## A CHARACTERIZATION OF PRUFER DOMAINS

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The purpose of this note is to give a new characterization of Prufer domains using the concept of ring epimorphism, and to indicate some connections with well-known properties of Prufer domains. All rings are commutative and have a unit element.

An extension  $R \subseteq S$  of rings is called epimorphic, if the injection map is an epimorphism in the category of commutative rings, i.e. cancellable on the right.  $R \subseteq S$  is epimorphic if and only if the natural map  $S \otimes_R S \to S$  is an isomorphism (see e.g. [8]).

We will say, that  $R \subseteq S$  is a completely epimorphic extension, if  $R \subseteq T$  is epimorphic for all rings T "between" R and S, i.e.  $R \subseteq T \subseteq S$ . This is clearly equivalent to the following condition: for all  $s \in S$ , we have  $s \otimes 1 = 1 \otimes s$  in  $R[s] \otimes_R R[s]$ .

A homomorphism  $R \to S$  is called flat, if S is a flat R-module, and an extension  $R \subseteq S$  is called essential, if each non-zero ideal of S has non-zero intersection with R.

LEMMA 1. A completely epimorphic extension  $R \subseteq S$  is essential.

<u>Proof.</u> If  $0 \neq s \in S$ , then  $s \otimes 1 = 1 \otimes s$  in  $R[s] \otimes_R R[s]$ . By [1, Chapter 1, §2, No. 11, Lemma 10], there exist elements  $x_j \in R[s]$  and  $a_{jk} \in R$  (j = 0, ..., m, k = 0, ..., n) such that, among other relations,

$$s = \sum_{j=0}^{m} x_j a_{j0} \quad \text{and}$$

$$\sum_{k=0}^{n} a_j s^k = 0 \quad (j = 0, \dots, m).$$

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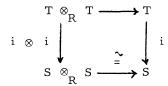
Since  $s \neq 0$ , there is at least one index i, such that  $a_{i0} \neq 0$  and we can write

$$-a_{i0} = s(\sum_{k=1}^{n} a_{ik}^{k}),$$

which implies the statement of the lemma.

LEMMA 2 (Lazard [5]). If  $R \subseteq S$  is epimorphic and flat, and if T is a ring between R and S, such that  $R \subseteq T$  is flat, then  $R \subseteq T$  is epimorphic.

Proof. The following diagram is commutative:



Since T and S are flat, i  $\otimes$  i is a monomorphism, hence T  $\underset{\mathsf{R}}{\otimes}$  T  $\rightarrow$  T is an isomorphism.

LEMMA 3. If  $R \subseteq S$  is completely epimorphic, then every T between R and S is integrally closed.

<u>Proof.</u> Suppose there exists a T which is not integrally closed and let  $s \not \in T$  be integral over T. Then T[s] is a finitely generated T-module and  $T \subseteq T[s]$  is not epimorphic by [4, Proposition 1.5]. This implies of course, that  $R \subseteq T[s]$  is not epimorphic.

If R is an integral domain with quotient field K, then a ring between R and K is called an overring of R. Every essential epimorphic extension, and in particular every completely epimorphic extension of an integral domain is isomorphic to an overring [8, Corollary 9.11].

However, not every overring of an integral domain is an epimorphic extension, as it will be shown in the next proposition.

A Prufer domain is an integral domain, such that every finitely generated ideal is invertible. A Dedekind domain is a noetherian Prufer domain.

PROPOSITION. Let R be an integral domain with quotient field K. Then R is a Prufer domain if and only if R  $\subseteq$  K is completely epimorphic.

<u>Proof.</u> If R is Prufer, then the tensor product of torsion-free modules is torsion-free [2, VII, Proposition 4.5], hence, for any  $s = a/b \in K$ ,  $b(s \otimes 1 - 1 \otimes s) = 0$  in  $R[s] \otimes_R R[s]$  implies  $s \otimes 1 = 1 \otimes s$ . The converse follows from Lemma 3 and a result of Davis [3].

The proof gives us a little bit more information.

COROLLARY 1. If R is an integral domain with quotient field K, then the following conditions are equivalent:

- (i) R is a Prufer domain;
- (ii) the tensor-product of torsion-free modules is torsion-free;
- (iii)  $R[s] \otimes_{D} R[s]$  is torsion-free for all  $s \in K$ .

COROLLARY 2. If R is a Prufer domain, then the following conditions for a ring  $T \supseteq R$  are equivalent:

- (i) T is isomorphic (over R) to an overring of R;
- (ii) R ⊆ T is essential and epimorphic;
- (iii) R ⊆ T is completely epimorphic.

This follows from the proposition and [8, Corollary 9.11]. With the appropriate definition of "overring", this corollary generalizes to semihereditary rings.

Some well-known properties of Prufer domains can now be deduced from the preceding results; we list two examples.

COROLLARY 3. Every overring of a Prufer domain is a Prufer domain.

COROLLARY 4 (Richman [7]). An integral domain is Prufer if and only if every overring is flat.

This is a consequence of Lemma 2 and the fact, that a torsion-free module over a Prufer domain is flat [2, VII, Proposition 4.2.].

We finally give a new proof of a result, which was first established in [6].

COROLLARY 5. Every overring of a Dedekind domain is a Dedekind domain.

<u>Proof.</u> By Corollary 3, every overring T is a Prufer domain, and it remains to show, that every T is noetherian. This follows from [4, Corollary 2.3.], since R is noetherian and  $R \subseteq T$  is a flat epimorphism.

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