

ON A CLASS OF SUPERSOLUBLE GROUPS

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Abstract

A subgroup H of a finite group G is said to be *S-semipermutable* in G if H permutes with every Sylow q -subgroup of G for all primes q not dividing $|H|$. A finite group G is an MS-group if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G . The aim of the present paper is to characterise the finite MS-groups.

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1. Introduction

In the following, G always denotes a finite group. Recall that subgroups H and K of G are said to *permute* if HK is a subgroup of G and that a subgroup H of G is said to be *permutable* in G if H permutes with all subgroups of G .

Various generalisations of permutability have been defined and studied and, in particular, we mention the S-semipermutability. A subgroup H is said to be *S-semipermutable* in G if H permutes with every Sylow q -subgroup of G for all primes q not dividing $|H|$. This subgroup embedding property has been extensively studied recently (see for instance [1, 4, 7, 9]). Most of these papers concern situations where many subgroups (for instance all maximal subgroups of the Sylow subgroups) have the stated property. Thus we say that a group G is an MS-group if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G .

The main aim of this paper is to characterise the MS-groups.

2. Preliminary results

In this section, we collect the definitions and results which are needed to prove our main theorems.

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We shall adhere to the notation used in [2]: this book will be the main reference for terminology and results on permutability.

A subgroup H is permutable in a group G if and only if H permutes with every p -subgroup of G for every prime p (see for instance [2, Theorem 1.2.2]). A less restrictive subgroup embedding property is the S-permutability introduced by Kegel [5] and defined in the following way.

DEFINITION 2.1. A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p -subgroup of G for every prime p .

Note that we are not considering all p -subgroups, but just the maximal ones, that is, the Sylow p -subgroups.

In recent years there has been widespread interest in the transitivity of normality, permutability and S-permutability.

DEFINITION 2.2.

- (1) A group G is a *T-group* if normality is a transitive relation in G , that is, if every subnormal subgroup of G is normal in G .
- (2) A group G is a *PT-group* if permutability is a transitive relation in G , that is, if H is permutable in K and K is permutable in G , then H is permutable in G .
- (3) A group G is a *PST-group* if S-permutability is a transitive relation in G , that is, if H is S-permutable in K and K is S-permutable in G , then H is S-permutable in G .

If H is S-permutable in G , it is known that H must be subnormal in G [2, Theorem 1.2.14(3)]. Therefore, a group G is a PST-group (respectively, a PT-group) if and only if every subnormal subgroup is S-permutable (respectively, permutable) in G .

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular p -groups) and PST does not imply PT (nonmodular p -groups).

A less restrictive class of groups is the class of T_0 -groups which has been studied in [3, 6, 8].

DEFINITION 2.3. A group G is called a *T_0 -group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

The group in Example 4.2 below is a soluble T_0 -group which is not a PST-group. Soluble T_0 -groups are closely related to PST-groups, as the following result shows.

THEOREM 2.4 [6, Theorems 5 and 7 and Corollary 3]. *Let G be a soluble T_0 -group with nilpotent residual $L = \gamma_\infty(G)$. Then:*

- (1) G is supersoluble;
- (2) L is a nilpotent Hall subgroup of G ;
- (3) if L is abelian, then G is a PST-group.

Here the *nilpotent residual* $\gamma_\infty(G)$ of a group G is the smallest normal subgroup N of G such that G/N is nilpotent, that is, the limit of the lower central series of G defined by $\gamma_1(G) = G$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$.

It is known that S-semipermutability is not transitive. Hence it is natural to consider the following class of groups.

DEFINITION 2.5. A group G is called a *BT-group* if S-semipermutability is a transitive relation in G , that is, if H is S-semipermutable in K and K is S-semipermutable in G , then H is S-semipermutable in G .

This class was introduced and characterised by Wang *et al.* in [9]. Further contributions were presented in [1].

THEOREM 2.6 [9, Theorem 3.1]. *Let G be a group. The following statements are equivalent:*

- (1) G is a soluble BT-group;
- (2) every subgroup of G is S-semipermutable;
- (3) G is a soluble PST-group and if p and q are distinct prime divisors of the order of G not dividing the order of the nilpotent residual of G , then $[G_p, G_q] = 1$, where $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$.

The group presented in Example 4.1 below is an MS-group which is not a soluble BT-group. Furthermore, Example 4.2 shows that the classes of T_0 -groups and MS-groups are not closed under taking subgroups.

The first remarkable fact concerning the structure of an MS-group can be found in [7]. It is proved there that every MS-group is supersoluble.

THEOREM 2.7 [7, Corollary 9]. *Let G be an MS-group. Then G is supersoluble.*

More recently, the second and fourth authors proved the following theorem.

THEOREM 2.8 [4, Theorems A, B and C]. *Let G be an MS-group with nilpotent residual $L = \gamma_\infty(G)$. Then:*

- (1) if N is a normal subgroup of G , then G/N is an MS-group;
- (2) L is a nilpotent Hall subgroup of G ;
- (3) G is a soluble T_0 -group.

It is well known that the nilpotent residual of a supersoluble group is nilpotent. Hence the nilpotency of L in Theorem 2.8 is a consequence of Theorem 2.7.

Throughout this paper we will use the following notation. Let G be a group whose nilpotent residual $L = \gamma_\infty(G)$ is a Hall subgroup of G . Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that if P is a Sylow p -subgroup of G , then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that if Q is a Sylow q -subgroup of G , then Q has only one maximal subgroup, or equivalently, Q is cyclic.

3. The main results

Our first main result is a characterisation theorem.

THEOREM 3.1. *Let G be a group with nilpotent residual $L = \gamma_\infty(G)$. Then G is an MS-group if and only if G satisfies the following properties.*

- (1) G is a T_0 -group.
- (2) L is a nilpotent Hall subgroup of G .
- (3) If $p \in \pi$ and $P \in \text{Syl}_p(G)$, then a maximal subgroup of P is normal in G .
- (4) Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$, then $[P, Q] = 1$.
- (5) Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ and M is the maximal subgroup of P , then $QM = MQ$ is a nilpotent subgroup of G .

PROOF. Let G be an MS-group. By Theorems 2.7 and 2.8, G is a supersoluble T_0 -group whose nilpotent residual L is a nilpotent Hall subgroup of G . Thus properties (1) and (2) hold.

Let $\pi = \pi(L)$ and let $p \in \pi$. Further, let P be a Sylow p -subgroup of G and let M be a maximal subgroup of P . Then $M \leq P \trianglelefteq L$ and M is normal in L and subnormal in G . Let $q \in \theta = \pi'$ and note that MQ is a subgroup of G for a given Sylow q -subgroup Q of G . Moreover M is a Sylow p -subgroup of MQ and so M is a normal subgroup of MQ . Consequently M is normalised by P and each Sylow q -subgroup Q of G , so M is a normal subgroup of G and property (3) holds.

Let X be a Hall θ -subgroup of G and note that $G = L \rtimes X$, the semidirect product of L by X , and X is nilpotent. Let t be a prime from θ_N and r be a prime from θ . Also let $T \in \text{Syl}_t(G)$ and $R \in \text{Syl}_r(G)$. Let M_1 and M_2 be two distinct maximal subgroups of $T = \langle M_1, M_2 \rangle$. Since G is an MS-group, $M_1R = RM_1$ and $M_2R = RM_2$. Applying [2, Theorem 1.2.2], we have $RT = TR$. Observe that TR is a θ -subgroup of G and so TR is nilpotent since TR is a subgroup of some conjugate of X . Therefore, $[T, R] = 1$ and property (4) holds.

Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. Further, let $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$. If M is the maximal subgroup of P , then $QM = MQ$ is a nilpotent θ -subgroup of G . Thus property (5) holds.

Let G be a group satisfying properties (1)–(5). We are to show that G is an MS-group. By properties (1) and (2), G is a soluble T_0 -group, and by Theorem 2.4, G is thus supersoluble.

Let $p \in \pi = \pi(L)$, let P be a Sylow p -subgroup of G , and let M be a maximal subgroup of P . Then M is a normal subgroup of G by property (3) and clearly P is a normal subgroup of G . This means that M permutes with every Sylow subgroup of G and P permutes with every maximal subgroup of any Sylow subgroup of G .

Let p and q be distinct primes from θ and let $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$. We consider a maximal subgroup M of P . Note that $\theta = \theta_N \cup \theta_C$ and $\theta_N \cap \theta_C = \emptyset$, the

empty set. If $p \in \theta_N$, then by property (4), $[P, Q] = 1$, so that $MQ = QM$. Hence assume $p \in \theta_C$. Then, by property (5), $MQ = QM$.

Therefore, every maximal subgroup of any Sylow subgroup of G is S-semi-permutable in G and G is an MS-group. □

The second and fourth authors in [4] posed the following two questions.

- (1) When is a soluble PST-group an MS-group?
- (2) When is a soluble PST-group which is also an MS-group a BT-group?

Using Theorem 3.1 we are able to answer the first question and provide a partial answer to the second.

THEOREM 3.2. *Let G be a soluble PST-group. Then G is an MS-group if and only if G satisfies (4) and (5) of Theorem 3.1.*

PROOF. Let G be a soluble PST-group with nilpotent residual $L = \gamma_\infty(G)$. By [6, Lemma 5], $G/\Phi(G)$ is a T-group and so G is a T_0 -group. Notice that (1), (2) and (3) of Theorem 3.1 are satisfied for the group G .

Assume that G is an MS-group. By Theorem 3.1, (4) and (5) are satisfied by G .

Conversely, assume that (4) and (5) of Theorem 3.1 are satisfied by G . By Theorem 3.1, G is an MS-group.

This completes the proof. □

The group given in Example 4.1 below is a soluble PST-group which is not an MS-group and the group given in Example 4.2 is an MS-group which is not a soluble PST-group.

THEOREM 3.3. *Let G be a soluble PST-group which is also an MS-group. If θ_C is the empty set, then G is a BT-group.*

PROOF. Let G be a soluble PST-group which is also an MS-group. Let $L = \gamma_\infty(G)$ be the nilpotent residual of G . By the Theorem of Agrawal [2, Theorem 2.1.8], L is an abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms. Recall that $\theta = \pi'$, where $\pi = \pi(L)$. Moreover $\theta = \theta_N$ if θ_C is empty. Let p and q be distinct primes from θ and let $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$. Note that since G is an MS-group, we have that G satisfies properties (4) and (5) of Theorem 3.1. Then $[G_p, G_q] = 1$ by property (4) of that theorem. Therefore, G is a BT-group by Theorem 2.6. This completes the proof of Theorem 3.3.

We remark that if θ contains only one prime, then G is a BT-group by [9, Corollary 3.4]. □

4. Examples

The following examples appear in [4]. For the sake of completeness, we list them here.

EXAMPLE 4.1. Let $G = \langle y, z, x \mid y^3 = z^2 = x^7 = 1, [y, z] = 1, x^y = x^2, x^z = x^{-1} \rangle$. Then $[(y)^x, z] \neq 1$ and G is a soluble group which is not a BT-group. However, G is an MS-group.

EXAMPLE 4.2. Let $G = \langle a, x, y \mid a^2 = x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1, x^a = x^{-1}, y^a = y^{-1} \rangle$. Then $H = \langle x, y \rangle$ is an extraspecial group of order 27 and exponent 3. Let $z = [x, y]$, so $z^a = z$. Then $\Phi(G) = \Phi(H) = \langle z \rangle = Z(G) = Z(H)$. Note that $G/\Phi(G)$ is a T-group so that G is a T_0 -group. The maximal subgroups of H are normal in G and it follows that G is an MS-group. Let $K = \langle x, z, a \rangle$. Then $\langle xz \rangle$ is a maximal subgroup of $\langle x, z \rangle$, the Sylow 3-subgroup of K . However, $\langle xz \rangle$ does not permute with $\langle a \rangle$ and hence $\langle xz \rangle$ is not an S-semipermutable subgroup of K . Therefore, K is not an MS-subgroup of G . Also note that $\Phi(K) = 1$ and so K is not a T-subgroup of G and K is not a T_0 -subgroup of G . Hence the class of soluble T_0 -groups is not closed under taking subgroups. Note that G is a soluble group which is not a PST-group.

EXAMPLE 4.3. Let $G = \langle y, z, x \mid y^9 = z^2 = x^{19^2} = 1, [y, z] = 1, x^y = x^{62}, x^z = x^{-1} \rangle$. Then the soluble group G is a PST-group, but G is not an MS-group since $[(y^2)^x, z] \neq 1$.

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