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A FRACTIONAL DIFFERENTIATION THEOREM FOR THE LAPLACE TRANSFORM

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1. **Introduction.** In certain systems analysis ([1], [2], [3]), it is essential to invert the n-dimensional Laplace transform and specify the inverse image at a single variable t. Let $F(s_1, \ldots, s_n)$ be a Laplace transform. The desired image function is then given by

$$g(t) = \mathscr{L}_n^{-1}[F]\,|_{t_1=t_2=\cdots=t_n=t},$$

where $\mathscr{L}_n^{-1}[F]=(1/(2\pi i)^n)\int_{\alpha_1-i\infty}^{\alpha_1+i\infty} \cdots \int_{\alpha_n-i\infty}^{\alpha_n+i\infty} \exp(\sum_{i=1}^n s_i t_i)F\,ds_1 \cdots ds_n$. An alternative approach to finding $g(t)$ is to collapse $F(s_1, s_2, \ldots, s_n)$ into a function $G(s)$ of one variable from which an application of the one-dimensional inverse transformation yields $g(t)$. $G(s)$ is said to be the "associated transform" of $F(s_1, \ldots, s_n)$, viz., $G(s)=A_nF(s_1, \ldots, s_n)$. Thus A_n is defined so as to make the following diagram commutative:

$$\begin{array}{ccc} F(s_1, \ldots, s_n) & \xrightarrow{\mathscr{L}_n^{-1}} & f(t_1, \ldots, t_n) \\ \Big\downarrow{\scriptstyle A_n} & & \Big\downarrow{\scriptstyle t_1=t_2=\ldots=t_n=t} \\ G(s) & \xrightarrow{\mathscr{L}_1^{-1}} & g(t) \end{array}$$

In this note we generalize a result given in [4].

2. **Fractional differentiation.** Let ν be a complex number with $\operatorname{Re}\nu>0$, and let D_∞^ν be the Weyl fractional derivative operator of order ν, defined by

$$D_\infty^\nu f = \frac{d^k}{dx^k}\left\{\frac{-1}{\Gamma(k-\nu)}\int_x^\infty f(y)(y-x)^{k-\nu-1}\,dy\right\} \tag{1}$$

where k is an integer satisfying $k-1<\operatorname{Re}\nu<k$, (see [5], pp. 181–212). Note if $\nu=k-1$, then $D_\infty^\nu f=d^{k-1}f/dx^{k-1}$.

THEOREM. *If* $G(s)=A_nF(s_1, \ldots, s_n)$, $G_1(s)=A_{n-1}F_1(s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_n)$, *and* $F(s_1, \ldots, s_n)=(1/s_m^{\nu+1})F_1(s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_n)$, *where* ν *is a complex number with* $\operatorname{Re}\nu>0$, *then* $G(s)=((-1)^{k-1}/\Gamma(\nu+1))D_\infty^\nu G_1(s)$, *where* k*=smallest integer greater than* $\operatorname{Re}\nu$.

The proof will be based on the following lemma, which is of interest in itself.

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LEMMA. *Let $f(x)$ be a function such that $f(x)=0$ for $x<0$, and $f(x)e^{-cx}$ is absolutely integrable on* $[0, \infty)$ *for some c. If $F(x)=\mathscr{L}[f](x)=\int_0^\infty f(y)e^{-xy}\,dy$, and ν and k are as in the theorem, then*

$$\mathscr{L}[t^\nu f(t)] = (-1)^{k-1}D_\infty^\nu[F(s)], \qquad s > c.$$

Proof.

$$D_\infty^\nu\left[F(s)\right] = \frac{-1}{\Gamma(k-\nu)}\frac{d^k}{ds^k}\left\{\int_s^\infty (y-s)^{k-\nu-1}\left[\int_0^\infty f(t)e^{-yt}\,dt\right]dy\right\}$$

$$= \frac{-1}{\Gamma(k-\nu)}\frac{d^k}{ds^k}\left\{\int_0^\infty f(t)\left[\int_s^\infty (y-s)^{k-\nu-1}e^{-yt}\,dy\right]dt\right\}.$$

By a change of variable, and the definition of the gamma function, the inner integral is seen to be equal to $e^{-ts}t^{\nu-k}\Gamma(k-\nu)$. Hence

$$D_\infty^\nu\,[F(s)] = -\frac{d^k}{ds^k}\int_0^\infty f(t)t^{\nu-k}e^{-ts}\,dt$$

$$= (-1)^{k-1}\int_0^\infty f(t)t^\nu e^{-ts}\,dt = (-1)^{k-1}\mathscr{L}[t^\nu f(t)].$$

Proof of the theorem. From the preceding diagram,

$$g(t) = f(t_1, \ldots, t_n)\Big|_{t_1=\cdots=t_n=t} = \mathscr{L}_n^{-1}[F] = \frac{t^\nu}{\Gamma(\nu+1)}\,g_1(t),$$

where $g_1(t)=\mathscr{L}_{n-1}^{-1}[F_1]|_{t_1=\cdots=t_{m-1}=t_{m+1}=\cdots=t_n=t}$.

The proof follows by taking the Laplace transform of both sides, and applying the lemma to $\mathscr{L}[t^\nu g_1(t)]$.

Note that when ν is a natural number, the theorem reduces to Theorem 1 of [4].

EXAMPLE. Given $F(s_1, s_2)=(1/s_1^{3/2})\cdot(1/s_2^n)$ (n=positive integer). Here $G_1(s)=1/s^n$. By our theorem $G(s)=(D_\infty^{1/2}/\Gamma(\frac{3}{2}))(1/s^n)=(\Gamma(n+\frac{1}{2})/\Gamma(\frac{3}{2})\Gamma(n))(1/s^{n+1/2})$. Inversion of this last expression yields $g(t)=2t^{n-1/2}/(n-1)!\sqrt{\pi}$.

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