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OSCILLATION PROPERTIES OF WEAKLY TIME DEPENDENT HYPERBOLIC EQUATIONS

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ABSTRACT. A Sturmian comparison theorem is established for a pair of linear hyperbolic differential equations. While the equations may be time dependent (in the sense of not allowing a separation of variables), a measure of the strength of such time dependence enters into the hypotheses of the theorem.

While it is possible to generalize the classical Sturm comparison theorem to hyperbolic equations [4], substantive difficulties are encountered in generalizing such results to equations having different principal parts. In [5] Travis overcomes such difficulties by restricting consideration to equations allowing a separation of variables (see also [2]). Specifically, the basic Theorem 2 of [5] deals with functions $u(x, t)$ and $v(x, t)$ which are, respectively, non-trivial solutions of

$$u_{tt} = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) - cu \tag{1}$$

$$v_{tt} = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv) - Cv \tag{2}$$

in a cylindrical domain $D \subset \mathbb{R}^{n+1}$. The separation of variables follows from the assumptions that the coefficients a_{ij}, A_{ij}, c, and C are functions of $\underline{x} = (x_1, \ldots, x_n)$ only and that D is a cylinder of the form $G \times (t_0, t_1)$, where G is a smooth, bounded domain in $\mathbb{R}^n$. Thus the problem considered in [5] is "time independent" in the sense that neither the coefficients nor the cross-sections

$$G_t = \{(\underline{x}, s) \in D : s = t\}$$

depend on t.

The purpose of this note is to show that Travis's technique can be generalized to establish Sturmian comparison theorems for time dependent problems, as long as such time dependence is sufficiently "weak", in a sense to

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be made precise below. We shall allow the coefficients of (1) and (2) to be functions of $\underline{x}$ and t, continuous in a cylindrical domain $D = G \times (t_0, t_1)$. We also assume that the right sides of (1) and (2) define formally selfadjoint elliptic operators

(3) $$l_t[u] \equiv -\sum_{i,j=1}^{n} D_i(a_{ij}(\underline{x}, t)D_j u) + c(\underline{x}, t)u$$

(4) $$L_t[v] \equiv -\sum_{i,j=1}^{n} D_i(A_{ij}(\underline{x}, t)D_j v) + C(\underline{x}, t)v$$

to which classical variational theory can be applied and that the solutions u and v of

(1) $$u_{tt} + l_t[u] = 0$$

(2) $$v_{tt} + L_t[v] = 0$$

satisfy boundary conditions of the form

(5) $$\frac{\partial u}{\partial \nu} + \sigma(\underline{x}, t)u = 0$$

(6) $$\frac{\partial v}{\partial \nu} + \tau(\underline{x}, t)v = 0$$

for $\underline{x} \in \partial G$, $t_0 < t < t_1$, where $\partial u/\partial \nu$ and $\partial v/\partial \nu$ are transverse derivatives defined by

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \nu}{\partial x_j}$$

$$\frac{\partial v}{\partial \nu} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \nu}{\partial x_j}$$

and $\partial \nu/\partial x_j$ denotes the cosine of the angle between the exterior normal to ∂G and the positive x_j-axis. Our comparison theorem will involve the functions $\lambda_0(t)$ and $\mu_0(t)$ which are, respectively, the smallest eigenvalues of

(7) $$l_t[u] = \lambda u \quad \text{in} \quad G; \qquad \frac{\partial u}{\partial \nu} + \sigma u \quad \text{on} \quad \partial G$$

(8) $$L_t[v] = \mu v \quad \text{in} \quad G; \qquad \frac{\partial u}{\partial \nu} + \tau v \quad \text{on} \quad \partial G.$$

Denoting by $\varphi(\underline{x}, t)$ and $\psi(\underline{x}, t)$ the positive normalized eigenfunctions of l_t and L_t corresponding to the eigenvalues $\lambda_0(t)$ and $\mu_0(t)$, respectively, we introduce the functions

(9) $$\alpha(t) = \sup_{x \in G} \left| \frac{\psi_t}{\psi} \right| \quad \text{and} \quad \beta(t) = \sup_{x \in G} \left| \frac{\psi_{tt}}{\psi} \right|$$

as measures of the weakness of the time dependence of (2), (6). (Note that in case (2), (6) allows a separation of variables we have $\alpha(t)\equiv 0$ and $\beta(t)\equiv 0$ for $t_0 \le t \le t_1$).

For the sake of simplicity, we first consider the case where (1), (5) is time independent, so that φ is independent of t and λ_0 is constant. However, we make no such assumptions about (2), (6).

THEOREM. *Suppose $u(x, t)$ is a solution of the time independent problem* (1), (5) *in a cylinder $D = G\times(t_0, t_1)$ satisfying $u(\underline{x}, t_0) = u(x, t_1) = 0$. If*

(i)
$$0 < \sum_{i,j=1}^{n} a_{ij}(\underline{x})\xi_i\xi_j \le \sum_{i,j=1}^{n} A_{ij}(\underline{x}, t)\xi_i\xi_j$$

(ii)
$$c(\underline{x}) + 3\alpha^2(t) + \beta(t) \le C(\underline{x}, t)$$

for all $(\underline{x}, t)\in D$ and all non-zero n-tuples $(\xi_1, \ldots, \xi_n)$, and if

(iii)
$$\sigma(\underline{x}) \le \tau(\underline{x}, t) \le +\infty$$

for all $(\underline{x}, t)\in \partial G\times(t_0, t_1)$, then every solution $v(\underline{x}, t)$ of (2), (6) *has a zero in* $G\times(t_0, t_1]$.

Proof. Defining

$$U(t) = \langle u(\underline{x}, t), \varphi(\underline{x}, t)\rangle = \int_G u(\underline{x}, t)\varphi(\underline{x}, t)\, d\underline{x}$$

$$V(t) = \langle v(\underline{x}, t), \psi(\underline{x}, t)\rangle = \int_G v(\underline{x}, t)\psi(\underline{x}, t)\, d\underline{x}$$

we have

$$\frac{dV}{dt} = \langle v_t, \psi\rangle + \langle v, \psi_t\rangle$$

and

$$\frac{d^2V}{dt^2} = \langle v_{tt}, \psi\rangle + 2\langle v_t, \psi_t\rangle + \langle v, \psi_{tt}\rangle.$$

From (9) we obtain

(10)
$$\left|\frac{dV}{dt} - \langle v_t, \psi\rangle\right| \le \alpha(t)\,|V|$$

and

(11)
$$\left|\frac{d^2V}{dt^2} - \langle v_{tt}, \psi\rangle\right| \le 2\alpha(t)\,|\langle v_t, \psi\rangle| + \beta(t)\,|V|.$$

Since (10) implies that

$$|\langle v_t, \psi\rangle| \le \alpha\,|V| + \left|\frac{dV}{dt}\right|$$

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we have from (11)

$$\left|\frac{d^2V}{dt^2} - \langle v_{tt}, \psi\rangle\right| \le 2\alpha(t)\left|\frac{dV}{dt}\right| + (2\alpha^2(t)+\beta(t))\,|V|$$

and that there exist continuous functions $F(t)$, $H(t)$ satisfying

$$|F(t)| \le 2\alpha(t);\ |H(t)| \le 2\alpha^2(t)+\beta(t)$$

for which

$$\frac{d^2V}{dt^2} + F(t)\frac{dV}{dt} + H(t)V = \langle v_{tt}, \psi\rangle.$$

Recalling the definition of ψ, we have

$$\langle v_{tt}, \psi\rangle = \langle -L_t v, \psi\rangle = -\langle v, L_t\psi\rangle = -\mu_0(t)V$$

so that $V(t)$ satisfies

$$\frac{d^2V}{dt^2} + F(t)\frac{dV}{dt} + (H(t)+\mu_0(t))V = 0. \tag{12}$$

In order to show that $v(x, t)$ has a zero in D, it is sufficient to show that $V(t)$ has a zero in $(t_0, t_1]$. To that end, we note that $U(t)$ satisfies

$$\frac{d^2U}{dt^2} + \lambda_0(t)U = 0, \tag{13}$$

and $U(t_0) = U(t_1) = 0$. Applying a known Sturmian comparison theorem [3, Theorem 2.1] to (12) and (13), we see that

$$\mu_0 + H \ge \frac{F^2}{4} + \lambda_0 \tag{14}$$

is a sufficient condition for $V(t)$ to have a zero in $(t_0, t_1]$. Clearly (14) is satisfied if $\mu_0 \ge \lambda_0 + 3\alpha^2 + \beta$. However, from classical variational theory for elliptic eigenvalue problems [1] we have

$$\lambda_0 = \inf_{\gamma\in\Gamma}\langle l_t\gamma, \gamma\rangle = \inf_{\gamma\in\Gamma}\int_G\left[\sum_{i,j=1}^n a_{ij}D_i\gamma D_i\gamma D_j\gamma + c\gamma^2\right]dx$$

$$\le \int_G\left[\sum_{i,j=1}^n A_{ij}D_i\psi D_j\psi + c\psi^2\right]d\underline{x} = c - C + \mu_0(t)$$

where Γ is the appropriate "admissible" class for the variational characterization of eigenvalues of L_t. Thus if $C - c \ge 3\alpha^2 + \beta$ it follows that (14) is satisfied, and this completes the proof.

Remarks and Examples. 1. In case (1), (5) is not time independent $U(t)$ satisfies a differential equation of the form

$$\frac{d^2U}{dt^2} + f(t)\frac{dU}{dt} + (h(t)+\lambda_0(t))U = 0 \tag{13'}$$

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where $|f|$ and $|g|$ are bounded by measures of time dependence analogus to $\alpha(t)$ and $\beta(t)$. In this case we can apply [3; Theorem 2.2] to (12) and (13′), with (14) replaced by

$$\mu_0 + G \geq \frac{f' - F'}{2} + \frac{F^2}{4} + \lambda_0 + h. \tag{14'}$$

For practical purposes, however, it is most convenient to compare (2), (6) with a time independent problem (1), (5).

2. As observed in [5], the techniques of Theorem 1 also apply in the case of equations of the form

$$(m(\underset{\sim}{x}, t)u_t)_t + l_t[u] = 0$$

and

$$(M(\underset{\sim}{x}, t)v_t)_t + L_t[v] = 0$$

if $0 < M \leq m$ in D.

3. As an example of a time dependent problem (2), (6) for which $\alpha(t)$ and $\beta(t)$ can be estimated directly, consider

$$v_{tt} - \left(\frac{1}{1 + t\cos x} v_x\right)_x + (k - t\cos x)v = 0$$

$$v(0, t) = v(\pi, t) = 0.$$

Here we may choose

$$l_t[v] = -\left(\frac{1}{1 + t\cos x} v_x\right)_x - (1 + t\cos x)v$$

so that

$$\psi(x, t) = k \sin(x + t \sin x)$$

for an appropriate normalizing constant k. This expression for ψ readily yields

$$\left|\frac{\psi_t}{\psi}\right| \leq \left|\frac{\sin x}{\sin(x + t\sin x)}\right| \quad \text{and} \quad \left|\frac{\psi_{tt}}{\psi}\right| \leq 1,$$

yielding the choice $\beta(t) \equiv 1$. For $0 \leq t < 1$ the argument $x + t \sin x$ is strictly increasing for $0 \leq x \leq \pi$ and we can use L'hospital's rule to evaluate

$$\lim_{x \to 0^+} \frac{\sin x}{\sin(x + t\sin x)} = \frac{1}{1+t} = \lim_{x \to \pi^-} \frac{\sin x}{\sin(x + t \sin x)}.$$

This shows that $\alpha(t)$ is bounded for $0 \leq t \leq 1$, and a specific bound can be found by finding the relative maxima of $\dfrac{\sin x}{\sin(x + t\sin x)}$ for $0 \leq x \leq \pi$.

4. With examples such as the one above we are able to estimate α and β by solving the elliptic boundary value problem associated with (2), (6). It would be of interest, however, to find techniques for determining *a priori* bounds for α and β which do not require an explicit determination of $\psi(\underset{\sim}{x}, t)$.

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