ON A RESULT IN YOUNG'S QUANTITATIVE SUBSTITUTIONAL ANALYSIS

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In (**4**) Young investigates the representation in terms of his exact seminormal units $\varpi_{\mu\nu}$ of substitutional expressions which are unchanged on premultiplication by any permutation of a given set of consecutive letters, or which are changed in sign on premultiplication by an odd permutation of those letters. He illustrates his results by deducing the forms of the matrices representing the positive and negative symmetric groups on a set of consecutive letters.

We shall derive Young's results on the positive and negative symmetric groups in a slightly different form, and in terms of his *orthogonal* representation of the symmetric group $\mathscr{S}_n$ of degree n corresponding to a partition (λ) of n. The standard tableaux of shape (λ) will be denoted by $S_1, S_2, \ldots, S_f$.

Theorem 1. *Suppose $\pi(p, q)$ is the positive symmetric group on the letters p, $p+1, \ldots, q$, where $1 \leqq p<q \leqq n$, and let $U = [u_{ij}]$ be the matrix representing $\pi(p, q)$ in Young's orthogonal representation corresponding to partition (λ). Then $u_{ij} = 0$, unless S_j is obtainable from S_i by a permutation involving the letters p, $p+1, \ldots, q$ only, in which case*

$$u_{ij} = \sqrt{\omega_i\omega_j} \tag{1}$$

where

$$\omega_i = \prod_{p \leqq \alpha<\beta \leqq q} (1+\rho_{\alpha\beta}^{(i)}), \tag{2}$$

$\rho_{\alpha\beta}^{(i)}$ denoting the reciprocal of the axial distance from β to α in S_i (see (**2**), *p.* 41).

Proof. The following proof, although essentially similar to that in (**4**), differs in detail.

We shall denote by σ a permutation involving the letters $p, p+1, \ldots, q$ only, and by $D(\sigma)$ the matrix representing σ. Thus

$$U = \sum_{\sigma} D(\sigma). \tag{3}$$

Clearly, U is symmetric; for, using a prime to indicate transposition, we have, since $D(\sigma)$ is orthogonal,

$$U' = \sum_{\sigma} D'(\sigma) = \sum_{\sigma} D(\sigma^{-1}) = U.$$

The tableaux $S_1, S_2, \ldots, S_f$ can be divided into subsets, each consisting of tableaux obtainable from one another by permutations involving the letters $p, p+1, \ldots, q$ only. We shall suppose the tableaux arranged so that all the members of any subset come together. From (3), Theorem V, if τ is a transposition of a pair of consecutive letters in the set $p, p+1, \ldots, q$, $D(\tau)$ can be expressed as a direct sum

$$D(\tau) = D_1(\tau) \dotplus D_2(\tau) \dotplus \ldots \dotplus D_v(\tau), \tag{4}$$

where each term corresponds to a subset; the symbol $\dotplus$ indicates a direct sum, and v is the number of subsets. Since $\pi(p, q)$ is generated by the transpositions just described it follows that

$$U = U_1 \dotplus U_2 \dotplus \ldots \dotplus U_v. \tag{5}$$

Thus $u_{ij} = 0$ if S_i and S_j belong to different subsets, verifying the first part of the theorem.

For the remainder of the proof we consider the diagonal submatrices U_r of U individually. We may without loss of generality consider $U_1 = [u_{ij}]$, of order $g \times g$, say. We use the fact that, for all transpositions τ of pairs of consecutive letters in the set $p, p+1, \ldots, q$,

$$\tau\pi(p, q) = \pi(p, q).$$

Hence

$$D_1(\tau)U_1 = U_1. \tag{6}$$

Now suppose S_i and S_j are obtainable from each other by interchanging a pair of consecutive letters $x-1$ and x, where $p < x \leqq q$. Then, letting $\tau = (x-1, x)$, we deduce from (6) and (3), Theorem V, that

$$[1-\rho_{x-1,x}^{(i)}]u_{i\mu} = \sqrt{[1-(\rho_{x-1,x}^{(i)})^2]}u_{j\mu}$$

for all μ, or, since $\rho_{x-1,x}^{(i)} = -\rho_{x-1,x}^{(j)}$,

$$u_{i\mu} = \sqrt{\frac{1+\rho_{x-1,x}^{(i)}}{1+\rho_{x-1,x}^{(j)}}}\, u_{j\mu}. \tag{7}$$

If S_h and S_k are any two tableaux in the subset we can construct a sequence of standard tableaux $S_h, S_{i_1}, \ldots, S_{i_t}, S_k$ in the subset, each of which is obtained from its predecessor by interchanging two consecutive letters. We shall construct this sequence, in fact, for the *distorted* tableaux ((4), p. 454), $T_1, T_2, \ldots, T_g$ obtained by omitting the letters $1, \ldots, p-1$ and $q+1, \ldots, n$ in $S_1, S_2, \ldots, S_g$. Clearly q lies at the right-hand end of a row and the foot of a column in both T_h and T_k. Suppose a letter x occupies in T_h the position occupied by q in T_k. If $x<q$, we replace it by q by first interchanging x and $x+1$ (which must lie in a different row and column from x), then (if $x+1<q$) interchanging $x+1$ and $x+2$, and so on, until q finally occupies the same position as in T_k. We then use the same process to move $q-1$ to its final position, and so on. A relationship of the form (7) arises from each consecutive pair of tableaux in the sequence $S_h, S_{i_1} \ldots, S_k$, and this enables us to express $u_{h\mu}$ uniquely as a multiple of $u_{k\mu}$.

Two cases must now be considered separately. First, suppose each tableau in the subset contains two or more of the letters $p, p+1, \ldots, q$ in one column. Then at least one of them, S_j, say, contains a pair of *consecutive* letters in one column—e.g. the tableau obtained by writing the letters $p, p+1, \ldots, q$ in order down the successive columns of the corresponding distorted shape. If τ denotes the interchange of the two consecutive letters, Theorem V of (3) shows that $D_1(\tau)$ contains an element -1 in position (j, j). Substituting in (6) we deduce that $u_{j\mu} = 0$ for all μ. If now S_i is any other tableau in the subset, we have just seen that $u_{i\mu}$ can be expressed as a multiple of $u_{j\mu}$, and so must be zero. That is,

$$U_1 = 0.$$

This is consistent with the theorem, since, for each tableau in the subset, at least one zero factor will occur on the right in (2).

Consider now those tableaux S_i for which no two of the letters $p, p+1, \ldots, q$ lie in the same column. We shall call such tableaux, and the corresponding subsets, *admissible*. If S_i is admissible none of the factors $1+\rho_{\alpha\beta}^{(i)}$ in (2) can be zero. If S_i and S_j are obtainable from each other by interchanging $k-1$ and k we have, since $\rho_{k-1,\mu}^{(i)} = \rho_{k\mu}^{(j)}$ and $\rho_{k\mu}^{(i)} = \rho_{k-1,\mu}^{(j)}$ for all $\mu \neq k-1$ or k,

$$\frac{\omega_j}{\omega_i} = \frac{1+\rho_{k-1,k}^{(j)}}{1+\rho_{k-1,k}^{(i)}}. \tag{8}$$

Hence from (7) we must have, for all i and all μ,

$$u_{i\mu} = c_\mu \sqrt{\omega_i \omega_\mu},$$

where c_μ is a constant. The symmetry of U_1 implies that $c_1 = c_2 = \ldots = c_g = c$, say. Hence

$$u_{ij} = c\sqrt{\omega_i \omega_j}. \tag{9}$$

It remains to prove that $c = 1$. We use the fact that

$$[\pi(p, q)]^2 = (q-p+1)!\pi(p, q)$$

so that

$$U_1^2 = (q-p+1)!U_1. \tag{10}$$

Now from (9) the element in position (i, j) in U_1^2 is

$$c^2 u_{ij} \sum_{k=1}^{g} \omega_k.$$

Hence either $c = 0$ or

$$c \sum_{k=1}^{g} \omega_k = (q-p+1)! \tag{11}$$

Now c cannot be zero. To show this we note first that for a given value of $q-p+1$ the number of admissible subsets is independent of p. For if we replace each of the letters $p, p+1, \ldots, q$ in an admissible tableau by p, the result is an admissible tableau with repetitions in the sense of (**1**). All tableaux of the same subset lead to the same tableau with repetitions, while different subsets lead to different tableaux. Thus, by (**1**), the number $N(q-p+1)$ of admissible subsets is the coefficient of $x_1 x_2 \ldots x_{p-1} x_p^{q-p+1} x_{q+1} \ldots x_n$ in the expansion of the bialternant $h_{(\lambda)}(x_1, x_2, \ldots, x_n)$ and this is clearly independent of p. Now for fixed $q-p+1$ all the expressions $\pi(p, q)$ can be transformed into one another,

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and consequently tr U is independent of p. In the special case $p = 1$ the admissible subsets each consist of a single tableau, in which the letters $1, 2, \ldots, q$ all lie in row 1. For the jth such subset, $D_j(\sigma) = 1$ for all σ, and so

$$U_j = (q-p+1)!.$$

Thus tr $U = N(q-p+1) \,.\, (q-p+1)!$ for all p, and this is possible only if c is non-zero for each admissible subset.

We now prove by induction that for the admissible subsets

$$\sum_{k=1}^{q} \omega_k = (q-p+1)! \tag{12}$$

This is easily verified if $q-p+1 = 2$. Suppose it has been proved for $q-p+1 = 2, 3, \ldots, s-1$. We then consider the case $q-p+1 = s$. Suppose the row-lengths of the distorted tableaux T_i are $s_1, s_2, \ldots, s_t$, and denote by α_j the axial distance from the right-hand node of row 1 to the right-hand node of row j, so that, in particular, $\alpha_1 = 0$. In T_i, q must lie at the right-hand end of some row—say, row j. If we denote by Γ^r the product of the factors in ω_i due to the axial distances from q to the letters in row r, we have

$$\Gamma^r = \prod_{u=0}^{s_r-1} \left(1 + \frac{1}{u+\alpha_r-\alpha_j}\right),$$

the first factor being omitted if $r = j$. It is easy to show then that

$$\Gamma^r = \begin{cases} 1 + \dfrac{s_r}{\alpha_r - \alpha_j} & \text{if } r \neq j \\ s_j & \text{if } r = j. \end{cases} \tag{13}$$

If we now denote by T_i^* the distorted tableau obtained by omitting q from T_i, and by ω_i^* the value defined by (2), with q replaced by $q-1$, for T_i^*, we have

$$\omega_i = \omega_i^* \prod_{r=1}^{t} \Gamma^r.$$

By the induction hypothesis

$$\sum_i \omega_i^* = (s-1)!,$$

the summation being over all tableaux T_i for which q lies in row j. Hence, summing over j,

$$\sum_{i=1}^{g} \omega_i = K(s-1)! \tag{14}$$

where

$$K = \sum_{j=1}^{t} s_j \prod_{\substack{r=1 \\ r \neq j}}^{t} \left(1 + \frac{s_r}{\alpha_r - \alpha_j}\right). \tag{15}$$

Expanding K as a polynomial in $s_1, s_2, \ldots, s_t$,

$$K = (s_1 + s_2 + \ldots + s_t) + \sum_{1 \leq j_1 < j_2 \ldots < j_\kappa \leq t} a(j_1, j_2, \ldots, j_\kappa) s_{j_1} s_{j_2} \ldots s_{j_\kappa} \tag{16}$$

where, on writing β_1 for α_{j_1}, etc.,

$$a(j_1, j_2, \ldots, j_\kappa) = \sum_{i=1}^{\kappa} \prod_{\substack{r=1 \\ r \neq i}}^{\kappa} \frac{1}{\beta_r - \beta_i}$$

$$= \frac{1}{\Delta(\beta_1, \beta_2, \ldots, \beta_\kappa)} [\Delta(\beta_2, \beta_3, \ldots, \beta_\kappa) - \Delta(\beta_1, \beta_3, \ldots, \beta_\kappa) + \ldots + (-1)^{\kappa-1} \Delta(\beta_1, \beta_2, \ldots, \beta_{\kappa-1})] \tag{17}$$

where

$$\Delta(\beta_1, \beta_2, \ldots, \beta_\kappa) = \prod_{1 \leqq i < j \leqq \kappa} (\beta_j - \beta_i).$$

Since the expression in square brackets in (17) is the Laplace expansion on column 1 of the determinant

$$\begin{vmatrix} 1 & 1 & \beta_1 & \beta_1^2 \ldots \beta_1^{\kappa-2} \\ 1 & 1 & \beta_2 & \beta_2^2 \ldots \beta_2^{\kappa-2} \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1 & 1 & \beta_\kappa & \beta_\kappa^2 \ldots \beta_\kappa^{\kappa-2} \end{vmatrix}$$

it follows that

$$a(j_1, j_2, \ldots, j_\kappa) = 0,$$

and so, by (16),

$$K = s_1 + s_2 + \ldots + s_t = s.$$

Substituting in (14) gives the required result (12). Thus $c = 1$ in (11), and

$$u_{ij} = \sqrt{\omega_i \omega_j}$$

as required.

This completes the proof of the Theorem.

Theorem 2. *Suppose $\pi'(p, q)$ is the negative symmetric group on the letters p, $p+1, \ldots, q$, where $1 \leqq p < q \leqq n$, and let $V = [v_{ij}]$ be the matrix representing $\pi'(p, q)$ in Young's orthogonal representation corresponding to partition (λ). Then $v_{ij} = 0$ unless S_j is obtained from S_i by a permutation involving the letters $p, p+1, \ldots, q$ only, in which case*

$$v_{ij} = (-1)^{i+j} \sqrt{\omega_i' \omega_j'} \tag{18}$$

where

$$\omega_i' = \prod_{p \leqq \alpha < \beta \leqq q} (1 - \rho_{\alpha\beta}^{(i)}). \tag{19}$$

The proof is very similar to that of Theorem 1. Note that now the subsets for which two or more of the letters $p, p+1, \ldots, q$ lie in the same *row* lead to zero diagonal submatrices in V.

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