A NOTE ON THE INDEPENDENCE OF THE AXIOMS FOR A VECTOR SPACE

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It is customary to define a vector space in some such manner as: A *vector space* over a field F is a set V of elements $a, b, \cdots$ called vectors, having the following properties: For arbitrary $a, b, c \in V$; $\lambda, \mu \in F$.

A1. There is a mapping of $V \times V$ into V which is called *addition* of vectors. The image of the pair $(a, b) \in V \times V$ is called the *sum* of a and b, and is denoted by $a+b$.

A2. There is a mapping of $F \times V$ into V which is called *multiplication* of vectors by scalars. The image of the pair $(\lambda, a) \in F \times V$ is called the *product* and is denoted by λa.

A3. $a+(b+c) = (a+b)+c$

A4. $a+b = b+a$

A5. $1 \cdot a = a$

A6. $\lambda(\mu a) = (\lambda\mu)a$

A7. $(\lambda+\mu)a = \lambda a+\mu a$

A8. $\lambda(a+b) = \lambda a+\lambda b$

A9. There is a vector $o \in V$ so that for every $a \in V$

$$o+a = a$$

A10. For each $a \in V$ there is a vector $-a \in V$ so that

$$a+(-a) = o$$

It is not at all evident that A9 and A10 are respectively independent of the preceding axioms, and it is to this question that we address ourselves. We observe that authors of elementary texts wherein one finds vector spaces defined *in extenso* are often vague on the question of independence. Thus P. R. Halmos [2] observes that 'These axioms are not claimed to be independent, they are merely a convenient characterisation'; on the other hand an apparent excess of economy leads N. H. Kuiper [3] to define finite dimensional vector spaces by axioms A1, $\cdots$, A8; these axioms he then asserts force the existence of a set of generators $\{a_i\}_{i\in I}$ so that every $a \in V$ is of the shape $\sum_{i\in I} \lambda_i a_i$ $(\lambda_i \in F) \cdots$ a further axiom that there be a finite set of generators is then shown to imply A9 and A10.

Interestingly we will see that not only is A9 independent of A1, $\cdots$, A8 but indeed A10 is independent of A1, $\cdots$, A9. Further it is not the case that A1, $\cdots$, A8 suffice to force the existence of a set of generators in the required sense, so that this property is also independent of A1, $\cdots$, A8. In particular we see that it is not the case that for arbitrary $a \in V$ the vector $0 \cdot a$ is the zero vector in the sense of A9.

LEMMA. *In the presence of* A1, $\cdots$, A8,

A11: *for arbitrary* $a, b \in V$ $0 \cdot a = 0 \cdot b$ *is equivalent to* A9, A10.

For if A1, $\cdots$, A8, A11 then

$$0 \cdot a + b = 0 \cdot b + b = 0 \cdot b + 1 \cdot b = (0+1)b = 1 \cdot b = b$$
$$a + (-1)a = 1 \cdot a + (-1)a = (1 + (-1))a = 0 \cdot a$$

and we have A9 with $0 \cdot a = o = 0 \cdot b$, and A10 with $-a = (-1)a$.

Conversely if A1, $\cdots$, A8, A9, A10

$$\begin{aligned} 0 \cdot a + b &= o + (0 \cdot a + b) = (o + 0 \cdot a) + b = \big((a + (-a)) + 0 \cdot a\big) + b \\ &= \big((a + 0 \cdot a) + (-a)\big) + b = \big((1 \cdot a + 0 \cdot a) + (-a)\big) + b \\ &= \big((1+0)a + (-a)\big) + b = \big(1 \cdot a + (-a)\big) + b \\ &= \big(a + (-a)\big) + b = o + b = b \end{aligned}$$

hence in particular for arbitrary $a \in V$

$$o = 0 \cdot a + o = 0 + 0 \cdot a = 0 \cdot a$$

and we have A11.

It is of course an immediate consequence of the lemma that to prove the independence of A9, A10 from A1, $\cdots$ A8 we need only construct a system which has A1, $\cdots$ A8 but lacks A11. Such a system S, which we call a *semi-vector space*, will be an additive commutative semigroup by A1, A3, A4 with 'idempotents' $0 \cdot a$ for each $a \in V$ (for $0 \cdot a + 0 \cdot a = 0 \cdot a$) by A2, A7.

THEOREM 1. *A semi-vector space S is the disjoint union of maximal sub-vector spaces V_a, V_b, $\cdots$ called the components of S, with the property that a, b are in the same component V of S if and only if* $0 \cdot a = 0 \cdot b$.

By Zorn's Lemma we may easily prove this result directly. We however observe that it is an immediate result of Theorem 1.11 of A. H. Clifford and G. B. Preston 'Algebraic Theory of Semigroups' [1] whereby each (distinct) idempotent lies in its own maximal subgroup of the semigroup

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S. Conversely by A2, A5, A7 each $a \in S$ lies in the maximal subgroup containing the idempotent $0 \cdot a$. Finally each subgroup is by A4, $\cdots$, A8 a sub-vector space of S.

We may represent each component $V_a, V_b, \cdots$ of S by an arbitrary element $a, b, \cdots$ therein, and let Y be the set of such representatives. We observe that the sum $a'+b'$ of arbitrary $a' \in V_a$, $b' \in V_b$ always lies in the unique component V_c whose idempotent is $0 \cdot c = 0 \cdot a + 0 \cdot b$. Then Y becomes a (lower) semilattice by defining the meet $a \wedge b$ of two elements, $a, b \in Y$ as that element $c \in Y$ such that $V_a + V_b \subseteqq V_c$; the partial ordering so induced in Y is (isomorphic to) the *natural partial ordering* of the idempotents of S; i.e. $0 \cdot c \leqq 0 \cdot a$ means $0 \cdot a + 0 \cdot c = 0 \cdot c$ (see Section 18 of [1]). We thus have

THEOREM 1'. *$S = U\{V_a : a \in Y\}$ is the union of the semilattice Y of maximal sub-vector spaces (components) of S; Y is the semilattice of idempotents of S (more exactly, isomorphic thereto).*

For completeness we mention the properties of the 'fine-structure' of S. We write $a \geqq b$ $a, b \in Y$ whenever $0 \cdot a \geqq 0 \cdot b$ or equivalently $V_a + V_b \subseteqq V_b$.

We define for $a \geqq b$ the mapping $\omega_{b,a} a' = 0 \cdot b + a'$ $(a' \in V_a)$. Then $\omega_{b,a}$ is a linear transformation of the vector space V_a into V_b; further if $a \geqq b \geqq c$ then $\omega_{c,b}\omega_{b,a} = \omega_{c,a}$, and $\omega_{a,a}$ is the identy mapping of V_a. Next we note that if $a' \in V_a$, $b' \in V_b$ then $a'+b' = \omega_{c,a}a' + \omega_{c,b}b'$ where $c = a \wedge b$. As an immediate consequence of Theorem 4.11 of [1] we then have

THEOREM 2. *Let Y be any semilattice and to each element a of Y assign a vector space V_a over the field F, such that V_a and V_b are disjoint if $a \neq b$ in Y. To each pair $a, b \in Y$ such that $a > b$ assign a linear transformation $\omega_{b,a}$ of V_a into V_b such that if $a > b > c$ then*

$$\omega_{c,b}\omega_{b,a} = \omega_{c,a}.$$

Let $\omega_{a,a}$ be the identity automorphism of V_a. Let S be the union of all the vector spaces $V_a (a \in Y)$ and define the sum of any two elements a', b' $(a' \in V_a, b' \in V_b)$ by

$$a'+b' = \omega_{c,a}a' + \omega_{c,b}b'$$

where c is the meet $a \wedge b$ of a, b in the semilattice Y.

Then S is a semi-vector space over F. Conversely every such semi-vector space can be constructed in this manner.

Clearly every semi-vector space with more than one component, and with semilattice Y not containing z such that $z \geqq a$ for all $a \in Y$, is an algebraic structure with properties A1, $\cdots$, A8 but lacking A9 and A10. On the other hand if Y contains a maximal element z then the idempotent

$0 \cdot z'$ $(z' \in V_z)$ is in fact a unique o of S in the sense of A9, and thus we obtain structures with properties A1, $\cdots$ A9 but without A10. Thus the axioms A9, A10 are independent of A1, $\cdots$ A8 and axiom A10 is independent of A1, $\cdots$, A9.

As trivial examples of semi-vector spaces we mention the space S_1 with elements $0 \cdot a$, $0 \cdot b$, $0 \cdot c$ and addition $0 \cdot a + 0 \cdot b = 0 \cdot c$, which has A1, $\cdots$ A8 but lacks A9, A10. On the other hand the subspace S_2 with elements $0 \cdot a$, $0 \cdot c$ and $0 \cdot a + 0 \cdot c = 0 \cdot c$ is a semi-vector space with zero $0 \cdot a$ but without additive inverse for $0 \cdot c$, and thus has A1, $\cdots$ A9 but lacks A10. S_1 and S_2 are of course semi-vector spaces over any field F with zero 0.

We also observe that a semi-vector space with more than one component does not have a set of generators $\{a_i\}_{i \in I}$ such that every $a \in S$ is of the shape $\sum_{i \in I} \lambda_i a_i$ $(\lambda_i \in F)$, for every such sum lies in the unique component containing the idempotent $\sum_{i \in I} 0 \cdot a_i$. Thus a semivector space with a set of generators is a vector-space.

Finally we mention that the Weber-Huntington axioms for a group [4] assure that the property: for any a, $b \in S$ there is a $c \in S$ such that $a + c = b$; makes a semi-vector space S a vector space. We can see that this is so by observing that this property cannot hold in a space with more than one component; in the presence of A1, $\cdots$, A8 (indeed in the presence of A1, A3, A4) this property is thus equivalent to A9, A10.

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References

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[4] E. V. Huntington, 'Simplified definition of a group' *Bull. Amer. Math. Soc.* 8 (1901—1902), 296—300.

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