2), we have

$$\begin{aligned} N(|\lambda_k|, \lambda_k, V) &= \int_0^{|\lambda_k|} \frac{n(t, \lambda_k, V) - n(0, \lambda_k, V)}{t}\, dt + n(0, \lambda_k, V)\log|\lambda_k| \\ &= \sum_{\nu=1}^{s} \log \frac{|\lambda_k|}{|\lambda_{k_\nu} - \lambda_k|} + \log|\lambda_k| \\ &= \log \prod_{\nu=1}^{s} \frac{|\lambda_k|}{|\lambda_{k_\nu} - \lambda_k|} + \log|\lambda_k| \\ &\leq \log\left\{\left(\frac{|\lambda_k|}{c^{\frac{1}{\alpha}}}\right)^s \prod_{\nu=1}^{s} \frac{1}{|k_\nu - k|^{\frac{1}{\alpha}}}\right\} + \log|\lambda_k|, \end{aligned}$$

where $s = n(|\lambda_k|, \lambda_k, V) - n(0, \lambda_k, V) = n(|\lambda_k|, \lambda_k, V) - 1$ since $|\lambda_{k_\nu} - \lambda_k|^\alpha \geq c$ for $k_\nu \neq k$ and the λ_{k_ν}'s are the points of V in $\{z : |z - \lambda_k| \leq |\lambda_k|, z \neq \lambda_k\}$. Obviously in the product $\prod_{\nu=1}^{s} \frac{1}{|k_\nu - k|}$ at most two terms $|k_\nu - k|$ can be equal. Let $[x]$ denote the integral part of x. Then we deduce that

$$\prod_{\nu=1}^{s} |k_\nu - k| \geq \left(\left[\frac{1}{2}(s-1)\right]!\right)^2,$$

since there are at least $[\frac{s-1}{2}]$ consecutive integers $|k_\nu - k|$ occurring in the product. Thus we have that

$$N(|\lambda_k|, \lambda_k, V) \leq \log\left\{\left(\frac{|\lambda_k|}{c^{\frac{1}{\alpha}}}\right)^s \frac{1}{\left([\frac{1}{2}(s-1)]!\right)^{\frac{2}{\alpha}}}\right\} + \log|\lambda_k|.$$

By using Stirling's formula in the form $n! \geq e^{n \ln n - n}$ for $[\frac{1}{2}(s-1)]!$, we obtain

$$\begin{aligned} N(|\lambda_k|, \lambda_k, V) &\leq \log\left\{\left(\frac{|\lambda_k|}{c^{\frac{1}{\alpha}}}\right)^s e^{\frac{3}{\alpha}s - \frac{s}{\alpha}\ln s}\right\} + \log|\lambda_k| \\ &= \left(\frac{3}{\alpha} + \log\frac{1}{c^{\frac{1}{\alpha}}}\right)s + s\log\frac{|\lambda_k|}{s^{\frac{1}{\alpha}}} + \log|\lambda_k|. \end{aligned} \tag{5.6}$$

Let $0 < \epsilon < e^{-1}$, then for $k >> 1$ we have that

$$s \leq n(|\lambda_k|, \lambda_k, V) \leq n(2|\lambda_k|, V) \leq \epsilon|\lambda_k|^\alpha. \tag{5.7}$$

The only term of (5.6) that needs estimation is the middle one. Let

$$G(x) = x\log\frac{|\lambda_k|}{x^{\frac{1}{\alpha}}}, \quad 0 < x \leq \epsilon|\lambda_k|^\alpha.$$

Then, in this range, we have

$$\begin{aligned} G'(x) &= \log|\lambda_k| - \frac{1}{\alpha}\log x - \frac{1}{\alpha} \\ &\geq \log|\lambda_k| - \frac{1}{\alpha}\log(\epsilon|\lambda_k|^\alpha) - \frac{1}{\alpha} \\ &= \frac{1}{\alpha}\left(\log\frac{1}{\epsilon} - 1\right) > \frac{1}{\alpha}(\log e - 1) = 0 \end{aligned}$$

so that G is increasing and

$$G(x) \leq \frac{1}{\alpha}\epsilon\log\left(\frac{1}{\epsilon}\right)|\lambda_k|^\alpha.$$

Thus, by (5.6) and (5.7), we deduce that

$$N(|\lambda_k|, \lambda_k, V) \leq \left(\frac{3}{\alpha} + \log\frac{1}{c^{\frac{1}{\alpha}}}\right)\epsilon|\lambda_k|^\alpha + \frac{1}{\alpha}\epsilon\log\left(\frac{1}{\epsilon}\right)|\lambda_k|^\alpha + \log|\lambda_k|.$$

It follows that

$$N(|\lambda_k|, \lambda_k, V) = o\{|\lambda_k|^\alpha\}$$

as $k \longrightarrow \infty$. This together with (5.5) yields, by Theorem 3.1, that V is an interpolating variety for $A^0_{|z|^\alpha}$. We conclude the proof by using Theorem 4.1. ∎

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