This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to W.O.J. Moser, University of Manitoba, Winnipeg, Manitoba.

ON THE ANALYTIC CONTINUATION OF THE ζ-FUNCTION

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It is well known that the Riemannian ζ-function can be extended into the whole region $s \neq 1$ by means of the Euler-MacLaurin summation formula. The author wishes to present a proof of this fact which is rather more elementary than those he is acquainted with.

Let $\mathcal{R} s > 1$ and put

$$\phi(s) = \zeta(s) - 1.$$

Repeated integration by parts yields

$$\begin{aligned}\phi(s) = {} & \frac{1}{s-1} - \frac{s}{2!}\phi(s+1) - \frac{s(s+1)}{3!}\phi(s+2) - \dots \\ & - \frac{s(s+1)\dots(s+k-2)}{k!}\phi(s+k-1) \\ & - \frac{s(s+1)\dots(s+k-1)}{k!}\sum_1^\infty \int_0^1 \frac{x^k\,dx}{(n+x)^{s+k}} .\end{aligned}$$

Replacing s by $s + m$ and k by $k - m$ $[m = 0, 1, \dots, k-1]$, we obtain the following system of equations.

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$$\phi(s) + \frac{s}{2!}\phi(s+1) + \frac{s(s+1)}{3!}\phi(s+2) + \dots + \frac{s(s+1)\dots(s+k-2)}{k!}\phi(s+k-1) = R_1,$$

$$\phi(s+1) + \frac{(s+1)}{2!}\phi(s+2) + \dots + \frac{(s+1)\dots(s+k-2)}{(k-1)!}\phi(s+k-1) = R_2,$$

$$\vdots$$

$$\phi(s+k-1) = R_k.$$

Here

$$R_1 = \frac{1}{s-1} - \frac{s(s+1)\dots(s+k-1)}{k!}\Sigma_1^\infty \int_0^1 \frac{x^k dx}{(n+x)^{s+k}},$$

$$R_2 = \frac{1}{s} - \frac{(s+1)\dots(s+k-1)}{(k-1)!}\Sigma_1^\infty \int_0^1 \frac{x^{k-1} dx}{(n+x)^{s+k}},$$

$$\vdots$$

$$R_k = \frac{1}{s+k-2} - (s+k-1)\Sigma_1^\infty \int_0^1 \frac{x dx}{(n+x)^{s+k}}.$$

Hence

$$\phi(s) = \begin{vmatrix} R_1 & \frac{s}{2!} & \frac{s(s+1)}{3!} & \dots & \frac{s(s+1)\dots(s+k-2)}{k!} \\ R_2 & 1 & \frac{s+1}{2!} & \dots & \frac{(s+1)\dots(s+k-2)}{(k-1)!} \\ R_3 & 0 & 1 & \dots & \frac{(s+2)\dots(s+k-2)}{(k-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_k & 0 & 0 & \dots & 1 \end{vmatrix}.$$

We now expand this determinant with respect to the first column. This yields

$$\phi(s) = R_1 - R_2 \cdot \frac{s}{2!} + R_3 \cdot \begin{vmatrix} \frac{s}{2!} & \frac{s(s+1)}{3!} \\ 1 & \frac{s+1}{2!} \end{vmatrix} \pm \dots$$

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$$+ (-1)^k R_{k-1} \cdot \begin{vmatrix} \frac{s}{2!} & \frac{s(s+1)}{3!} & \cdots & \frac{s(s+1)\ \ldots\ (s+k-3)}{(k-1)!} \\ 1 & \frac{s+1}{2!} & \cdots & \frac{(s+1)\ \ldots\ (s+k-3)}{(k-2)!} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \frac{s+k-3}{2!} \end{vmatrix}$$

$$+ (-1)^{k+1} R_k \cdot \begin{vmatrix} \frac{s}{2!} & \frac{s(s+1)}{3!} & \cdots & \frac{s(s+1)\ldots(s+k-3)}{(k-1)!} & \frac{s(s+1)\ldots(s+k-2)}{k!} \\ 1 & \frac{s+1}{2!} & \cdots & \frac{(s+1)\ldots(s+k-3)}{(k-2)!} & \frac{(s+1)\ldots(s+k-2)}{(k-1)!} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & \frac{s+k-2}{2!} \end{vmatrix}$$

or

$$\phi(s) = R_1 - s \cdot \frac{1}{2} R_2 + s(s+1) \cdot \begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} \\ 1 & \frac{1}{2!} \end{vmatrix} R_3$$

$$- s(s+1)(s+2) \begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & \frac{1}{2!} \end{vmatrix} R_4 \pm \ldots$$

$$+ (-1)^{k-1} \cdot s(s+1)\ldots(s+k-2) \begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{k!} \\ 1 & \frac{1}{2!} & \cdots & \frac{1}{(k-1)!} \\ 0 & 0 & \cdots & \frac{1}{2!} \end{vmatrix} R_k$$

$$= R_1 + \frac{B_1}{1!} s R_2 + \frac{B_2}{2!} s(s+1) R_3 + \frac{B_3}{3!} s(s+1)(s+2) R_4 + \ldots$$

$$+ \frac{B_{k-1}}{(k-1)!} s(s+1)\ \ldots\ (s+k-2) R_k .$$

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Here the B_n are the Bernoulli numbers. We thus obtain the required formulas

$$(1) \qquad \phi(s) = \frac{1}{s-1} + \frac{B_1}{1!} + \frac{B_2}{2!} s + \frac{B_3}{3!} s(s+1) + \dots$$

$$+ \frac{B_{k-1}}{(k-1)!} s(s+1) \dots (s+k-3)$$

$$- s(s+1) \dots (s+k-1) \Sigma_1^{\infty} n \int_0^1 \frac{P_k(x)dx}{(n+x)^{s+k}}$$

where

$$P_k(x) = \frac{B_0}{k!0!} x^k + \frac{B_1}{(k-1)!1!} x^{k-1} + \dots + \frac{B_{k-1}}{1!(k-1)!} x$$

is the k-th Bernoulli polynomial. The last term in (1) converges for $\mathcal{R} s > 1 - k$.

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