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FINITE DIMENSIONAL REPRESENTATIONS OF $U_t(\mathrm{sl}(2))$ AT ROOTS OF UNITY

XIAO JIE

ABSTRACT. All finite dimensional indecomposable representations of $U_t(\mathrm{sl}(2))$ at roots of 1 are determined.

1. **Introduction.** Quantum group or quantum enveloping algebra $U_q(\mathfrak{g})$ is a certain (Hopf algebra) deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a complex simple finite-dimensional Lie algebra $\mathfrak{g}$, introduced by Drinfeld [Dr1], [Dr2], Jimbo [Ji] and Kulish-Reshetikhin [Kr] in their study of the quantum Yang-Baxter equation. The simplest and most important example is that of the simple Lie algebra $\mathrm{sl}(2)$. An important problem is to describe finite dimensional representations of the algebra $U_q(\mathfrak{g})$. Now it is well-known that: (1) when q is not a root of 1 the (finite dimensional) representation theory of $U_q(\mathfrak{g})$ is essentially the same as that of $U(\mathfrak{g})$, namely representations of $U_q(\mathfrak{g})$ are deformations of representations of $U(\mathfrak{g})$, so that the latter are obtained as $q \longrightarrow 1$ [Lu], [Ro]; (2) when q is a root of 1, then the situation changes dramatically and finite dimensional representations of $U_q(\mathfrak{g})$ are not completely reducible in general, however, all simple modules of $U_q\big(\mathrm{sl}(2)\big)$ are classified (for example, see [DCK]); (3) a very profound application of the representation theory of $U_q\big(\mathrm{sl}(2)\big)$ is that Reshetikhin *et al.* (see [KiR], [RT1], [RT2], [KM]) construct some new topological invariants of compact oriented 3-manifolds and of framed links in those manifolds. The aim of the present note is to determine all finite dimensional restrictable modules of $U_q\big(\mathrm{sl}(2)\big)$. Thanks to [RT2], all projective and injective objects in the category of finite dimensional restrictable modules of $U_q\big(\mathrm{sl}(2)\big)$ are implicitly given. Therefore, we can apply the BGG philosophy [BGG] to reduce the problem of classifying all restrictable modules of $U_q\big(\mathrm{sl}(2)\big)$ into that of modules over a finite dimensional algebra. In our situation, it is not difficult to see that the corresponding algebra is just one of tame representation type. It is interesting to us that this gives us a close relationship between the restrictable representations of $U_q\big(\mathrm{sl}(2)\big)$ and those of some tame quivers and their trivial extensions (see [Ri], [Ha]). The approach in this note not only allows us to construct all finite dimensional restrictable modules of $U_q\big(\mathrm{sl}(2)\big)$, but also to arrive at a position to understand their category completely; for example, we provide an Auslander-Reiten formula to compute $\mathrm{Ext}^1_{U_q(\mathrm{sl}(2))}(\, , \,)$.

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2. **Simples and projectives.** We mainly adopt the notations as in [RT2]. For given $q \in \mathbb{C}$, the quantum group $U_q\big(\mathrm{sl}\,(2)\big)$ is the associative algebra over the cyclotomic field $\mathbb{Q}(q^{1/2})$ with 4 generators K, K^{-1}, X, Y subject to the following relations:

$$XY - YX = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}} \tag{2.1.1}$$

$$XK = q^{-1/2}KX, \quad YK = q^{1/2}KY, \quad KK^{-1} = K^{-1}K = 1. \tag{2.1.2}$$

Since we want to consider the restrictable representations of $U_q\big(\mathrm{sl}\,(2)\big)$ at q a root of 1, let $t = \exp(\pi\sqrt{-1}m/2r)$, where m, r are coprime integers with odd m and $m \geq 1$, $r \geq 2$, $q = t^4$. We define the quotient algebra $U_t\big(\mathrm{sl}\,(2)\big)$ of $U_q\big(\mathrm{sl}\,(2)\big) \otimes \mathbb{Q}(t)$ over the cyclotomic field $\mathbb{Q}(t)$ with generators K, K^{-1}, X, Y subject to the following relations

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}} \tag{2.1.3}$$

$$XK = t^{-2}KX, \quad YK = t^2KY, \quad KK^{-1} = K^{-1}K = 1. \tag{2.1.4}$$

$$K^{4r} = 1, \quad X^r = Y^r = 0 \tag{2.1.5}$$

where $q^r = t^{4r} = 1$. A representation of $U_q\big(\mathrm{sl}\,(2)\big)$ over $\mathbb{Q}(t)$ is called *restrictable* if it satisfies the relations (2.1.3), (2.1.4) and (2.1.5). The algebra $U_t\big(\mathrm{sl}\,(2)\big)$ also has the structure of a Hopf algebra; the action of comultiplication Δ, counit ε and the antipode μ are given by the following formulas:

$$\begin{gathered}\Delta(X) = X \otimes K + K^{-1} \otimes X, \quad \Delta(Y) = Y \otimes K + K^{-1} \otimes Y, \quad \Delta(K) = K \otimes K \\ \mu(K) = K^{-1}, \quad \mu(X) = -t^2X, \quad \mu(Y) = -t^{-2}Y \\ \varepsilon(K) = 1, \quad \varepsilon(X) = \varepsilon(Y) = 0.\end{gathered} \tag{2.1.6}$$

However we don't need to use the Hopf structure.

The following notation is often used in consideration for representations of quantum groups

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}} = \frac{\sin(\pi mn/r)}{\sin(\pi m/r)} \quad \text{and } [n]! = [n][n-1]\cdots[1].$$

For $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ and $0 \leq i \leq r-1$, we define a $(i+1)$-dimensional $U_t\big(\mathrm{sl}\,(2)\big)$-module $V^i(\alpha)$ as follows. This module has a basis $e_0^i(\alpha), e_1^i(\alpha), \ldots, e_i^i(\alpha)$ and the actions of the generators are given by the following rules

$$\begin{gathered}Ke_n^i(\alpha) = at^{i-2n}e_n^i(\alpha) \\ Xe_n^i(\alpha) = \alpha^2[n][i+1-n]e_{n-1}^i(\alpha) \\ Ye_n^i(\alpha) = e_{n+1}^i(\alpha)\end{gathered} \tag{2.1.7}$$

where $n = 0, 1, \ldots, i$ and $e_{-1}^i(\alpha) = e_{i+1}^i(\alpha) = 0$. It is easy to see that $V^i(\alpha)$ for $0 \leq i \leq r-1$ is a simple $U_t\big(\mathrm{sl}\,(2)\big)$-module. It is well-known now that all

$\bigl\{V^i(\alpha) \mid \alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ and $0 \le i \le r-1\bigr\}$ form a complete non-redundant list of simple $U_t\bigl(\mathrm{sl}(2)\bigr)$-modules over $\mathbb{Q}(t)$.

Let U_t^+ and U_t^- be the subalgebras of $U_t\bigl(\mathrm{sl}(2)\bigr)$ generated by K, X and K, Y respectively. We also have the Verma modules $W^j(\alpha)$ and $\tilde{W}^j(\alpha)$ which are free over U_t^- and U_t^+ respectively, where $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ and $0 \le j \le r-1$; so $\dim_{\mathbb{Q}(t)} W^j(\alpha) = \dim_{\mathbb{Q}(t)} \tilde{W}^j(\alpha) = r$. They are given by the following rules:

$$
W^j(\alpha):\quad
\begin{aligned}
Ke_n^j(\alpha) &= \alpha t^{j-2n} e_n^j(\alpha)\\
Xe_n^j(\alpha) &= \alpha^2 [n][j+1-n] e_{n-1}^j(\alpha)\\
Ye_n^j(\alpha) &= e_{n+1}^j(\alpha)
\end{aligned}
\tag{2.1.8}
$$

$$
\tilde{W}^j(\alpha):\quad
\begin{aligned}
Kf_n^j(\alpha) &= \alpha t^{-j+2n} f_n^j(\alpha)\\
Xf_n^j(\alpha) &= f_{n+1}^j(\alpha)\\
Yf_n^j(\alpha) &= \alpha^2 [n][j+1-n] f_{n-1}^j(\alpha)
\end{aligned}
\tag{2.1.9}
$$

where $n = 0, 1, \ldots, r-1$ and $f_r^j(\alpha) = e_r^j(\alpha) = 0$, $e_{-1}^j(\alpha) = f_{-1}^j(\alpha) = 0$.

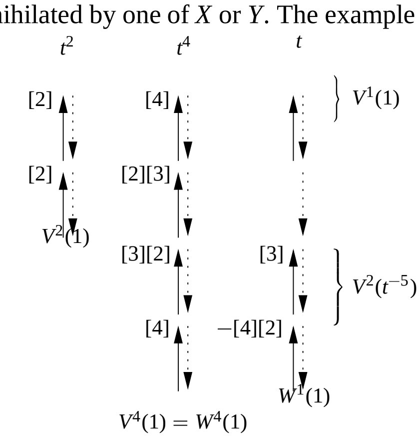
We have the following extensions:

$$
\begin{gathered}
0 \longrightarrow V^{r-j-2}(\alpha t^{-r}) \longrightarrow W^j(\alpha) \longrightarrow V^j(\alpha) \longrightarrow 0\\
0 \longrightarrow V^{r-j-2}(\alpha t^{r}) \longrightarrow \tilde{W}^j(\alpha) \longrightarrow V^j(\alpha) \longrightarrow 0.
\end{gathered}
\tag{2.1.10}
$$

It is obvious that

$$
W^{r-1}(\alpha) \simeq \tilde{W}^{r-1}(\alpha) \simeq V^{r-1}(\alpha).
$$

It is convenient to use a graphical representation for the structure of $U_t\bigl(\mathrm{sl}(2)\bigr)$-modules. Every vertex stands for a vector from our chosen basis; arrows and dotted ones show the actions of X and Y respectively; more precisely, an arrow may be labeled by a scalar corresponding to the action of X or Y and a vertex labeled by its weight (*i.e.* eigenvalue for K); the absence of arrows coming out of a vertex means that the corresponding vector is annihilated by one of X or Y. The example below is for $r = 5$.



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LEMMA 2.1.11. *$V^{r-1}(\alpha) \simeq W^{r-1}(\alpha)$ is a projective $U_t\big(\mathrm{sl}(2)\big)$-module.*

PROOF. Assume M to be a finite dimensional $U_t\big(\mathrm{sl}(2)\big)$-module and a surjective morphism $F\colon M \longrightarrow V^{r-1}(\alpha)$. Then M has a decomposition $M = \oplus_{\lambda\in\Lambda} M_\lambda$, where $M_\lambda = \{x \in M \mid (K-\lambda)^n x = 0 \text{ for some } n > 0\}$ (we don't need to assume that M has a decomposition into a direct sum of its weight spaces under K-action). Take $x \in M$ such that $Fx = e_0^{r-1}(\alpha)$, then $x \in M_{\alpha t^{r-1}}$ by (2.1.8). Because the maximal common factor of $(T-\alpha t^{r-1})^n$ and $T^{4r}-1$ is $T-\alpha t^{r-1}$, $Kx = \alpha t^{r-1}x$. Take $y = Y^{r-1}x$; then $Fy = Y^{r-1}Fx = Y^{r-1}e_0^{r-1}(\alpha) = e_{r-1}^{r-1}(\alpha)$, $Yy = 0$ and $Ky = KY^{r-1}x = \alpha t^{-(r-1)}y$ by (2.1.4); also $Ke_{r-1}^{r-1}(\alpha) = \alpha t^{-(r-1)}e_{r-1}^{r-1}(\alpha)$. Therefore there exists a unique $U_t\big(\mathrm{sl}(2)\big)$-morphism $G\colon V^{r-1}(\alpha) \longrightarrow M$ such that $Ge_{r-1}^{r-1}(\alpha) = y$ since $W^{r-1}(\alpha)$ is a Verma module. So F is a split surjective morphism, *i.e.*, $V^{r-1}(\alpha)$ is a projective $U_t\big(\mathrm{sl}(2)\big)$-module. ∎

Also $V^{r-1}(\alpha)$ is an injective $U_t\big(\mathrm{sl}(2)\big)$-module by a similar discussion. We denote: $P^{r-1}(\alpha) = V^{r-1}(\alpha)$.

2.2. The indecomposable extensions of the Verma modules $W^j(\alpha)$ for $j \neq r-1$, constructed in [RT2], are fundamentally important to generate other modules. For any $0 \leq j \leq r-2$ we define these modules, denoted by $P^{r-j-2}(\alpha)$, by the following rules. The basis of $P^{r-j-2}(\alpha)$ is $\{b_n^{r-j-2}(\alpha), a_n^{r-j-2}(\alpha), n = 0,\dots,r-j-2, \text{ and } e_n^j(\alpha), f_n^j(\alpha), n = 0,\dots j\}$ and the actions of K, X, Y are given by the following rules:

$$\begin{gathered} Kb_n^{r-j-2}(\alpha) = \alpha t^{r-j-2-2n} b_n^{r-j-2}(\alpha) \\ Xb_n^{r-j-2}(\alpha) = \alpha^2[n][r-j-1-n]b_{n-1}^{r-j-2}(\alpha) + a_{n-1}^{r-j-2}(\alpha) \\ Yb_n^{r-j-2}(\alpha) = b_{n+1}^{r-j-2}(\alpha) \\ n = 0,\dots,r-j-2 \end{gathered} \tag{2.2.1}$$

$$\begin{gathered} Kf_n^j(\alpha) = \alpha t^{-r-j+2n} f_n^j(\alpha) \\ Xf_n^j(\alpha) = f_{n+1}^j(\alpha) \\ Yf_n^j(\alpha) = \alpha^2 t^{2r}[n][j+1-n]f_{n-1}^j(\alpha) \\ n = 0,\dots,j \end{gathered} \qquad \begin{gathered} Ke_n^j(\alpha) = \alpha t^{r+j-2n} e_n^j(\alpha) \\ Xe_n^j(\alpha) = \alpha^2 t^{2r}[n][j+1-n]e_{n-1}^j(\alpha) \\ Ye_n^j(\alpha) = e_{n+1}^j(\alpha) \\ n = 0,\dots,j \end{gathered}$$

$$\begin{gathered} Ka_n^{r-j-2}(\alpha) = \alpha t^{r-j-2-2n} a^{r-j-2}(\alpha) \\ Xa^{r-j-2}(\alpha) = \alpha^2[n][r-j-1-n]a_{n-1}^{r-j-2}(\alpha) \\ Ya^{r-j-2}(\alpha) = a_{n+1}^{r-j-2}(\alpha) \\ n = 0,\dots,r-j-2 \end{gathered}$$

where $b_{r-j-1}^{r-j-2}(\alpha) = f_j^j(\alpha)$, $f_{j+1}^j(\alpha) = a_{r-j-2}^{r-j-2}(\alpha)$, $e_{j+1}^j(\alpha) = a_0^{r-j-2}(\alpha)$ and $a_{-1}^{r-j-2}(\alpha) = e_j^j(\alpha)$.

Obviously $\dim P^{r-j-2}(\alpha) = 2r$.

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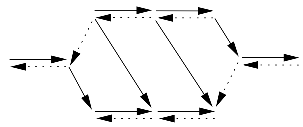
(2.2.2)



$P^2(1)$

The structure of $P^{r-j-2}(\alpha)$ $(0 \le j \le r-2)$ is indicated above (we take $r = 5, j = 1$, $\alpha = 1$).

$P^{r-j-2}(\alpha)$ has a unique maximal submodule and the quotient of $P^{r-j-2}(\alpha)$ modulo this submodule is

$$\operatorname{top} P^{r-j-2}(\alpha) = v^{r-j-2}(\alpha) \tag{2.2.3}$$

and $P^{r-j-2}(\alpha)$ has a unique minimal submodule, it is

$$\operatorname{soc} P^{r-j-2}(\alpha) = V^{r-j-2}(\alpha). \tag{2.2.4}$$

We also have the following extensions

$$\begin{gathered} 0 \longrightarrow W^j(\alpha t^r) \longrightarrow P^{r-j-2}(\alpha) \longrightarrow W^{r-j-2}(\alpha) \longrightarrow 0 \\ 0 \longrightarrow \tilde{W}^j(\alpha t^{-r}) \longrightarrow P^{r-j-2}(\alpha) \longrightarrow \tilde{W}^{r-j-2}(\alpha) \longrightarrow 0 \end{gathered} \tag{2.2.5}$$

LEMMA 2.2.6. *$P^{r-j-2}(\alpha)$ for $0 \le j \le r-2$, $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ are projective $U_t\big(\mathrm{sl}(2)\big)$-modules.*

PROOF. We want to prove $\operatorname{Ext}^1\big(P^{r-j-2}(\alpha), V\big) = 0$ for any simple $U_t\big(\mathrm{sl}(2)\big)$-module V. Assume there is a non-split exact sequence

$$0 \longrightarrow V \hookrightarrow M \longrightarrow P^{r-j-2}(\alpha) \longrightarrow 0.$$

Then M has a decomposition $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where $M_\lambda = \{x \in M \mid (K-\lambda)^n x = 0$ for some $n > 0\}$ and $\lambda \in \Lambda$ satisfies $\lambda^{4r} = 1$. Moreover, for any $x \in M_\lambda$, the maximal common factor of $(K-\lambda)^n$ with $\lambda^{4r} = 1$ and $K^{4r} - 1$ is $K - \lambda$, therefore $Kx = \lambda x$, and $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is the weight space decomposition. Consider M as $\mathbb{Q}(t)[K]$-module we have $M = V \oplus P^{r-j-2}(\alpha)$. We use the basic fact that non-split extensions of simple $U_t\big(\mathrm{sl}(2)\big)$-modules must be Verma modules $\tilde{W}^l$ or W^l $(0 \le l \le r-1)$. Fix the basis of $P^{r-j-2}(\alpha)$ as in (2.2.1). So we only have the following cases in M:

(1) $0 \ne Xb_0^{r-j-2}(\alpha) - e_j^j(\alpha) \in V$; this implies that $V = V^j(\alpha t^r)$ and $M/\tilde{W}^j(\alpha t^{-r})$ is indecomposable. However now $V^j(\alpha t^r) \oplus V_j(\alpha t^r)$ is a submodule of $M/\tilde{W}^j(\alpha t^{-r})$ with the quotient $V^{r-j-2}(\alpha)$; this is a contradiction to our basic fact.

(2) $0 \ne Yb_{r-j-2}^{r-j-2}(\alpha) - f_j^j(\alpha) \in V$. Similar discussion as in (1).

(3) $0 \ne Xe_0^j(\alpha) \in V$; because $e_0^j(\alpha) = X^{r-1}b_{r-j-2}^{r-j-2}(\alpha)$, this contradicts $X^r = 0$.

(4) $0 \ne Yf_0^j(\alpha) \in V$. Similar discussion as in (3).

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(5) $0 \neq Xf_j^j(\alpha) - a_{r-j-2}^{r-j-2}(\alpha) \in V$; this implies that $V = V^{r-j-2}(\alpha)$; now M has a submodule N generated by $f_0^j(\alpha)$; but $V^{r-j-2}(\alpha) \oplus V^{r-j-2}(\alpha)$ is a submodule of N with the quotient $V^j(\alpha t^{-r})$, a contradiction to the basic fact.

(6) $0 \neq Ye_j^j(\alpha) - a_0^{r-j-2}(\alpha) \in V$. Similar discussion as in (5).

(7) $0 \neq Xa_0^{r-j-2}(\alpha) \in V$ or $0 \neq Ya_{r-j-2}^{r-j-2} \in V$, contradicting $X^r = 0$ or $Y^r = 0$ again.

Therefore the extension is split, hence $P^{r-j-2}(\alpha)$ is projective. ∎

By a similar discussion we know that $P^{r-j-2}(\alpha)$ is an injective $U_t\big(\mathrm{sl}\,(2)\big)$-module, too.

REMARK. Because $U_t\big(\mathrm{sl}\,(2)\big)$ is finite dimensional over $\mathbb{Q}(t)$, every indecomposable projective $U_t\big(\mathrm{sl}\,(2)\big)$-module is finite dimensional; and since the tops of $P^j(\alpha)$ ($0 \leq j \leq r-1, \alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$) are just all simple modules $V^j(\alpha)$, so $P^j(\alpha)$, $0 \leq j \leq r-1$, $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ is a complete list of indecomposable projective $U_t\big(\mathrm{sl}\,(2)\big)$-modules up to isomorphism.

COROLLARY 2.2.7. *Every $U_t\big(\mathrm{sl}\,(2)\big)$-module (possibly infinite dimensional) has a decomposition into a direct sum of its weight spaces under the action of K.*

PROOF. Every $U_t\big(\mathrm{sl}\,(2)\big)$-module N has a projective cover $P \longrightarrow N \longrightarrow 0$. Since P is a direct sum of $P^j(\alpha)$'s, so P, hence N, has such decomposition. ∎

3. **Blocks of $U_t\big(\mathrm{sl}\,(2)\big)$.**

3.1. Before we decompose $U_t\big(\mathrm{sl}\,(2)\big)$ into the direct sum of blocks (up to Morita equivalence), we should introduce some basic notions widely used in the representation theory of finite dimensional algebras (see [G] or [Ri]).

Given a finite dimensional algebra A over a field k, $\mathrm{mod}\,A$ denotes the category of all finite dimensional A-modules. A non-split exact sequence in $\mathrm{mod}\,A$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

with L, N decomposable, is called an *Auslander-Reiten sequence* provided: for any morphism $h: L \longrightarrow L'$ which is not a split injection, there exists $i: M \longrightarrow L'$ such that $i \circ f = h$; and for any morphism $j: N' \longrightarrow N$ which is not a split surjection, there exists $l: N' \longrightarrow N$ such that $g \circ l = j$. It is easy to see that Auslander-Reiten sequences, if they exist, essentially are unique for given L or given N; so we denote $L = \tau N$ and $N = \tau^- L$ (in fact $\tau = DTr$ and $\tau^- = TrD$; these functors are defined in [AR]). We say that A has Auslander-Reiten sequences provided that for any indecomposable non-injective module L there exists an Auslander-Reiten sequence starting with L, and to any indecomposable non-projective module N there exists an Auslander-Reiten sequence ending in N. According to a famous theorem due to Auslander-Reiten [Ar], we know that A always has an Auslander-Reiten sequence. A morphism $f: M \longrightarrow N$ with M, N indecomposable is said to be *irreducible* if f is not an isomorphism and given any factorization $f = f' \circ f''$:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle f''}\searrow & & \nearrow{\scriptstyle f'} \\ & V & \end{array}$$

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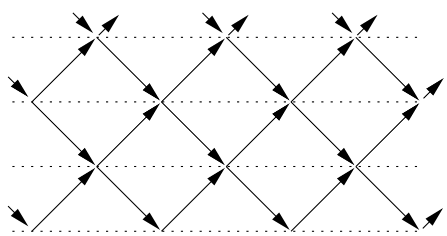
then f' is a split surjection or f'' is a split injection.

Given an Auslander-Reiten sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$, the irreducible morphism starting with L or ending in N can easily be determined: those starting with L are of the form $f' \colon L \longrightarrow M'$ where M' is a non-zero direct summand of M, say $M = M' \oplus M''$, and $f = \binom{f'}{f''}$ for some f''; those ending in N are of the form $g' \colon M' \longrightarrow N$, where again $M = M' \oplus M''$ and $g = (g', g'')$, for some g''.

Let M, N be indecomposable A-modules; denote by $\operatorname{rad}(M,N)$ the set of non-isomorphisms from M to N. If M, N are not necessarily indecomposable, say with decompositions $M = \bigoplus_i M_i$, $N = \bigoplus_j N_j$ where M_i, N_j are indecomposable, define $\operatorname{rad}(M,N) = \bigoplus_{i,j} \operatorname{rad}(M_i, N_j)$ and $\operatorname{rad}^2(M,N) = \{f \in \operatorname{Hom}_A(M,N) \mid f = f' \circ f'', f' \in \operatorname{rad}(I,N), f'' \in \operatorname{rad}(M,I)$ for some A-module $I\}$. $f \colon M \longrightarrow N$ is irreducible if and only if $\bar{f}$ is non-zero in $\operatorname{rad}(M,N)/\operatorname{rad}^2(M,N)$. Now we could define the Auslander-Reiten quiver Γ_A of A: Γ_A has $[M]$ of the isomorphic class of indecomposable module $M \in \operatorname{mod} A$ as a vertex. Two vertices $[M]$ and $[N]$ are linked together by n arrows $[M] \longrightarrow [N]$ if $n = \dim_k \operatorname{rad}(M,N)/\operatorname{rad}^2(M,N) \geq 1$. Let us denote by $\mathbb{P}_\Lambda$ and $\mathbb{I}_\Lambda$ the subset of Γ_A corresponding to projective and injective modules respectively; we have the Auslander-Reiten translation $\tau \colon \Gamma_A \setminus \mathbb{P}_A \xrightarrow{\sim} \Gamma_A \setminus \mathbb{I}_A$ such that $\tau[N] = [\tau N]$. In this sense, Γ_A is a translation quiver.

A possible form of a component of Γ_A is $\mathbb{Z}A_\infty/n$, which is called a *tube*; it is given by the following translation quiver and by making an identification along x with τ_x^n for any vertex x.



where dotted lines stand for τ-orbits. n is called the *rank* of the tube. A rank 1 tube is said to be *homogeneous*.

If A is a hereditary algebra of tame type, the classification of the indecomposable modules in $\operatorname{mod} A$ is finished (see [DR] or [Ri]); $\operatorname{mod} A$ is divided into three parts, the first is the component of preprojective modules; the second part consists of a $\mathbb{P}_1 k$-family of components which are tubes and among those almost are homogeneous tubes; the third is the component of preinjective modules. The corresponding Auslander-Reiten quiver can be drawn by hand.

3.2. In representation theory, a quiver Δ is just a directed graph. Write $\Delta = (\Delta_0, \Delta_1)$; here Δ_0 is the set of vertices and Δ_1 the set of arrows. A representation $V = (V_x, V_\beta)$ of Δ over k is given by finite dimensional vector spaces V_x, for all $x \in \Delta_0$, and linear maps $V_\beta \colon V_x \longrightarrow V_y$, for any arrow $\beta \colon x \longrightarrow y$. If V, V' are two representations of Δ over k, a map $f = (f_x): V \longrightarrow V'$ is given by maps $f_x: V_x \longrightarrow V'_x$ ($x \in \Delta_0$) such that $V'_\beta f_x = f_y V_\beta$. In this way we obtain the category of representations of Δ. Now assume there is given a representation V of Δ over k. If $p = (a|\beta_1, \ldots, \beta_l|b)$ is a directed path in Δ, we denote by V_p the composition $V_p = V_{\beta_l} \circ \cdots \circ V_{\beta_1}: V_a \longrightarrow V_b$. We say that V satisfies the relation $r = \sum_p \lambda_p p$ ($\lambda \in k$), provided $\sum_p \lambda_p V_p = 0$ (Note that we may require that all paths p occurring in one relation have a fixed starting point, say a, and a fixed end point, say b; thus all V_p are linear maps from V_a to V_b and we form the linear combination $\sum_p \lambda_p V_p$ in $\mathrm{Hom}_k(V_a, V_b)$). A basic theorem (due to Gabriel) in the representation theory of finite dimensional algebras claims that, if k is algebraically closed, $\mathrm{mod}\,A$ is always equivalent to the category of representation of a finite quiver Δ with a certain set I of relations. We also say that A is given by quiver Δ and relations I.

Operating an algebra A by the dual functor $D = \mathrm{Hom}_k(\ ,k)$, we get a new algebra $T(A)$, called the *trivial extension* of A. The underlying vector space of $T(A) = A \oplus D(A)$ and the multiplication is given by

$$(a, d)(a', d') = (aa', da' + ad')$$

for $a, a' \in A$, $d, d' \in D(A)$, since $D(A)$ admits an $A-A$-bimodule structure in an obvious way.

Because $T(A)$ is a selfinjective algebra, we could form the stable category $\underline{\mathrm{mod}}T(A)$ like this: objects of $\underline{\mathrm{mod}}T(A)$ are those of $\mathrm{mod}\,T(A)$ and given two objects M, N, the set of morphisms from M to N is defined as $\underline{\mathrm{Hom}}_{T(A)}(M, N) = \mathrm{Hom}_{T(A)}(M, N)/P(M, N)$ where $P(M, N) = \{f \in \mathrm{Hom}_{T(A)}(M, N) \mid$ there exists projective $T(A)$-module P and morphisms $g: M \longrightarrow P$, $h: P \longrightarrow N$ such that $f = h \circ g\}$. So $\underline{\mathrm{mod}}T(A)$ is a quotient category of $\mathrm{mod}\,T(A)$. The structure of $\underline{\mathrm{mod}}T(A)$ can be derived from that of $\mathrm{mod}\,A$ if A is a tilted algebra (see [Ha]). Particularly, if A is a hereditary algebra of tame type, not only $\underline{\mathrm{mod}}T(A)$, but also $\mathrm{mod}\,T(A)$ is clearly displayed. The aim of this section is just to fit $\mathrm{mod}\,U_t\big(\mathrm{sl}\,(2)\big)$ into this kind of category.

REMARK. The meaning of "graphical representations" and "representation of quiver" is totally different. The first is used to represent the structure of an $U_t\big(\mathrm{sl}\,(2)\big)$-module under the actions of K, X, Y firstly by Kirillov-Reshetikhin [KR], the latter was introduced by Gabriel and is widely used in the representation theory of finite dimensional algebras.

3.3. We come back investigating the structure blocks of $U_t\big(\mathrm{sl}\,(2)\big)$. Following methods introduced by Brauer in the modular representation theory of finite group (see [A]) and by Bernstein-Gelfand-Gelfand in the study of the category O for complex semisimple Lie algebra (see [BGG]), we can decompose $U_t\big(\mathrm{sl}\,(2)\big)$ into the direct sum of its blocks. The structure of every block is determined by that of the corresponding projective $U_t\big(\mathrm{sl}\,(2)\big)$-modules which are linked to each other. Let $P_1, P_2, \ldots, P_l$ be all non-isomorphic projective $U_t\big(\mathrm{sl}\,(2)\big)$-modules which are linked; then the corresponding block is defined as $B = \mathrm{End}_{U_t(\mathrm{sl}\,(2))}\,(P_1 \oplus \cdots \oplus P_l)$. Since any endomorphism of a Verma module or simple

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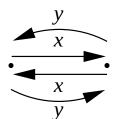
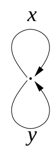
module is a multiplication by a scalar, so, by the proof of the Gabriel theorem (see 4.3 of [G]), the algebra of every block can be given by its quiver and relations over $\mathbb{Q}(t)$.

3.3.1. (1) Because the blocks corresponding to projective modules $P^{r-1}(\alpha)$, $\alpha \in \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ are always trivial (Lemma 2.1.11), $U_t\big(\mathrm{sl}(2)\big)$ contains 4 blocks isomorphic to $\mathbb{Q}(t)$. In view of (2.2), the structure of $P^{r-j-2}(\alpha)$ $(0 \le j \le 2)$ is clearly displayed: Note that, along the way of the canonical morphisms $P^{r-j-2}(\alpha) \longrightarrow P^{j}(\alpha t^{-r}) \longrightarrow P^{r-j-2}(\alpha)$ and $P^{r-j-2}(\alpha) \longrightarrow P^{j}(\alpha t^{r}) \longrightarrow P^{r-j-2}(\alpha)$, the vector $b_0^{r-j-2}(\alpha)$ goes to $a_0^{r-j-2}(\alpha)$. By a standard technique for representations of quivers with relations, we have: (2) if $t^r = 1$, then the block corresponding to the projective modules $P^{r-j-2}(\alpha)$ and $P^{j}(\alpha)$ is isomorphic to the algebra Λ_2 given by the following quiver and relations



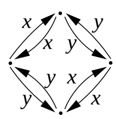
$$x^2 - y^2 = 0 \qquad xy = yx = 0 \qquad \text{if } 2j \neq r-2.$$

or isomorphic to the algebra Λ_1 given by



$$x^2 - y^2 = 0 \qquad xy = yx = 0 \qquad \text{if } 2j = r-2;$$

(3) if $t^r = -1$, then the block corresponding to the projective modules $P^{r-j-2}(\alpha)$ and $P^{j}(-\alpha)$ $(0 \le j \le r-2)$ is isomorphic to Λ_2 too; (4) if $t^r = \sqrt{-1}$ or $-\sqrt{-1}$, then the block corresponding to the projective modules $P^{r-j-2}(\alpha)$, $P^{j}(\alpha t^{r})$, $P^{j}(\alpha t^{-r})$ and $P^{r-j-2}(\alpha t^{2r})$ is isomorphic to Λ_4, given by the following quiver and relations



$$x^2 - y^2 = 0 \qquad xy = yx = 0.$$

By a detailed counting we have the following:

THEOREM 3.3.2. *The blocks of $U_t\big(\mathrm{sl}(2)\big)$ over $\mathbb{Q}(t)$ consist of* 4 *blocks isomorphic to $\mathbb{Q}(t)$ and one of the following situations:*

(i) $2(r-1)$ *blocks isomorphic to* Λ_2 *if* $t^r = 1$ *and* r *is odd.*

(ii) $2(r-2)$ *blocks isomorphic to* Λ_2 *and* 4 *blocks isomorphic to* Λ_1 *if* $t^r = 1$ *and* r *is even.*

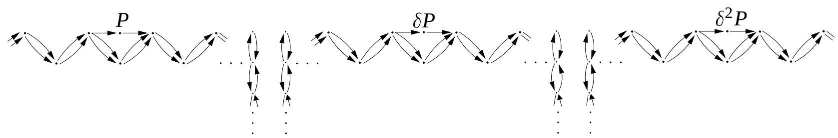
(iii) $2(r-1)$ *blocks isomorphic to* Λ_2 *if* $t^r = -1$.

(iv) $r-1$ *blocks isomorphic to* Λ_4 *if* $t^r = \sqrt{-1}$ *or* $-\sqrt{-1}$.

We easily know that there are good coverings $\Lambda_4 \longrightarrow \Lambda_2$ and $\Lambda_2 \longrightarrow \Lambda_1$. Λ_4 is the trivial extension of the hereditary algebra

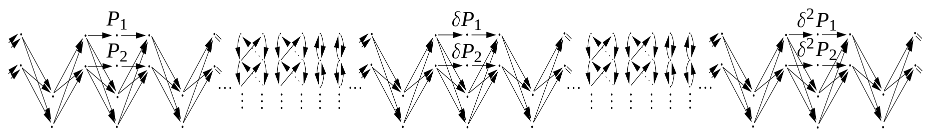
$$\tilde{A}_3:\ \begin{array}{ccccc} & & \cdot & & \\ & \nearrow & & \nwarrow & \\ \cdot & & & & \cdot \\ & \searrow & & \swarrow & \\ & & \cdot & & \end{array}$$

and Λ_2 is the trivial extension of the Kronecker algebra $\cdot \rightrightarrows \cdot$. After the work of Tachikawa-Wakamatsu and Happel (see [Ha3]), the categories of finite dimensional modules over those algebras Λ_1, Λ_2 and Λ_4 can be very well displayed. Their Auslander-Reiten quivers are pictured above.



A $\mathbb{P}_1\mathbb{Q}(t)$ family of homogeneous tubes

The Auslander-Reiten quiver Γ_{Λ_1} of Λ_1 is obtained by identifying along P with δP, and Γ_{Λ_2} of Λ_2 by identifying along P with $\delta^2 P$.



Two tubes of $\mathbb{Z}A_\infty/2$ and a $\mathbb{P}_1\mathbb{Q}(t)^*$ family of homogeneous tubes

The Auslander-Reiten quiver Γ_{Λ_4} of Λ_4 is obtained by identifying along $\binom{P_1}{P_2}$ with $\binom{\delta^2 P_1}{\delta^2 P_2}$. Up to now we have realized all finite dimensional indecomposable $U_t\big(\mathrm{sl}\,(2)\big)$-modules as representations of the corresponding quivers.

4. **Constructing of indecomposables.** Reducing the problem of representations of $U_t\big(\mathrm{sl}\,(2)\big)$ to those of Λ_1, Λ_2 and Λ_4 means that the category $\mathrm{mod}\, U_t\big(\mathrm{sl}\,(2)\big)$ is very clear now. However we will give the structure of all finite dimensional indecomposable $U_t\big(\mathrm{sl}\,(2)\big)$-modules by their chosen basis and the actions of generators K, X, Y.

We consider only the case of q a primitive root of 1 *i.e.*, the case (iv) of Theorem 3.3.2. The other cases can be easily deduced from this one.

4.1 Indecomposable modules $V^j(\alpha, n)$. The basis of $V^j(\alpha, n)$ is

$$\{a_u^{r-j-2}(\alpha, m-1), e_v^j(\alpha, m) \mid 0 \le m \le n, 0 \le u \le r-j-2, 0 \le v \le j\}$$

and the actions are given by:

$$Ke_v^j(\alpha, m) = \alpha t^{mr} t^{j-2v} e_v^j(\alpha, m)$$
$$Xe_v^j(\alpha, m) = \alpha^2 t^{2mr}[v][j+1-v]e_{v-1}^j(\alpha, m) + \delta_{v0} a_{r-j-2}^{r-j-2}(\alpha, m+1)$$
$$Ye_v^j(\alpha, m) = e_{v+1}^j(\alpha, m)$$

and

$$Ka_u^{r-j-2}(\alpha, m-1) = \alpha t^{(m-1)r} t^{r-j-2-2u} a_u^{r-j-2}(\alpha, m-1)$$
$$Xa_u^{r-j-2}(\alpha, m-1) = \alpha^2 t^{2(m-1)r}[u][r-j-1-u]a_{u-1}^{r-j-2}(\alpha, m-1)$$
$$Ya_u^{r-j-2}(\alpha, m-1) = a_{u+1}^{r-j-2}(\alpha, m-1)$$

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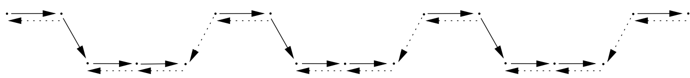
where $a_u^{r-j-2}(\alpha,-1)=0$, $a_u^{r-j-2}(\alpha,n)=0$, $a_{-1}^{r-j-2}(\alpha,m-1)=a_{r-j-1}^{r-j-2}(\alpha,m-1)=0$ and $e_{j+1}^{j}(\alpha,m)=a_0^{r-j-2}(\alpha,m-1)$. The graphical representations of $V^j(\alpha,n)$ are as follows:



$r=5, j=1, n=3.$

4.2 Indecomposable modules $\tilde{V}^j(\alpha,n)$. The basis $\tilde{V}^j(\alpha,n)$ is $\{a_u^{r-j-2}(\alpha,m-1), e_u^j(\alpha,m) \mid 0\le m\le n, 0\le u\le r-j-2, 0\le v\le j\}$ and the actions are given by:

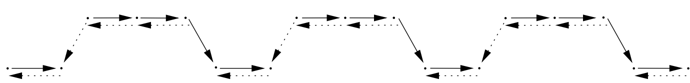
$$\begin{gathered}
Ke_v^j(\alpha,m)=\alpha t^{mr}t^{j-2v}e_v^j(\alpha,m)\\
Xe_v^j(\alpha,m)=\alpha^2 t^{2mr}[v][j+1-v]e_{v-1}^j(\alpha,m)\\
Ye_v^j(\alpha,m)=e_{v+1}^j(\alpha,m)
\end{gathered}$$

and

$$\begin{gathered}
Ka_u^{r-j-2}(\alpha,m-1)=\alpha t^{(m-1)r}t^{r-j-2-2u}a_u^{r-j-2}(\alpha,m-1)\\
Xa_u^{r-j-2}(\alpha,m-1)=\alpha^2 t^{2(m-1)r}[u][r-j-1-u]a_{u-1}^{r-j-2}(\alpha,m-1)+\delta_{u0}e_j^j(\alpha,m)\\
Ya_u^{r-j-2}(\alpha,m-1)=a_{u+1}^{r-j-2}(\alpha,m-1)
\end{gathered}$$

where $a_u^{r-j-2}(\alpha,-1)=a_u^{r-j-2}(\alpha,n)=0$, $e_{-1}^j(\alpha,m)=e_{j+1}^j(\alpha,m)=0$ and $a_{r-j-1}^{r-j-2}(\alpha,m-1)=e_0^j(\alpha,m-1)$.

The graphical representations of $\tilde{V}^j(\alpha n)$ are as follows:



$r=5, j=1, n=3.$

The induced Auslander-Reiten sequences are:

$$\begin{aligned}
&0\longrightarrow V^j(\alpha,n)\longrightarrow V^j(\alpha,n+1)\oplus V^j(\alpha t^{2r},n+1)\longrightarrow V^j(\alpha t^{2r},n+2)\longrightarrow 0\\
&0\longrightarrow \tilde{V}^j(\alpha t^{2r},n+2)\longrightarrow \tilde{V}^j(\alpha,n+1)\oplus \tilde{V}^j(\alpha t^{2r},n+1)\longrightarrow \tilde{V}^j(\alpha,n)\longrightarrow 0
\end{aligned}\tag{4.3}$$

and $0\longrightarrow \tilde{V}^j(\alpha,1)\longrightarrow V^j(\alpha)\oplus P^{r-j-2}(\alpha t^r)\oplus V^j(\alpha t^{2r})\longrightarrow V^j(\alpha,1)\longrightarrow 0$.

The Auslander-Reiten translation is defined by

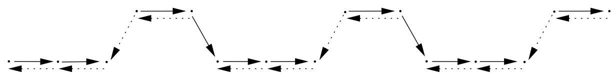
$$\tau V^j(\alpha t^{2r},n+2)=V^j(\alpha,n),\ \tau\tilde{V}^j(\alpha,n)=\tilde{V}^j(\alpha t^{2r},n+2),\quad n\ge 0,$$

and $\tau V^j(\alpha,1)=\tilde{V}^j(\alpha,1)$, where $V^j(\alpha,0)=\tilde{V}^j(\alpha,0)=V^j(\alpha)$.

4.4 The indecomposable modules $W^j(\alpha,n)$ *and* $\tilde{W}^j(\alpha,n)$. The basis of $W^j(\alpha,n)$ is $\{e_u^j(\alpha,m)\mid 0\le m\le r-1, 1\le m\le n\}$ and the actions of K, X, Y as follows:

$$\begin{gathered}
Ke_u^j(\alpha,m)=\alpha t^{(1+(-1)^m)r}t^{j-2u}e_u^j(\alpha,m)\\
Xe_u^j(\alpha,m)=\alpha^2[u][j+1-u]e_{u-1}^j(\alpha,m)+\delta_{u0}e_{r-1}^j(\alpha,m+1)\\
Ye_u^j(\alpha,m)=e_{u+1}^j(\alpha,m)
\end{gathered}$$

where $e_u^j(\alpha, n+1) = 0$ and $e_r^j(\alpha, m) = 0$ for $1 \leq m \leq n$. The graphical representation of $W^j(\alpha, n)$ is as follows

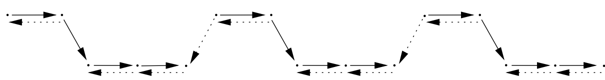


$r = 5, j = 1, n = 3.$

The basis of $\tilde{W}^j(\alpha, n)$ is $\{f_u^j(\alpha, m) \mid 0 \leq u \leq r-1, 1 \leq m \leq n\}$ and the actions of K, X, Y as follows

$$Kf_u^j(\alpha, m) = \alpha t^{(1+(-1)^m)r} t^{-j+2u} f_u^j(\alpha, m)$$
$$Xf_u^j(\alpha, m) = f_{u+1}^j(\alpha, m)$$
$$Xf_u^j(\alpha, m) = \alpha^2[u][j+1-u] f_{u-1}^j(\alpha) + \delta_{u,0} f_{r-1}^j(\alpha, m-1)$$

where $f_u^j(\alpha, 0) = 0$ for any u and $f_r^j(\alpha, m) = 0$ for $1 \leq m \leq n$. The graphical representation of $\tilde{W}^j(\alpha, n)$ is as follows



$r = 5, j = 1, n = 3.$

The induced Auslander-Reiten sequences are

$$0 \longrightarrow W^j(\alpha t^{2r}, n) \longrightarrow W^j(\alpha, n+1) \oplus W^j(\alpha t^{2r}, n-1) \longrightarrow W^j(\alpha, n) \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{W}^j(\alpha t^{2r}, n) \longrightarrow \tilde{W}^j(\alpha, n+1) \oplus \tilde{W}^j(\alpha t^{2r}, n-1) \longrightarrow \tilde{W}^j(\alpha, n) \longrightarrow 0$$

and $\tau W^j(\alpha, n) = W^j(\alpha t^{2r}, n)$, $\tau \tilde{W}^j(\alpha, n) = \tilde{W}^j(\alpha t^{2r}, n)$ where $W^j(\alpha t^{2r}, 0) = \tilde{W}^j(\alpha t^{2r}, 0) = 0.$

4.5 The indecomposable modules $T^j(\alpha, \lambda, n)$ and $\tilde{T}^j(\alpha, \lambda, n)$. The basis of $T^j(\alpha, \lambda, n)$ is $\{e_u^j(\alpha, m), \hat{e}_u^j(\alpha, m) \mid 0 \leq u \leq r-1, 1 \leq m \leq n\}$ and the actions of K, X, Y are given as follows:

$$Ke_u^j(\alpha, m) = \alpha t^{j-2u} e_u^j(\alpha, m)$$
$$Xe_u^j(\alpha, m) = \alpha^2[u][j+1-u] e_{u-1}^j(\alpha, m) + \lambda_1 \delta_{u,0} \hat{e}_{r-1}^j(\alpha, m-1)$$
$$Ye_u^j(\alpha, m) = e_{u+1}^j(\alpha, m)$$

and

$$K\hat{e}_u^j(\alpha, m) = \alpha t^{2r} t^{j-2u} \hat{e}_u^j(\alpha, m)$$
$$X\hat{e}_u^j(\alpha, m) = \alpha^2[u][j+1-u] \hat{e}_{u-1}^j(\alpha, m) + \lambda_2 \delta_{u,0} e_{r-1}^j(\alpha, m) + \delta_{u,0} e_{r-1}^j(\alpha, m-1)$$
$$Y\hat{e}_u^j(\alpha, m) = \hat{e}_{u+1}^j(\alpha, m)$$

where $e_u^j(\alpha, 0) = \hat{e}_u^j(\alpha, 0) = 0$, $e_r^j(\alpha, m) = \hat{e}_r^j(\alpha, m) = 0$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Q}(t)^* \times \mathbb{Q}(t)^*$.

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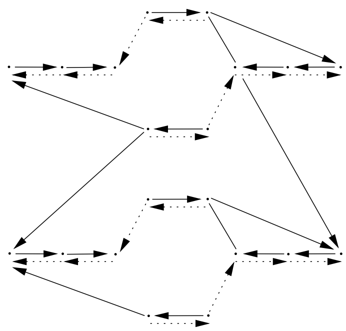


The graphical representation of $T^j(\alpha,\lambda,n)$ is shown above, $r=5, j=1, n=2$.

The basis of $\tilde{T}^j(\alpha,\lambda,n)$ is $\{f_u^j(\alpha,m), \hat{f}_u^j(\alpha,m) \mid 0 \le u \le r-1, 1 \le m \le n\}$. The actions of K, X, Y as follows:

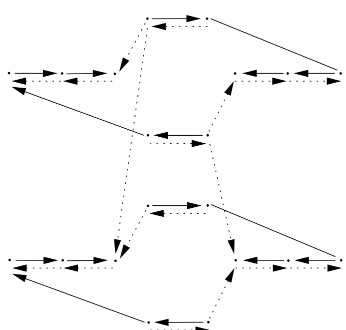
$$Kf_u^j(\alpha,m) = \alpha t^{-j+2u} f_u^j(\alpha,m)$$
$$Xf_u^j(\alpha,m) = f_{u+1}^j(\alpha,m)$$
$$Yf_u^j(\alpha,m) = \alpha^2[u][j+1-u]f_{u-1}^j(\alpha,m) + \lambda_1\delta_{u0}\hat{f}_{r-1}^j(\alpha,m) + \delta_{u0}\hat{f}_{r-1}^j(\alpha,m-1)$$

and

$$K\hat{f}_u^j(\alpha,m) = \alpha t^{2r} t^{-j+2u} \hat{f}_u^j(\alpha,m)$$
$$X\hat{f}_u^j(\alpha,m) = \hat{f}_{u+1}^j(\alpha,m)$$
$$Y\hat{f}_u^j(\alpha,m) = \alpha^2[u][j+1-u]\hat{f}_{u-1}^j(\alpha,m) + \lambda_2\delta_{u0}f_{r-1}^j(\alpha,m) + \delta_{u,0}f_{r-1}^j(\alpha,m-1)$$

where $f_u^j(\alpha,0) = \hat{f}_u^j(\alpha,0) = 0$, $f_r^j(\alpha,m) = \hat{f}_r^j(\alpha,m) = 0$ and $\lambda = (\lambda_1,\lambda_2) \in \mathbb{Q}(t)^* \times \mathbb{Q}(t)^*$.

The graphical representation of $\tilde{T}^j(\alpha,\lambda,n)$ is as follows:



$r=5, j=1, n=2$.

It is easy to see that $T^j(\alpha,\lambda,n) \simeq T^j(\alpha,\lambda' n)$ if and only if there is a $c \in \mathbb{Q}(t)$ with $\lambda = c\lambda'$; so is it for $\tilde{T}^j(\alpha,\lambda,n)$. So we write $\lambda \in \mathbb{P}_1\mathbb{Q}(t)^*$ to denote those λ.

REMARK. In the definition of $T^j(\alpha,\lambda,n)$, one can change the action of X on $\hat{e}_u^j(\alpha,m)$ by $X\hat{e}_u^j(\alpha,m) = \alpha^2[u][j+1-u]\hat{e}_{u-1}^j(\alpha,m) + \lambda_2\delta_{u0}e_{r-1}^j(\alpha,m)$, also obtain an indecomposable $U_t\big(\mathrm{sl}(2)\big)$-module; however, we claim that this module is isomorphic to $T^j(\alpha,\lambda,n)$. Similarly for $\tilde{T}^j(\alpha,\lambda,n)$.

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The induced Auslander-Reiten sequences are

$$0 \longrightarrow T^j(\alpha,\lambda,n) \longrightarrow T^j(\alpha,\lambda,n+1) \oplus T^j(\alpha,\lambda,n-1) \longrightarrow T^j(\alpha,\lambda,n) \longrightarrow 0$$
$$0 \longrightarrow \tilde{T}^j(\alpha,\lambda,n) \longrightarrow \tilde{T}^j(\alpha,\lambda,n+1) \oplus \tilde{T}^j(\alpha,\lambda,n-1) \longrightarrow \tilde{T}^j(\alpha,\lambda,n) \longrightarrow 0$$

and $\tau T^j(\alpha,\lambda,n) = T^j(\alpha,\lambda,n)$, $\tau\tilde{T}^j(\alpha,\lambda,n) = \tilde{T}^j(\alpha,\lambda,n)$, where $T^j(\alpha,\lambda,0) = \tilde{T}^j(\alpha,\lambda,0) = 0$.

Now we have given all finite dimensional indecomposable modules of $U_t\big(\mathrm{sl}\,(2)\big)$; however we find some modules in Subsection 4.5 are isomorphic, so we give the following identifications.

PROPOSITION 4.6. *$T^j(\alpha t^{2r},\lambda^{-1},n) \simeq T^j(\alpha,\lambda,n) \simeq \tilde{T}^j(\alpha,\lambda^{-1},n) \simeq \tilde{T}^j(\alpha t^{2r},\lambda,n)$ for any $\lambda \in \mathbb{P}_1\mathbb{Q}(t)^*$ and $n \geq 1$, $0 \leq j \leq r-2$.*

REMARK. If $\lambda = (\lambda_1,\lambda_2) \in \mathbb{P}_1\mathbb{Q}(t)^*$, we denote $(\lambda_2,\lambda_1) \in \mathbb{P}_1\mathbb{Q}(t)^*$ by λ^{-1}; also $\lambda^{-1} = (\lambda_1^{-1},\lambda_2^{-1})$ in $\mathbb{P}_1\mathbb{Q}(t)^*$.

PROOF. (1) $T^j(\alpha,\lambda,n) \simeq T^j(\alpha t^{2r},\lambda^{-1},n)$. Define $\varphi\colon T^j(\alpha,\lambda,1) \longrightarrow T^j(\alpha t^{2r},\lambda^{-1},1)$ as: $\varphi\big(e^j_u(\alpha,m)\big) = \hat{e}^j_u(\alpha t^{2r},m)$ and $\varphi\big(\hat{e}^j_u(\alpha,m)\big) = e^j_u(\alpha t^{2r},m)$. We only check that, $\varphi\big(Xe^j_u(\alpha,m)\big) = \varphi\big(\alpha^2[u][j+1-u]e^j_{u-1}(\alpha,m)+\lambda_1 d_{u,0}\hat{e}^j_{r-1}(\alpha,m)+\delta_{u,0}\hat{e}^j_{r-1}(\alpha,m-1)\big) = \alpha^2[u][j+1-u]\hat{e}^j_{u-1}(\alpha t^{2r},m)+\lambda_1\delta_{u,0}e^j_{r-1}(\alpha t^{2r},m)+\delta_{u,0}e^j_{r-1}(\alpha t^{2r},m-1) = X\hat{e}^j_u(\alpha t^{2r},m)$. So φ induces $T^j(\alpha,\lambda,n) \simeq T^j(\alpha t^{2r},\lambda^{-1},n)$. Similarly $\tilde{T}^j(\alpha,\lambda^{-1},n) \simeq \tilde{T}^j(\alpha t^{2r},\lambda,n)$.

(2) Define $\psi\colon \tilde{T}^j(\alpha,\lambda^{-1},1) \longrightarrow T^j(\alpha,\lambda,1)$ as follows:

$$\psi\big(f^j_{j+1+i}(\alpha,1)\big) = (-1)^i\alpha^{2i}\frac{[r-1]!\,[r-j-2]!}{[r-1-i]!\,[r-j-2-i]!}\hat{e}^j_{r-i-1}(\alpha,1)$$

for $0 \leq i \leq r-j-2$.

$$\psi\big(f^j_{j-i}(\alpha,1)\big) = \frac{1}{\lambda_1\alpha^{2i}}\frac{[j-i]!}{[j]!\,[i]!}e^j_i(\alpha,1)$$

for $0 \leq i \leq j$.

$$\psi\big(\hat{f}^j_{r-1-i}(\alpha,1)\big) = (-1)^i\frac{[r-1-i]!\,[r-j-2-i]!}{\alpha^{2(j+1)}[j]!\,[j]!\,[r-1]!\,[r-j-2]!}e^j_{j+i+1}(\alpha,1)$$

for $0 \leq i \leq r-j-2$.

$$\psi\big(\hat{f}^j_i(\alpha,1)\big) = \frac{(-1)^{r-j-2}\alpha^{2(r+i-j-2)}[r-1]!\,[r-j-2]!\,[i]!\,[j]!}{\lambda_2[j+1]!\,[j-i]!}\hat{e}^j_{j-i}(\alpha,1)$$

for $0 \leq i \leq j$.

It can be checked that ψ preserves the actions of X and Y. Therefore we have $\tilde{T}^j(\alpha,\lambda^{-1},1) \simeq T^j(\alpha,\lambda,1)$.

(3) Since $T^j(\alpha,\lambda,1)$ and $\tilde{T}^j(\alpha,\lambda^{-1},1)$ are corresponding to the simple regular modules over Λ_4 and $\dim_{\mathbb{Q}(t)}\mathrm{Ext}^1\big(T^j(\alpha,\lambda,1),T^j(\alpha,\lambda,1)\big) = 1$, $T^j(\alpha,\lambda,n) \simeq \tilde{T}^j(\alpha,\lambda^{-1},n)$ for $n \geq 1$ in view of (2).

Comparing with the Auslander-Reiten quiver of Λ_4, we can summarize the work of this section into the following result.

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THEOREM 4.7. *The modules (i)* $P^i(\alpha)$, $\alpha \in \{1,-1,\sqrt{-1},-\sqrt{-1}\}$ *and* $0 \le i \le r-1$, *(ii)* $V^j(\alpha,n)$ *and* $\tilde{V}^j(\alpha,n)$, $\alpha \in \{1,-1,\sqrt{-1},-\sqrt{-1}\}$, $0 \le j \le r-2$, $n \ge 0$, *(iii)* $W^j(\alpha,n)$ *and* $\tilde{W}^j(\alpha,n)$, $\alpha \in \{1,-1,\sqrt{-1},-\sqrt{-1}\}$, $0 \le j \le r-2$, $n \ge 1$ *and (iv)* $T^j(\alpha,\lambda,n)$, $\alpha \in \{1,\sqrt{-1}\}$, $\lambda \in \mathbb{P}_1\mathbb{Q}(t)^*$, $n \ge 1$, *form a complete list of all finite dimensional indecomposable modules of* $U_t\big(\mathrm{sl}\,(2)\big)$ *over* $\mathbb{Q}(t)$ *up to isomorphism.*

By [AR], we have the following Auslander-Reiten formula.

COROLLARY. *There exist canonical isomorphisms* $\mathrm{Ext}_{U_t(\mathrm{sl}\,(2))}(M,N) \simeq D\underline{\mathrm{Hom}}_{U_t(\mathrm{sl}\,(2))}(N,\tau M)$ *for any indecomposable* $U_t\big(\mathrm{sl}\,(2)\big)$*-modules M and N, where* $D = \mathrm{Hom}_{\mathbb{Q}(t)}\big(\ ,\mathbb{Q}(t)\big)$.

REMARK. All statements in this note are valid over the field $\mathbb{C}$ of complex numbers.

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