Can. J. Math. Vol. 48 (2), 1996 pp. 302–315

A FAMILY OF GENERALIZED RIESZ PRODUCTS

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ABSTRACT. Generalized Riesz products similar to the type which arise as the spectral measure for a rank-one transformation are studied. A condition for the mutual singularity of two such measures is given. As an application, a probability space of transformations is presented in which almost all transformations are singular with respect to Lebesgue measure.

1. **Introduction.** Recently, Choksi and Nadkarni [6] gave a simple proof that the maximal spectral type of a rank-one transformation was, modulo some atoms, a kind of generalized Riesz product measure.

In [2], Bourgain examined a special space of rank-one transformations called class-one (originally due to Ornstein [14]) and derived, for each transformation in this space, the same kind of generalized Riesz product as part of the spectral measure of an L^2 function. Using this, together with some results of Bonami on the group $\mathcal{D}_\infty$, Bourgain demonstrated that almost all the mixing transformations of class-one have spectral measures which are singular with respect to Lebesgue measure.

The generalized Riesz products obtained by Bourgain and Choksi-Nadkarni differ in a significant way from the generalized Riesz products considered by Parreau [15], or from the G-measures considered in [3] in that they are no longer defined on independent sets, but on "blocks" of integers which are pairwise independent.

In this paper, we show how to generalize the dichotomy techniques of [4], [11], [17] to this setting. As a result, we obtain a Bourgain-type result based on a simple counting argument.

The plan of the paper is as follows: In Section 2, we define our generalized Riesz products, prove uniqueness and give a criterion for continuity. Section 3 contains the principal dichotomy results. In Section 4, we give a criterion for absolute continuity both with respect to Lebesgue measure and with respect to another generalized Riesz product. In the final section, we apply these results to the Ornstein class-one transformations.

2. **A family of generalized Riesz products.** In [6], Choksi and Nadkarni defined polynomials on the unit circle by putting

$$q_k(z) = \frac{1}{\sqrt{m_k}} \left(1 + \sum_{j=1}^{m_k-1} z^{-\left(jh_k+\sum_{i=1}^{j} a_i^k\right)}\right) \tag{2.0}$$

The research of the second author was partially supported by NSF Grant DMS-9303555.
Received by the editors July 13, 1994; revised May 25, 1995.
AMS subject classification: Primary: 28D03; secondary: 42A55, 47A35.

and set $p_k = |q_k|^2$. (Here h_k, m_k and a_j^k are given as part of the construction of a rank-one transformation—see Section 5). Then they considered the weak*-limit $\mu = \lim \prod_{k=1}^{n} p_k \lambda$, which exists, is unique, and is related to the maximal spectral type of the constructed rank-one transformation.

The definitions below are motivated by this example. First, some notation.

NOTATION 2.1. Let $\mathcal{S}$ denote a finite set of positive integers. Set

$$\mathcal{D}_1(\mathcal{S}) = \left\{ s_1 - s_2 : s_i \in \mathcal{S} \cup \{0\}, s_1 \neq s_2 \right\}.$$

Notice that we can also write the set

$$\mathcal{D}_1(\mathcal{S}) = \left\{ \epsilon_1 s_1 + \epsilon_2 s_2 : s_i \in \mathcal{S}, s_1 \neq s_2, \epsilon_i \in \{-1, 0, 1\}, 0 < \sum |\epsilon_i| \leq 2, \left|\sum \epsilon_i\right| \leq 1 \right\}.$$

Observe that if $q(z) = \sum_{s \in \mathcal{S}^0} b(s) z^{-s}$ is a polynomial with frequencies from $\mathcal{S}^0 = \mathcal{S} \cup \{0\}$, then $\mathcal{D}_1(\mathcal{S}) \cup \{0\}$ consists exactly of the frequencies which occur in $h(z) = |q(z)|^2$. Indeed, one can write

$$h(z) = \sum_{s \in \mathcal{S}^0} |b(s)|^2 + \sum_{u \in \mathcal{D}_1(\mathcal{S})} \gamma(u) z^{-u},$$

where

$$\gamma(u) = \sum_{\substack{u = s_1 - s_2 \\ s_i \in \mathcal{S}^0}} b(s_1) \overline{b(s_2)},$$

the sum being taken over all representatives $u = s_1 - s_2$ with $s_1 \neq s_2$. If, as will be the case with most of our examples, there is a unique such representation, then there is just a single term in this sum.

The function $b(s)$ defined on $\mathcal{S}^0$ is assumed to satisfy $\sum_{s \in \mathcal{S}^0} |b(s)|^2 \leq 1$.

In order to cover the Riesz product case, we normalize $h(z)$ so that it has constant term 1. We call this modified function p

$$p(z) = h(z) + \left(1 - \sum_{s \in \mathcal{S}^0} |b(s)|^2\right)$$

By the construction, we have

LEMMA 2.2. *With the above notation*

(i) $\int p \, d\lambda = 1$,

(ii) $p(z) = 1 + \sum_{u \in \mathcal{D}_1(\mathcal{S})} \left(\sum_{u = s_1 - s_2} b(s_1) \overline{b(s_2)} \right) z^{-u}$ *where the inner sum is taken over all representations* $u = s_1 - s_2$ *with* $s_1 \neq s_2 \in \mathcal{S}^0$. ∎

DEFINITION 2.3. We shall say that $\mathcal{S}$ is (order-one) *difference dissociate* if each of the elements of $\mathcal{D}_1(\mathcal{S})$ are obtained in a unique way as $s_1 - s_2$ with $s_1 \neq s_2 \in \mathcal{S}^0$.

For difference dissociate sets, the form of p is particularly simple, as the inner sum contains just one term.

Our generalized Riesz products will be constructed from a sequence $\{p_k\}$ of functions of the above form.

DEFINITION 2.4. Let $\{\mathcal{S}_k : k \in \mathbb{N}\}$ be a sequence of finite sets of positive integers. We will say that $\{\mathcal{S}_k\}$ is *dissociate across the k's* if the (nonempty) finite sums $\sum_{i=1}^{n} u_i$ with $u_i \in \mathcal{D}_1(\mathcal{S}_{n_i})$ and $n_i \neq n_j$ for $i \neq j$, are all distinct.

REMARK 2.5. If each $\mathcal{S}_k$ is a singleton, then to say that $\{\mathcal{S}_k\}$ is dissociate across the k's amounts to saying that $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$ is a dissociate set in the usual sense [8].

NOTATION 2.6. Now suppose that we have a sequence $\{\mathcal{S}_k : k \in \mathbb{N}\}$ of finite sets of positive integers. Suppose further that for each $k \in \mathbb{N}$, we have a trigonometric polynomial $q_k(z) = \sum_{s \in \mathcal{S}_k^0} b_k(s) z^{-s}$ with $\sum_{s \in \mathcal{S}_k^0} |b_k(s)|^2 \leq 1$. As above, set

$$p_k(z) = |q_k(z)|^2 + \Big(1 - \sum_{s \in \mathcal{S}_k^0} |b_k(s)|^2\Big).$$

Define the measure $\mu_n = \prod_{k=1}^{n} p_k \lambda$, where λ denotes Lebesgue measure.

It is easy to recover the usual Riesz product construction from this. Let $\mathcal{S}_k = \{s_k\}$ a singleton for each k, and assume we are given $\alpha_k \in (-1, 1)$. Choose a_k and b_k with $a_k^2 + b_k^2 \leq 1$ and $2a_k b_k = \alpha_k$. Then, in the above construction put $q_k(z) = a_k + b_k z^{s_k}$. This results in

$$d\mu_n(t) = \prod_{k=1}^{n} (1 + \alpha_k \cos s_k t)\, dt.$$

A slight variation of the usual proof gives

PROPOSITION 2.7. *If $\{\mathcal{S}_k\}$ is dissociate across the k's, then the μ_n have a unique weak*-limit.*

Notice that $\hat{\mu}_k(0) = 1$, and if $u \in \mathbb{Z} \setminus \{0\}$, $\hat{\mu}_k(u) = 0$ unless u has the form $\sum_{i=1}^{\infty} u_i$, $u_i \in \mathcal{D}_1(\mathcal{S}_i) \cup \{0\}$ with only finitely many u_i non zero. If u has this form, then by dissociativity across the k's,

$$\hat{\mu}_k(u) = \prod_{i=1}^{k} \hat{p}_i(u_i) = \prod_{i=1}^{k} \sum_{u_i = s_1 - s_2} b(s_1)\overline{b(s_2)},$$

the sum being over all such representations of u_i with $s_1 \neq s_2 \in \mathcal{S}_i^0$. Thus we have

COROLLARY 2.8. *Suppose that $\{\mathcal{S}_k\}$ is dissociate across the k's. Let μ be the unique weak*-limit of the μ_k. Then if $u \in \mathbb{N}$ has the form $u = \sum_{i=1}^{N} u_i$, where $u_i \in \mathcal{D}_1(\mathcal{S}_{n_i})$ and $n_i \neq n_j$ for $i \neq j$, we have*

$$\hat{\mu}(u) = \prod_{i=1}^{N} \hat{\mu}(u_i), \qquad \textit{where}$$

$$\hat{\mu}(u_i) = \sum b(s_1)\overline{b(s_2)},$$

the sum being taken over all representations of $u_i = s_1 - s_2$ with $s_1 \neq s_2 \in \mathcal{S}_i^0$. If u cannot be represented in this way, we have $\hat{\mu}(u) = 0$.

DEFINITION 2.9. The set of integers which have the form $\sum u_i$, $u_i \in \mathcal{D}_1(\mathcal{S}_{n_i})$, $n_i \neq n_j$ for $i \neq j$, is denoted $W_1\{\mathcal{S}_k\}$ and referred to as the *order*-1 *words* of $\mathcal{S}_k$.

Any measure formed by the above procedure will be called a *generalized Riesz product*.

https://doi.org/10.4153/CJM-1996-016-1 Published online by Cambridge University Press

3. **Mutual singularity.** The usual proofs of singularity for Riesz products involve heavy use of the identity $\hat{\mu}(u_1 - u_2) = \hat{\mu}(u_1)\overline{\hat{\mu}(u_2)}$, where the u_i belong to the dissociate set upon which the Riesz product is based. In our case, this equality no longer holds, but we can nevertheless control the sum of the terms $\hat{\mu}(u_1-u_2)-\hat{\mu}(u_1)\overline{\hat{\mu}(u_2)}, u_1, u_2 \in \mathcal{D}_1(\mathcal{S}_k)$. This will be done in Lemma 3.6, which is central to the proof of our main theorem.

In order to control our estimates, we need to assume that $\{\mathcal{S}_k\}$ is a bit more than difference dissociate.

NOTATION 3.1. For $\mathcal{S}$ a finite set of positive integers, we are interested in the set $\mathcal{D}_1^2(\mathcal{S}) = \mathcal{D}_1\big(\mathcal{D}_1(\mathcal{S})\big) = \{u - v : u \neq v \in \mathcal{D}_1(\mathcal{S})^0\}$, where $\mathcal{D}_1(\mathcal{S})^0 = \mathcal{D}_1(\mathcal{S}) \cup 0$. This set will arise when, in the sequel, we calculate $|p|^2$. A problem with this set is that many terms are obtained in more than one manner as a sum, $s - s' = (s - t) - (s' - t)$ for all $t \neq s, s'$.

The above set is contained in $\mathcal{D}_2(\mathcal{S})$ where we define

$$\mathcal{D}_K(\mathcal{S}) = \left\{\sum_{i=1}^{2K} \epsilon_i s_i : s_i \in \mathcal{S}, s_i \neq s_j, i \neq j, \epsilon_i \in \{-K, \dots, 0, \dots, K\}, \right.$$

$$\left. \sum |\epsilon_i| \neq 0, \left|\sum \epsilon_i\right| \leq K \right\}$$

DEFINITION 3.2. We will say that $\mathcal{S}$ is *order-2 dissociate*, if each number in $\mathcal{D}_2(\mathcal{S})$ is obtained as a unique sum.

Given a sequence $\{\mathcal{S}_k\}$, we can form the *order-2 words* $n = \sum_{i=1}^{N} w_i$ where $w_i \in \mathcal{D}_2(\mathcal{S}_{n_i})$, $n_i \neq n_j$ for $i \neq j$. Notice that every order-1 word is also an order-2 word.

DEFINITION 3.3. We say that the sequence $\{\mathcal{S}_k\}$ is *order-2 dissociate across the k's* if all order-2 words are obtained as unique sums.

These definitions are actually slightly stronger than needed in the sequel, but they are somewhat simpler than a collection of statements which give the best possible theorem. We leave to the interested reader the task of formulating the weakest possible conditions under which our techniques apply.

LEMMA 3.4. *With the above Definitions 3.2 and 3.3, for u_1, $u_2 \in \mathcal{D}_1(\mathcal{S}_k)$, the difference $u_1 - u_2$ cannot be expressed as $\sum u_{n_j}$, $u_{n_j} \in \mathcal{D}_1(\mathcal{S}_{n_j})$ except in the form $u_1 - u_2 = u_3$, $u_3 \in \mathcal{D}_1(\mathcal{S}_k)$.*

PROOF. The number $u_1 - u_2 = (s_1 - s_2) - (s_3 - s_4)$ is in $\mathcal{D}_2(\mathcal{S}_k)$. As such, there is only one way (this way) to express it as an order-2 word. For it to be an order-1 word there must be some cancellation. Either, $s_1 = s_3$ or $s_2 = s_4$. If there is no cancellation, then $u_1 - u_2$ is an order-2 word which is not an order-1 word and so cannot be any other sum.

We will now show how to adapt Peyrière's proof of the singularity of Riesz products to the generalized Riesz products considered here. The theorem we will prove is

THEOREM 3.5. *Suppose that $\{\mathcal{S}_k\}$ is order-2 dissociate across the k's, with each $\mathcal{S}_k$ order-2 difference dissociate. Suppose further that μ and μ' are generalized Riesz products based on $\{\mathcal{S}_k\}$, associated to the sequences $\{b_k(s) : s \in \mathcal{S}_k\}$ and $\{b'_k(s) : s \in \mathcal{S}_k\}$ respectively, as in Notation 2.6.*

If $\sum_{k=1}^{\infty} \sum_{s \neq t \in \mathcal{S}_k^0} |b_k(s)\overline{b_k(t)} - b'_k(s)\overline{b'_k(t)}|^2 = \infty$ then $\mu \perp \mu'$.
The above is equivalent, by each $\mathcal{S}_k$ order-1 dissociate, to

$$\sum_{n=1}^{\infty} \sum_{u \in \mathcal{D}_1(\mathcal{S}_k)} |\hat{\mu}(u) - \hat{\mu}'(u)|^2 = \infty.$$

PROOF. Choose a sequence $0 < n(1) < n(2) < \cdots$ with

$$\sum_{n(i)+1}^{n(i+1)} \sum_{s \neq t \in \mathcal{S}_k^0} |b_k(s)\overline{b_k(t)} - b'_k(s)\overline{b'_k(t)}|^2 \geq 1.$$

Define

$$f_R(z) = \frac{1}{R} \sum_{1}^{R} \sum_{k=n(i)+1}^{n(i+1)} \sum_{s,t \in \mathcal{S}_k^0} c_{s,t}^{(k)} \left(z^{s-t} - b_k(s)\overline{b_k(t)}\right),$$

$$g_R(z) = \frac{1}{R} \sum_{1}^{R} \sum_{k=n(i)+1}^{n(i+1)} \sum_{s,t \in \mathcal{S}_k^0} c_{s,t}^{(k)} \left(z^{s-t} - b'_k(s)\overline{b'_k(t)}\right),$$

where

$$c_{s,t}^{(j)} = \frac{\overline{b_j(s)\overline{b_j(t)} - b'_j(s)\overline{b'_j(t)}}}{\sum_{n(i)+1}^{n(i+1)} \sum_{s \neq t \in \mathcal{S}_k^0} |b_k(s)\overline{b_k(t)} - b'_k(s)\overline{b'_k(t)}|^2},$$

and $c_{s,s}^{(j)} = 0$ for $n(i) + 1 \leq j \leq n(i+1)$.

It follows that $g_R - f_R = 1$, $\int f_R \, d\mu = \int g_R \, d\mu' = 0$, and $\sum_{n(i)+1}^{n(i+1)} \sum_{s,t \in \mathcal{S}_k^0} \left|c_{s,t}^{(k)}\right|^2 \leq 1$.
Our aim is to calculate $\int |f_R|^2 \, d\mu$, and show it is small.
We calculate

$$\begin{aligned} |f_R|^2 = \frac{1}{R^2} \Bigg(& \sum\sum_{k \neq j} \sum_{\substack{s,t \in \mathcal{S}_k^0 \\ u,v \in \mathcal{S}_j^0}} c_{s,t}^{(k)} \overline{c_{u,v}^{(j)}} \left(z^{s-t} - b_k(s)\overline{b_k(t)}\right) \left(\overline{z^{u-v} - b_j(u)\overline{b_j(v)}}\right) \\ & + \sum\sum_{k=j} \sum_{\substack{(s,t) \neq (u,v) \\ s,t \in \mathcal{S}_k^0}} c_{s,t}^{(k)} \overline{c_{u,v}^{(k)}} \left(z^{s-t} - b_k(s)\overline{b_k(t)}\right) \left(\overline{z^{u-v} - b_k(u)\overline{b_k(v)}}\right) \\ & + \sum\sum_{k=j} \sum_{\substack{(s,t) = (u,v) \\ s,t \in \mathcal{S}_k^0}} |c_{s,t}^{(k)}|^2 \left(z^{s-t} - b_k(s)\overline{b_k(t)}\right) \left(\overline{z^{s-t} - b_k(s)\overline{b_k(t)}}\right) \Bigg) \end{aligned}$$

Expanding and integrating the above, we find that the first sum-triple is zero and the third sum-triple is easily bounded by $1/R$. The middle sum-triple reduces to

$$\frac{1}{R^2} \sum_{i=1}^{R} \sum_{k=n(i)+1}^{n(i+1)} \sum_{(s,t) \neq (u,v)} c_{s,t}^{(k)} \overline{c_{u,v}^{(k)}} \left(\hat{\mu}\big((s-t) - (u-v)\big) - b_k(s)\overline{b_k(t)b_k(u)}b_k(v)\right)$$

We claim that this is dominated by $3/R$. It will then follow that $\int |f_R|^2\, d\mu \leq 4/R$. By interchanging the roles of μ and μ', one sees that $\int |g_R|^2\, d\mu' \leq 4/R$.

The proof now follows exactly as in Peyrière. By the F. Riesz theorem, we can find a sequence R_n so that $f_{R_n} \longrightarrow 0\mu$ a.e. and $g_{R_n} \longrightarrow 0\mu'$ a.e. Recall also that $f_{R_n} - g_{R_n} \equiv 1$. Letting A be the set of points x so that $f_{R_n}(x) \longrightarrow 0$, and B be the set of points where $g_{R_n}(x) \longrightarrow 0$, we have $\mu(A) = 1, \mu'(B) = 1, \mu'(A) = \mu(B) = 0$ and A and B disjoint. From this follows $\mu \perp \mu'$.

The proof of the theorem will be complete once we show the following lemma.

LEMMA 3.6. *For each i*

$$\left|\sum_{n(i)+1}^{n(i+1)} \sum_{(s,t)\neq(u,v)} c_{s,t}^{(k)}\overline{c_{u,v}^{(k)}}\Big(\hat{\mu}\big((s-t)-(u-v)\big) - b_k(s)\overline{b_k(t)}\,\overline{b_k(u)}b_k(v)\Big)\right| \leq 3. \tag{3.6}$$

PROOF. Consider the first term of Lemma 3.6.

$$\sum_{n(i)+1}^{n(i+1)} \sum_{(s,t)\neq(u,v)} c_{s,t}^{(k)}\overline{c_{u,v}^{(k)}}\hat{\mu}\big((s-t)-(u-v)\big) =$$

$$\sum_{n(i)+1}^{n(i+1)} \sum_{\substack{(s,t)\neq(u,v)\\ s\neq t,u\neq v}} \frac{\big(\overline{b_k(s)}b_k(t) - \overline{b_k'(s)}b_k'(t)\big)\big(\overline{\overline{b_k(u)}b_k(v) - \overline{b_k'(u)}b_k'(v)}\big)}{\big(\sum\sum |b_k(s)\overline{b_k(t)} - b_k'(s)\overline{b_k'(t)}|^2\big)^2}\hat{\mu}\big((s-t)-(u-v)\big).$$

Since $\{S_k\}$ is order-2 dissociate across the k's we have

$$\hat{\mu}\big((s-t)-(u-v)\big) = \begin{cases} \hat{\mu}(s-u) & \text{if } t = v,\\ \hat{\mu}(v-t) & \text{if } s = u,\\ 0 & \text{otherwise.}\end{cases}$$

For the case $t = v$ this gives

$$\left|\sum_{n(i)+1}^{n(i+1)} \frac{\sum_{\substack{s\neq t\neq u\\ \in S_k^0}} \Big(\big(\overline{b_k(s)}b_k(t) - \overline{b_k'(s)}b_k'(t)\big)b_k(s)\Big)\Big(\big(\overline{\overline{b_k(u)}b_k(t) - \overline{b_k'(u)}b_k'(t)}\big)\overline{b_k(u)}\Big)}{\big(\sum\sum |b_k(s)\overline{b_k(t)} - b_k'(s)\overline{b_k'(t)}|^2\big)^2}\right|$$

$$\leq \sum_{n(i)+1}^{n(i+1)} \frac{\sum_{t\in S_k^0}\Big(\sum_{s\neq t}\big|\big(\overline{b_k(s)}b_k(t) - \overline{b_k'(s)}b_k'(t)\big)b_k(s)\big| \sum_{u\neq t}\big|\big(\overline{\overline{b_k(u)}b_k(t) - \overline{b_k'(u)}b_k'(t)}\big)\overline{b_k(u)}\big|\Big)}{\big(\sum\sum |b_k(s)\overline{b_k(t)} - b_k'(s)\overline{b_k'(t)}|^2\big)^2},$$

which by Hölder's inequality, applied to each of the innermost sums, is ≤ 1. Similarly, for the case $s = u$ and the second term of Lemma 3.6. ∎

In [5], Brown and Moran using similar techniques consider the problem of mutual singularity for two Riesz products based on different dissociate sequences. This will be explored in a future paper. The criteria of Brown and Moran is not valid for groups where there are large sets of characters whose square is one, see [12].

https://doi.org/10.4153/CJM-1996-016-1 Published online by Cambridge University Press

4. **Absolute continuity.** In this section we give a sufficient condition for two generalized Riesz products to be mutually absolutely continuous which generalizes Theorem 7.2.1 (ii) of [8] and provides a partial converse to Theorem 3.5. The method is a more or less straightforward extension of the proof given in [8]; we will just indicate the necessary changes. The theorem is

THEOREM 4.1. *Suppose that $\{\mathcal{S}_k\}$ is order-2 dissociate across the k's with each $\mathcal{S}_k$ order-2 dissociate. Let μ and μ' denote generalized Riesz products based on $\{\mathcal{S}_k\}$, associated to the sequences $\{b_k(s) : s \in \mathcal{S}_k\}_{n=1}^{\infty}$ and $\{b_k'(s) : s \in \mathcal{S}_k\}_{k=1}^{\infty}$ respectively. Then the two conditions $\sum_{s\in\mathcal{S}_k} |b_k(s)|^2 < 1$ for all k, and*

$$\sum_{k=1}^{\infty} \frac{\sum_{s\neq t\in\mathcal{S}_k^0} |b_k(s)\overline{b(t)} - b_k'(s)\overline{b_k'(t)}|^2}{2 - \left(\sum_{s\in\mathcal{S}_k^0} |b_k(s)|^2 + |b_k'(s)|^2\right)} < \infty$$

imply $\mu' < \mu$.

PROOF. As in Notation 2.6, we associate polynomials p_k and p_k' respectively to b_k and b_k'.

Define, for $n, r, s \in \mathbb{N} h(n,r,s) = \prod_{j=1}^{n} p_j' \prod_{k=n+r+1}^{n+r+s} p_k$, and set

$$I(n,r,s) = \int \prod_{k=n+1}^{n+r} (p_k p_k')^{1/2} h(n,r,s)\, d\lambda.$$

An application of Hölder's inequality shows that $0 \leq I(n,r,s) \leq 1$ for all n, r, s. A straightforward adaptation of Lemma 7.2.6 of [8] gives:

LEMMA. *Suppose that $\lim_{n\to\infty}\left(\inf_{r,s} I(n,r,s)\right) = 1$. Then μ' is absolutely continuous with respect to μ and*

$$\prod_{k=1}^{n} \frac{p_k'}{p_k} \longrightarrow \frac{d\mu'}{d\mu} \text{ in } L^1(d\mu).$$

PROOF. The proof is, indeed word-for-word the same as that of Lemma 7.2.6 of [8], except that at the top of Page 208 one should replace the phrase "just a Riesz product" with the phrase "just a generalized Riesz product". ∎

PROOF OF THE THEOREM. One has

$$\begin{aligned}\prod_{n+1}^{n+r} (p_k p_k')^{1/2} &= \prod_{n+1}^{n+r} \frac{1}{2}(p_k + p_k')\left(1 - (p_k - p_k')^2/(p_k + p_k')^2\right)^{1/2} \\ &\geq \prod_{n+1}^{n+r} \frac{1}{2}(p_k + p_k')\left(1 - (p_k - p_k')^2/(p_k + p_k')^2\right),\end{aligned}$$

since the term inside the square root is less than one.

Using the fact that for $0 \leq a_i \leq 1$, $\prod(1 - a_i) \geq 1 - \sum a_i$ we see that the right hand side of the above inequality is bounded below by

$$\prod_{k=n+1}^{n+r} \frac{1}{2}(p_k' + p_k') - \sum_{j=n+1}^{n+r} \left(\prod_{\substack{i=n+1 \\ i\neq j}}^{n+r} \frac{1}{2}(p_i + p_i')(p_j - p_j') \right)^2 \Big/ \inf_u \left(p_j(u) + p_j'(u)\right).$$

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(The third line in formula (16) of Graham and McGehee's proof has a typo—the second product should be a sum).

By Definition 2.1,

$$\inf_x p_j(x) + p_j'(x) \geq 2 - \Big(\sum_{s \in \mathcal{S}_k^0} |b_k(s)|^2 + |b_k'(s)|^2\Big).$$

Using the fact that $\{\mathcal{S}_k\}$ is order-2 dissociate across the k's, one has

$$\int \prod_{\substack{i=n+1 \\ i \neq j}}^{n+r} \frac{1}{2}(p_i + p_i')(p_j - p_j')h(n,r,s)\, d\lambda = \sum_{s \neq t \in \mathcal{S}_k^0} |b_k(s)\overline{b(t)} - b_k'(s)\overline{b_k'(t)}|^2.$$

It now follows from our hypothesis, as on Page 208 of [8], that $\liminf I(u,v,s) = 1$ and so $\mu' << \mu$. ∎

COROLLARY 4.2. *Let μ be a generalized Riesz product as above. Then $\sum_{k=1}^{\infty} \sum_{s \neq t \in \mathcal{S}_k^0} |b_k(s)\overline{b_k(t)}|^2 = \infty$ if and only if $\mu \perp \lambda$.*

PROOF. Note that Lebesgue measure λ is the generalized Riesz product defined by $b_k'(s) = 0 \forall s \in \mathcal{S}_k$. In this case the necessary condition of Theorem 4.1 and the sufficient condition of Theorem 3.5 coincide. ∎

DEFINITION 4.3. Let μ be a generalized Riesz product based on the sequence $\{\mathcal{S}_k\}$, which is dissociate across the k's and associated to the sequence $\{b_k(s) : s \in \mathcal{S}_k^0\}$ with $\sum |b_k(s)|^2 < \infty$ for all k. Let P be a subset of the integers and define

$$b_k'(s) = \begin{cases} b_k(s) & \text{if } k \in P, \\ 0 \forall s \in \mathcal{S}_k & \text{if } k \notin P. \end{cases}$$

Then form the measure μ' based on $\{\mathcal{S}_k\}$ and associated to $\{b_k'\}$. We refer to the process of going from μ to μ' as *thinning out μ with respect to P.*

Corollary 4.2 immediately yields

PROPOSITION 4.4. *(i) If μ is equivalent to Lebesgue measure then $\mu \sim \mu'$ for every μ' obtained by thinning out μ.*

(ii) If μ has the property that $\mu \sim \mu'$ for every measure μ' obtained from thinning out μ then μ is equivalent to Lebesgue measure. ∎

REMARK. By this proposition, in order to show that a certain generalized Riesz product is singular with respect to Lebesgue measure, it suffices to show that there exists a thinned out version of μ which is singular with respect to Lebesgue measure. This result and approach is the one taken by Bourgain in [2].

5. **An Ornstein-type space of transformations.** In [14], Ornstein introduced a collection of transformations which he dubbed class-one, and showed that this collection contains mixing transformations. In fact, his argument showed that in a certain sense, there is a set of positive measure of mixing transformations in this class. We will not be concerned with mixing properties here, but rather with singularity properties. To this end, Bourgain [2] showed that almost all of the Ornstein class-one transformations have singular spectrum. As an application of the previous results, we show how to construct an Ornstein type probability space of transformations and obtain a Bourgain type result.

EXAMPLE 5.1. We present here a specific example of a probability space of transformations. The construction is based on ideas in [14] and will in the sequel, satisfy the definition of a Class $1'$ construction. This is a special case of a rank-one construction.

Recall that in a rank-one construction, we start with the unit interval, cut it into m_1 subintervals, place spacers $a_{1,i}$, $1 \leq i \leq m_1$ on top of each subinterval, and stack these columns right-over-left. The transformation T is defined inductively as "going up the tower".

Here is the inductive step of our specific construction. We have a tower of height h_k. We cut this tower into $m_k = 10^k$ pieces. To determine the spacers, we first select the numbers $x_{k,i}$ for $i = 0, \ldots, 10^k$. Set $x_{k,0} = 0 = x_{k,10^k}$. Now randomly choose integers $x_{k,i}$, $1 \leq i < 10^k$ from $\{-h_{k-1}/2, \ldots, h_{k-1}/2\}$. The spacers are $a_{k,i} = h_{k-1} + x_{k,i} - x_{k,i-1}$ for $i = 1, \ldots, 10^k$ in that order.

It is immediate that the lowest a spacer can be is 0 and the highest is $2h_{k-1}$. Further the height of the next tower is

$$h_{k+1} = 10^k(h_k + h_{k-1}).$$

To clarify the beginning, we set the heights $h_0 = 0$ and $h_{-1} = 0$. The first height, that of the unit interval, is $h_1 = 1$. It then follows that $h_2 = 10^1(1+0) = 10$, and so on. With this, we see that at the first three stages all the x's are zero (*i.e.*, $x_{1,i} = 0$, $x_{2,i} = 0$ and $x_{3,i} = 0$).

Let $X_k = \{-h_{k-1}/2, \ldots, h_{k-1}/2\}$, with $\lambda_k \equiv (2h_{k-1}+1)^{-1}$ the uniform distribution. The first two happen to be $X_1 = \{0\}, X_2 = \{0\}$. Then $X_3 = \{-5, -4, \ldots, 4, 5\}$ because $h_2 = 10$.

Now form the product probability space

$$\Omega = \prod_{k=1}^{\infty} \prod_{1}^{m_k - 1} X_k = \prod_{k=1}^{\infty} X_k^{m_k - 1}.$$

Every point $\bar{\omega} = \left\{\bar{x}_k = \{x_{k,i}\} : 1 \leq k, 1 \leq i \leq m_k - 1\right\} \in \Omega$ completely defines the spacers and hence a transformation $T_{\bar{\omega}}$. We can thus speak of some property of the transformations as occuring for almost all points of Ω.

Specifically, we wish to show that, for almost all points $\bar{\omega}$ the transformation $T_{\bar{\omega}}$ has singular spectrum. In order to show this, it is sufficient to find a subsequence k_n so that the associated polynomials p_{k_n} are based on sets $\mathcal{S}_{k_n}$ which are order-2 dissociate in both

https://doi.org/10.4153/CJM-1996-016-1 Published online by Cambridge University Press

senses. That this is true for almost all points will follow from the results in the sequel. We point out here, that the probability argument is simply that at the k-th stage we are picking $10^k - 1$ integers from a set of size $h_{k-1}+1$, where $h_{k-1} = 10^{k-2}(h_{k-2}+h_{k-3}) \geq 10^{k-2}10^{k-3}\cdots 10 \cdot 1 = 10^{(k-1)(k-2)/2}$. So the probability is very high that they are all different, as are the necessary differences and sums.

NOTATION 5.2. In this section, we present the general construction of a space of Class $1'$ transformations. We recall Ornstein's original construction and set some notation. The construction is re-explained in [2], and some related constructions (rank-one) are found in [6] as well as N. Friedman's book [7]. Our notation and conventions are slightly different from these sources.

Let $\{m_k \nearrow \infty\}$ and $\{x_{k,i} \geq 0\}$ be fixed sequences of integers. Start with the unit interval C_0 and divide it into m_1 equal subintervals. Above each, $a_{1,j}$ spacers are added, and these are stacked, (the right subcolumns placed on top of the subcolumn to their immediate left), into a single column C_2 of height $h_1 = m_1 + \sum_{j=1}^{m_1} a_{1,j}$. This procedure is continued inductively. At the k-th stage, column C_{k-1} is divided into m_k equal parts and $a_{k,j}$ spacers are put above the j-th column of C_{k-1} to obtain a stack of height $h_k = m_k h_{k-1} + \sum_{j=1}^{m_k} a_{k,j}$.

For a general rank-one transformation there is no restrictions on the number of spacers. In the class-one case, however, one requires that $0 \leq a_{k,j} < 2h_{k-1}$ for $1 \leq j \leq m_{k-1}$.

More or less following the construction of Ornstein and Bourgain we choose numbers $x_{k,j} \in \{-h_{k-1}/2, \ldots, h_{k-1}/2\}$ at random and place, on top of the j-th column of C_{k-1}, $a_{k,j} = h_{k-1}+x_{k,j}-x_{k,j-1}$ spacers. By convention, we set $x_{k,0} = 0$. This is nearly Ornstein's construction. For a general (j,k) we have $0 \leq a_{k,j} \leq 2h_{k-1}$ and he further required the last spacer $a_{k,m_k} = x_{k,m_k} - x_{k,m_k-1}$ to be chosen between 1 and 4. We will say that the family of transformations so generated is of *class* $1'$ as the sequences $\{m_k\}$, $\{x_{k,i}\}$ are understood and fixed.

In the above construction, we have $h_{k+1} = m_k(h_k + h_{k-1}) + x_{k,m_k}$, and the sequence $\{h_k\}$ is completely determined by specifying the sequences $\{m_k\}$ and $\{x_{k,m_k}\}$.

As in the earlier example, a transformation of class $1'$ is given by a point in a probability space.

$$\Omega = \left(\prod_{k=1}^{\infty} \prod_{1}^{m_k-1} X_k, \bigotimes_{k=1}^{\infty} \otimes_1^{m_k-1} \lambda_k \right),$$

where $X_k = \{-h_{k-1}/2, \ldots, h_{k-1}/2\}$, equipped with the uniform probability distribution $\lambda_k = 1/(h_{k-1}+1)$.

The rest of this section is devoted to proving the following

THEOREM. *If $(9m_k)^8/h_{k-1} \longrightarrow 0$ as $k \longrightarrow \infty$ then for almost all points in Ω the transformation $T_{\bar\omega}$ has singular spectrum.*

5.3. Let $\bar\omega \in \Omega$. Define the sets

$$S_k = \{j(h_k + h_{k-1}) + x_{k,j} : 1 \leq j < m_k\}$$

and put

$$b_k(s) = \frac{1}{\sqrt{m_k}} \forall s \in \mathcal{S}_k^0.$$

We see immediately that $\sum_{s\in\mathcal{S}_k^0} |b_k(s)|^2 = 1$. We have from [6]

THEOREM. *The maximal spectral type of the transformation $T_{\bar{\omega}}$ is given by the generalized Riesz product associated with $\{\mathcal{S}_k\}$ and $\{b_k\}$, modulo some atoms.* ∎

5.4. As an immediate application of Theorem 3.5 we have

THEOREM. *If the sequence $\{\mathcal{S}_k\}$ is order-2 dissociate across the k's with each $\mathcal{S}_k$ order-2 dissociate, then the maximal spectral type of $T_{\bar{\omega}}$ is singular with respect to Lebesgue measure.*

PROOF. For such a sequence $\{\mathcal{S}_k\}$ of sets, let μ be the generalized Riesz product constructed above and let λ denote Lebesgue measure. Then

$$\sum_{s\neq t\in\mathcal{S}_k^0} |b_k(s)\overline{b_k(t)} - 0|^2 = \sum_{s\neq t\in\mathcal{S}_k^0} \frac{1}{m_k^2} = \frac{m_k - 1}{m_k}.$$

When summed over k this is clearly infinite. Since any atoms not included in the generalized Riesz product are already singular to Lebesgue, the conclusion follows from Theorem 3.5. ∎

In a similiar way, using Corollary 4.2 we have

THEOREM. *If there is a subsequence $\{\mathcal{S}_{k_n}\}$ which is order-2 dissociate across the k_n's with each $\mathcal{S}_{k_n}$ order-2 dissociate, then the maximal spectral type of $T_{\bar{\omega}}$ is singular with respect to Lebesgue measure.*

5.5. We now present conditions which imply the various dissociate conditions we need for a class 1′ transformation.

Given a point $\bar{\omega}$ define

$$\mathcal{D}_{K,k} = \mathcal{D}_{K,k}(\bar{\omega}) = \begin{cases} \sum_{n=1}^{2K} \epsilon_n x_{k,i_n} i_n \neq i_m & \text{if } n \neq m, \\ & \epsilon_n \in \{-K, -K+1, \ldots, K-1, K\}, \\ & \sum_1^{2K} |\epsilon_n| \neq 0, |\sum_1^{2K} \epsilon_n| \leq K. \end{cases}$$

Observe that $\mathcal{D}_{K,k} \subset \mathcal{D}_{K+1,k}$, and that $\mathcal{D}_{K,k}$ is symmetric, *i.e.*, $-d \in \mathcal{D}_{K,k}$ whenever $d \in \mathcal{D}_{K,k}$. Recalling, that $\mathcal{S}_k = \{j(h_k + h_{k-1}) + x_{k,j} : 1 \leq j < m_k\}$, we have

$$\mathcal{D}_1(\mathcal{S}_k) = \{(i-j)(h_k + h_{k-1}) + x_i^k - x_j^k\}$$

where $1 \leq i \neq j < m_k$. We will see that the dissociate properties for $\mathcal{S}_k$ will be controlled by the differences (order-1 and higher) of the $x_{k,i}$.

LEMMA. *The only way two of these sums in $\mathcal{D}_1(\mathcal{S}_k)$ can be equal is if $(j-i) = (j'-i')$ and $x_{k,j} - x_{k,i} = x_{k,j'} - x_{k,i'}$.*

PROOF. Suppose

$$\big((j-i)-(j'-i')\big)(h_k + h_{k-1}) = (x_{k,j} - x_{k,i}) - (x_{k,j'} - x_{k,i'}).$$

If the LHS is not zero, then it must be bounded below by $h_k + h_{k-1} = m_{k-1}(h_{k-1} + h_{k-2}) + h_{k-1} > 3h_{k-1}$ (by symmetry we can assume positivity). But the RHS is bounded above by $4 \cdot h_{k-1}/2 = 2h_{k-1}$. ■

As a corollary, we have

LEMMA. *If all the sums in $\mathcal{D}_{1,k}(\bar{\omega})$ are distinct then $\mathcal{S}_k$ is order-1 difference dissociate.*

Next we examine

$$\mathcal{D}_2(\mathcal{S}_k) = \Big\{\big((j-i)-(j'-i')\big)(h_k + h_{k-1}) + (x_{k,j} - x_{k,i}) - (x_{k,j'} - x_{k,i'})\Big\}.$$

By similiar arguments we have

LEMMA. *If $m_k > 4$ then the only way two of these sums can be equal is if $(j_1 - i_1) - (j_1' - i_1') = (j_2 - i_2) - (j_2' - i_2')$ and $(x_{k,j_1} - x_{k,i_1}) - (x_{k,j_1'} - x_{k,i_1'}) = (x_{k,j_2} - x_{k,i_2}) - (x_{k,j_2'} - x_{k,i_2'})$.*

LEMMA. *If all the sums in $\mathcal{D}_{2,k}(\bar{\omega})$ are distinct then $\mathcal{S}_k$ is order-2 difference dissociate.*

In order to show all the sums are distinct we use the following

LEMMA. *If $0 \notin \mathcal{D}_{4,k}(\bar{\omega})$ then $\mathcal{S}_k$ is order-2 dissociate.*

PROOF. If two of the sums in $\mathcal{D}_{2,k}(\bar{\omega})$ are the same, then their difference gives a sum in $\mathcal{D}_{4,k}(\bar{\omega})$ which adds to 0. ■

5.6. Now we can ask about the probability of this. The situation is that we are picking $m_k - 1$ numbers from a block of size $h_{k-1}+1$, and asking for an upper bound on the number of ways that $0 = \sum_1^8 \epsilon_n x_{k,i_n}$. In particular, each of the 8 ϵ_n can take on one of 9 values; the 8 indices i_n can occur in $\binom{m_k-1}{8}$ ways; 7 of these 8 x_{k,i_n} can roam through the $h_{k-1}+1$ different values - the 8-th then being determined; and the other $(m_k - 1) - 8x_{k,j}$ can take on any of the $h_{k-1} + 1$ values. Thus we have

LEMMA. $\Pr\big(0 \in \mathcal{D}_{4,k}(\bar{\omega})\big) < 9^8 \binom{m_k-1}{8} \frac{(h_{k-1}+1)^{m_k-2}}{(h_{k-1}+1)^{m_k-1}} < \frac{(9m_k)^8}{h_{k-1}}$.

From this it is an easy conclusion that

THEOREM. *If $(9m_k)^8 / h_{k-1} \to 0$ as $k \to \infty$ then for almost all points in Ω there is a sub-sequence k_j along which each $\mathcal{S}_{k_j}$ is order-2 dissociate.*

5.7. Now we study dissociativity across the k's.

Suppose we have a point $\bar{\omega}$, and a finite sequence $k_1, k_2, \ldots, k_{n-1}$ so that these $\mathcal{S}_k$ are order-2 dissociate in both senses. Let W be the set of order-2 words formed from the sequence $\{\mathcal{S}_{k_i}\}_1^{n-1}$. Look at the differences of these, which can be considered as order-4 words. Consider an $\mathcal{S}_{k_n}$. Look at the differences of the integers in $\mathcal{D}_2(\mathcal{S}_{k_n})$ (which could be considered in $\mathcal{D}_4(\mathcal{S}_{k_n})$). If the intersection of these two finite difference sets is empty, $W_4(\{\mathcal{S}_{k_i}\}_1^{n-1}) \cap \mathcal{D}_4(\mathcal{S}_{k_n}) = \emptyset$, then $\mathcal{S}_{k_n}$ is order-2 dissociate from the $\mathcal{S}_{k_j}$ for $1 \leq j \leq n-1$.

Define for a fixed $L \geq 0$

$$A_{K,k}(L) = \{\bar{\omega} = \{\bar{x}_k\} : \mathcal{D}_{K,k}(\bar{\omega}) \cap \{-L, \ldots, L\} = \emptyset\}.$$

LEMMA. *Assume $h_k > 10L$. If $\bar{\omega} \in A_{2K,k}(2L)$ then $\mathcal{D}_K(\mathcal{S}_k) \cap \{-L, \ldots, L\} = \emptyset$, and every sum $u + l$, $u \in \mathcal{D}_K(\mathcal{S}_k)$ $l \in \{-L, \ldots, L\}$ is unique.*

PROOF. A term $u \in \mathcal{D}_K(\mathcal{S}_k)$ is of the form $u = j(h_k + h_{k-1}) + d$ where $d \in \mathcal{D}_{K,k}$. Because h_k is so large in comparison to L, the only way this could be within the range $\{-L, \ldots, L\}$ is if $j = 0$. But by assumption $d \notin \{-L, \ldots, L\}$.

Suppose $u + l = u' + l'$, then $u - u' = l' - l$. Since $u - u' = j(h_k + h_{k-1}) + d - d'$, from the magnitude of h_k we must have $j = 0$. But by assumption $d - d' \neq l - l'$. ∎

Now we calculate the probability. For fixed l, $-L \leq l \leq L$ the probability that $l \in \mathcal{D}_K$, *i.e.*, $l = \sum_1^{2K} \epsilon_n x_{k,i_n}$ is analyzed in the same manner that we analyzed $0 = \sum_1^4 \epsilon_n x_{k,i_n}$ in 5.6. Thus

LEMMA. $\Pr\big(A_{K,k}(L)\big) > 1 - \frac{(L+1)\cdot((2K+1)m_k)^{2K}}{(h_{k-1}+1)}$.

Suppose for almost all points $\bar{\omega}$ there is a sequence $k_1(\bar{\omega}), \ldots, k_{n-1}(\bar{\omega})$ along which we have order-2 dissociate in both senses. Then for each of these $\bar{\omega}$ the differences of the order-2 words (*i.e.*, order-4 words) are contained in some $\{-L, \ldots, L\}$, $L = L(\bar{\omega})$. Thus we have partitioned the space Ω into a countable number of disjoint sets Ω_L. By the lemma, for almost all points we can find another $k_n(\bar{\omega})$ which, with the previous, is order-2 dissociate in both senses. Therefore we have the following from which our main theorem follows.

THEOREM. *If $(9m_k)^8 / h_{k-1} \longrightarrow 0$ as $k \longrightarrow \infty$ then for almost all points in Ω there is a subsequence $k_j = k_j(\bar{\omega})$ so that $\mathcal{S}_{k_j}$ is order-2 dissociate in both senses.*

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