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TETRAVALENT s-TRANSITIVE GRAPHS OF ORDER TWICE A PRIME POWER

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Abstract

A graph is *s-transitive* if its automorphism group acts transitively on s-arcs but not on $(s+1)$-arcs in the graph. Let X be a connected tetravalent s-transitive graph of order twice a prime power. In this paper it is shown that $s = 1, 2, 3$ or 4. Furthermore, if $s = 2$, then X is a normal cover of one of the following graphs: the 4-cube, the complete graph of order 5, the complete bipartite graph $K_{5,5}$ minus a 1-factor, or $K_{7,7}$ minus a point-hyperplane incidence graph of the three-dimensional projective geometry $PG(2, 2)$; if $s = 3$, then X is a normal cover of the complete bipartite graph of order 4; if $s = 4$, then X is a normal cover of the point-hyperplane incidence graph of the three-dimensional projective geometry $PG(2, 3)$. As an application, we classify the tetravalent s-transitive graphs of order $2p^2$ for prime p.

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1. Introduction

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set and full automorphism group. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X, and $X(u)$ is the neighborhood of u in X, that is, the set of vertices adjacent to u in X. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph such that v_{i-1} is adjacent to v_i for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. For a subgroup G of the automorphism group $\text{Aut}(X)$ of a graph X, X is said to be (G, s)*-arc-transitive* or (G, s)*-regular* if G acts transitively or regularly on the set of s-arcs of X, and (G, s)*-transitive* if G acts transitively on the set of s-arcs but not on the set of $(s+1)$-arcs of X; in particular,

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if $G = \mathrm{Aut}(X)$, then X is simply said to be *s-arc-transitive*, *s-regular* or *s-transitive*, respectively. In particular, 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is *edge-transitive* if $\mathrm{Aut}(X)$ is transitive on $E(X)$.

Let X be a connected symmetric graph, and let $G \leq \mathrm{Aut}(X)$ be arc-transitive on X. For a normal subgroup N of G, the *quotient graph* X_N of X relative to the orbit set of N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between vertices lying in these two orbits. Assume further that X is $(G, 2)$-arc-transitive. If N is intransitive then either $X_N \cong K_2$, or X_N and X have the same valency. For the former case, X_N is sometimes called a *trivial* normal quotient; for the latter, X is called a *G-normal cover* of X_N. In particular, if $G = \mathrm{Aut}(X)$ then X is said to be a *normal cover* of X_N. If X has no nontrivial normal quotient with respect to G, X is said to be *G-basic*. An $\mathrm{Aut}(X)$-basic graph is simply called a *basic* graph. Clearly, X is G-basic if and only if each nontrivial normal subgroup of G has at most two orbits on $V(X)$. A general approach to the characterization of 2-arc-transitive graphs is to investigate G-basic 2-arc-transitive graphs and their normal covers (see [24]).

Arc-transitive or s-transitive graphs with small valencies have received considerable attention in the literature. For instance, Tutte [26] initiated the investigation of cubic s-transitive graphs by proving that there exist no finite s-transitive cubic graphs for $s \geq 6$, and there has been much subsequent work in this area (see [5, 7, 9–13, 23]). Gardiner and Praeger [14, 15] generally explored tetravalent symmetric graphs by considering their automorphism groups. A lot of work has been done on tetravalent s-transitive Cayley graphs as a part of a more general problem dealing with the investigation of the tetravalent edge-transitive Cayley graphs (see [8, 21], for example). Recently, Li *et al.* [20] classified all vertex-primitive symmetric graphs of valency 3 or 4.

There also has been a lot of interest in classifications of s-transitive graphs of small valencies with given orders. Let p be a prime. The classification of s-transitive graphs of order np and of valency 3 or 4 can be obtained from [3, 4, 27], where $1 \leq n \leq 3$. Feng *et al.* [10, 12, 13] classified cubic s-transitive graphs of order np or np^2 with $n = 4$, 6, 8 or 10. In [10], Feng *et al.* investigated the automorphism groups of cubic s-transitive graphs of order $2p^n$ and, as an application, classified the cubic s-transitive graphs of order $2p^2$.

In this paper, we aim to study the tetravalent s-transitive graphs of order $2p^n$. Let X be a connected tetravalent s-transitive graph of order $2p^n$. It is shown that $s = 1, 2, 3$ or 4. Furthermore, if $s = 2$, then X is a normal cover of one of the following graphs: Q_4 (the 4-cube), K_5 (the complete graph of order 5), $K_{5,5} - 5K_2$ (the complete bipartite graph $K_{5,5}$ minus a 1-factor), or $B'(PG(2, 2))$ ($K_{7,7}$ minus the point-hyperplane incidence graph of a three-dimensional projective geometry $PG(2, 2)$); if $s = 3$, then X is a normal cover of the complete bipartite graph $K_{4,4}$; if $s = 4$, then X is a normal cover of $B(PG(2, 3))$ (the point-hyperplane incidence graph of the three-dimensional projective geometry $PG(2, 3)$). As an application, we classify connected tetravalent s-transitive graphs of order $2p^2$ for each prime p. It follows from this classification that, with the exception of $K_{4,4}$, all such graphs are 1-transitive.

To end this section, we define a Cayley graph. Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if G is transitive and semiregular on Ω. For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, for $x \in G$. The homomorphism R is the *right regular representation* of G; the image $R(G) = \{R(g) \mid g \in G\}$ of G is a regular permutation group acting on the elements of G. It is easy to see that $R(G)$ is isomorphic to G, which can therefore be regarded as a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Thus the Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [28, Proposition 1.5] proved that $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$. A graph is called a *circulant graph*, or a *circulant* for short, if it is a Cayley graph on a cyclic group.

2. Graph constructions and preliminaries

In this section, we introduce some tetravalent s-transitive graphs of order twice a prime power, and collect some preliminary results which will be used later in the paper. Throughout this paper we denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n, K_n and nK_1 the cycle, the complete graph and the null graph of order n, respectively. For two groups M and N, $N \le M$ means that N is a subgroup of M, $N < M$ means that N is a proper subgroup of M, $N \rtimes M$ denotes a semidirect product of N by M, and $\mathrm{Aut}(N)$ denotes the automorphism group of N.

The first example is the lexicographic product of C_{p^2} and $2K_1$.

EXAMPLE 2.1. Let p be a prime. The *lexicographic product* $C_{p^2}[2K_1]$ is defined as the graph with vertex set $V(C_{p^2}[2K_1]) = V(C_{p^2}) \times V(2K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_{p^2}[2K_1])$, u is adjacent to v in $C_{p^2}[2K_1]$ whenever $\{x_1, x_2\} \in E(C_{p^2})$.

Note that $C_4[2K_1] \cong K_{4,4}$ is 3-transitive and $\mathrm{Aut}(C_4[2K_1]) \cong (S_4 \times S_4) \rtimes \mathbb{Z}_2$ if $p = 2$, and the tetravalent graph $C_{p^2}[2K_1]$ is 1-transitive and $\mathrm{Aut}(C_{p^2}[2K_1]) \cong \mathbb{Z}_2^{p^2} \rtimes D_{2p^2}$ if $p > 2$. From [30, Table 1] we observe that $C_{p^2}[2K_1]$ is isomorphic to $\mathrm{Cay}(G, S)$, where

$$G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_2 \quad \text{and} \quad S = \{a, a^{-1}, ab, a^{-1}b\}.$$

Next, we introduce some tetravalent symmetric Cayley graphs on an abelian group of order $2p^2$ for prime p.

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EXAMPLE 2.2. Let p be a prime congruent to $1 \bmod 4$, and let $H = \langle h \rangle$ be the unique subgroup of order 4 of $\mathbb{Z}_{2p^2}^*$. Define $X_{2p^2}^0 = \mathrm{Cay}(G, \{a, a^{-1}, a^h, a^{h^3}\})$, where $G = \langle a \rangle \cong \mathbb{Z}_{2p^2}$.

Xu [29, Theorems 2 and 3] classified tetravalent symmetric circulant graphs, and one may deduce the following proposition.

PROPOSITION 2.3. *Let p be a prime. A connected tetravalent circulant graph X of order $2p^2$ is symmetric and normal if and only if $p \equiv 1 \bmod 4$ and $X \cong X_{2p^2}^0$. Furthermore,* $\mathrm{Aut}(X_{2p^2}^0) \cong \mathbb{Z}_{2p^2} \rtimes \mathbb{Z}_4$.

EXAMPLE 2.4. Let p be an odd prime and

$$G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_2.$$

Define $X_{2p^2}^1 = \mathrm{Cay}(G, \{ca, ca^{-1}, cb, cb^{-1}\})$.

From [30, Theorem 3.3 and Proposition 3.3(iv)], one may deduce the following proposition.

PROPOSITION 2.5. *Let p be an odd prime, and*

$$G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_2.$$

A connected tetravalent Cayley graph X on G is normal and symmetric if and only if $X \cong X_{2p^2}^1$. Furthermore, $X_{2p^2}^1$ is 1-transitive and $\mathrm{Aut}(X_{2p^2}^1) \cong G \rtimes D_8$.

The following three infinite families of graphs were constructed by Gardiner and Praeger [14, Definitions 3.2, 4.2 and 4.3].

EXAMPLE 2.6. Let p be an odd prime. The graph $\bar{X}_{2p^2}^1$ is defined to have vertex set $\mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$ with two vertices $(0, (x_1, y_1))$ and $(1, (x_2, y_2))$ being adjacent if and only if

$$(x_2, y_2) - (x_1, y_1) \in \{\pm(1, 0), \pm(0, 1)\}.$$

Note that the map defined by

$$(i, (x, y)) \mapsto c^i a^x b^y \quad \forall (i, (x, y)) \in \mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p),$$

is an isomorphism from $\bar{X}_{2p^2}^1$ to $X_{2p^2}^1$. Thus, $\bar{X}_{2p^2}^1 \cong X_{2p^2}^1$.

EXAMPLE 2.7. Let p be a prime congruent to 1 mod 4, and let $\pm\varepsilon$ be the two elements of order 4 of $\mathbb{Z}_p^*$. The graph $X_{2p^2}^2$ is defined to have vertex set $\mathbb{Z}_2 \times (\mathbb{Z}_p \times \mathbb{Z}_p)$ with two vertices $(0, (x_1, y_1))$ and $(1, (x_2, y_2))$ being adjacent if and only if

$$(x_2, y_2) - (x_1, y_1) \in \{(1, 1), (-1, \varepsilon), (1, -1), (-1, -\varepsilon)\}.$$

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By [31, Lemma 3.1], $X^2_{2p^2} \cong \mathrm{Cay}(G(2p^2), S)$, where

$$G(2p^2) = \langle a, b, c \mid a^p = b^p = c^2 = 1, cac = a^{-1}, cbc = b^{-1}, ab = ba\rangle$$

is the *generalized dihedral group* of order $2p^2$ and

$$S = \{cab, ca^{-1}b^{\varepsilon}, cab^{-1}, ca^{-1}b^{-\varepsilon}\}.$$

Furthermore, $\mathrm{Aut}(X^2_{2p^2}) \cong G(2p^2) \rtimes \mathbb{Z}_4$, implying that the tetravalent graph $X^2_{2p^2}$ is 1-regular.

EXAMPLE 2.8. Let p be an odd prime. The graph $X^3_{2p^3}$ has vertex set $\mathbb{Z}_2 \times \mathbb{Z}_p^3$, with two vertices $(0, (x_0, y_0, z_0))$ and $(1, (x_1, y_1, z_1))$ being adjacent if and only if

$$(x_1, y_1, z_1) - (x_0, y_0, z_0) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}.$$

LEMMA 2.9. *Let p be an odd prime, and let*

$$G(2p^3) = \langle a, b, c, d \mid a^p = b^p = c^p = d^2 = 1, dad = a^{-1}, dbd = b^{-1}, dcd = c^{-1}, ab = ba, ac = ca, bc = cb\rangle$$

be the generalized dihedral group *of order $2p^3$. Set $S = \{da, db, dc, d(abc)^{-1}\}$. Then $X^3_{2p^3} \cong \mathrm{Cay}(G(2p^3), S)$. Also, $X^3_{2p^3}$ is 2-transitive and $\mathrm{Aut}(X^3_{2p^3}) \cong G(2p^3) \rtimes S_4$.*

PROOF. Let $G = G(2p^3)$, $X = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}(X)$. Note that $X^3_{2p^3}$ has vertex set $\mathbb{Z}_2 \times \mathbb{Z}_p^3$ with two vertices $(0, (x_0, y_0, z_0))$ and $(1, (x_1, y_1, z_1))$ adjacent whenever

$$(x_1, y_1, z_1) - (x_0, y_0, z_0) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}.$$

It is easy to see that the map defined by

$$(i, (x, y, z)) \mapsto d^i a^x b^y c^z, \quad \text{for } (i, (x, y, z)) \in \mathbb{Z}_2 \times \mathbb{Z}_p^3,$$

is an isomorphism from $X^3_{2p^3}$ to X. Thus, $X \cong X^3_{2p^3}$.

Let α be the automorphism of G induced by $a \mapsto b$, $b \mapsto c$, $c \mapsto (abc)^{-1}$, $d \mapsto d$. Similarly, β and γ are the automorphisms of G induced by $a \mapsto b$, $b \mapsto c$, $c \mapsto a$, $d \mapsto d$, and by $a \mapsto b$, $b \mapsto a$, $c \mapsto c$, $d \mapsto d$. It is easy to see that α, β and γ fix S setwise and hence they are automorphisms of X. Furthermore, α cyclicly permutes the elements in S, β fixes $d(abc)^{-1}$ and cyclicly permutes the remaining three elements in S, and γ fixes c and $d(abc)^{-1}$ and interchanges da and db. Since S generates $G(2p^3)$, $\langle \alpha, \beta, \gamma\rangle$ acts faithfully on S. Thus, $\langle \alpha, \beta, \gamma\rangle \cong S_4$ and $R(G(2p^3)) \rtimes \langle \alpha, \beta, \gamma\rangle$ is 2-arc-transitive on X. If $p = 3$, it can be shown with the

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help of the computer software package MAGMA [2] that $A = \mathrm{Aut}(X)$ has order 1296, which implies that $A = R(G(2 \cdot 3^3)) \rtimes \langle \alpha, \beta, \gamma \rangle$ and X is 2-transitive. Let $p > 3$. Then X has girth 6, and there are exactly two girth cycles passing through any given 2-arc. Thus, X is not 3-arc-transitive because otherwise there are at least three girth cycles passing through any given 2-arc. Furthermore, $A_1^* = 1$, where A_1^* is the subgroup of A_1 fixing S pointwise. As a result, X is 2-transitive and $A = R(G(2p^3)) \rtimes \langle \alpha, \beta, \gamma \rangle$. □

The following proposition is taken from [14, Theorem 1.3].

PROPOSITION 2.10. *Let X be a connected tetravalent symmetric graph such that* $\mathrm{Aut}(X)$ *has an elementary abelian normal subgroup $N \cong \mathbb{Z}_p^n$, where $n > 1$ and p is an odd prime. If N has exactly two orbits on vertices, then X is isomorphic to $X^1_{2p^2}$, $X^2_{2p^2}$ or $X^3_{2p^3}$.*

EXAMPLE 2.11. Let n be an integer greater than 2, and q be a prime power. Let $PG(n-1, q)$ be the $(n-1)$-dimensional projective geometry over the field $GF(q)$. Denote by P and H the point set and the hyperplane set of $PG(n-1, q)$. The *point-hyperplane incidence graph* of $PG(n-1, q)$ is defined to have vertex set $P \cup H$ and edge set $\{\{x, y\} \mid x \in P, y \in H, x \in y\}$. The graph $B'(PG(n-1, q))$ is defined to have vertex set $P \cup H$ and edge set $\{\{x, y\} \mid x \in P, y \in H, x \notin y\}$.

The following proposition is taken from [4, Theorem 2.4].

PROPOSITION 2.12. *Let p be a prime. A connected tetravalent symmetric graph of order $2p$ is either 1-transitive or isomorphic to one of the following graphs.*

X	$\lvert X \rvert$	s-transitive	$\mathrm{Aut}(X)$
$K_{5,5} - 5K_2$	10	2-transitive	$S_5 \times \mathbb{Z}_2$
$B'(PG(2, 2))$	14	2-transitive	$\mathrm{PGL}(3, 2) \,.\, 2$
$B(PG(2, 3))$	26	4-transitive	$\mathrm{PSL}(3, 3) \rtimes \mathbb{Z}_2$

EXAMPLE 2.13. The 4-cube Q_4 may be viewed as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^4, S)$, where $\mathbb{Z}_2^4 = \langle s_1 \rangle \times \langle s_2 \rangle \times \langle s_3 \rangle \times \langle s_4 \rangle$ and $S = \{s_1, s_2, s_3, s_4\}$.

The following proposition can be obtained from [18], where the connected 2-arc-transitive graphs of order a prime power were characterized.

PROPOSITION 2.14. *Let X be a connected tetravalent s-transitive graph of order a prime power. Then $s = 1, 2$ or 3. Furthermore, if $s = 2$, then X is a normal cover of the graphs $K_{4,4}$, Q_4 or K_5; if $s = 3$, then X is a normal cover of the complete bipartite $K_{4,4}$.*

The following proposition is due to Praeger *et al.*; see [14, Theorem 1.1] and [24].

PROPOSITION 2.15. *Let X be a connected tetravalent $(G, 1)$-arc-transitive graph. For each normal subgroup N of G, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N acts transitively on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, the quotient graph X_N is a cycle of length r, and G induces the full automorphism group D_{2r} on X_N;*
(4) *N has $r \geq 5$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected tetravalent G/N-symmetric graph, and X is a G-normal cover of X_N.*

Moreover, if X is also $(G, 2)$-arc-transitive, then case (3) cannot happen.

The next proposition characterizes the vertex stabilizer of the connected tetravalent s-transitive graphs, which can be deduced from [21, Lemma 2.5], [20, Proposition 2.8], or [19, Theorem 2.2].

PROPOSITION 2.16. *Let X be a connected tetravalent (G, s)-transitive graph. Let G_v be the stabilizer of a vertex $v \in V(X)$ in G. Then $s = 1, 2, 3, 4$ or 7. Furthermore, either G_v is a 2-group for $s = 1$, or G_v is isomorphic to A_4 or S_4 for $s = 2$; $A_4 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times S_4$, $S_3 \times S_4$ for $s = 3$; $\mathbb{Z}_3^2 \rtimes \mathrm{GL}(2, 3)$ for $s = 4$; or $[3^5] \rtimes \mathrm{GL}(2, 3)$ for $s = 7$, where $[3^5]$ represents an arbitrary group of order 3^5.*

3. Main results

In this section, we shall characterize connected tetravalent s-transitive graphs of order twice a prime power. To do this, we need the following lemma.

LEMMA 3.1. *Let p be a prime and let $n > 1$ be an integer. Let X be a connected tetravalent graph of order $2p^n$. If $G \leq \mathrm{Aut}(X)$ is transitive on the arc set of X, then every minimal normal subgroup of G is solvable.*

PROOF. Let $v \in V(X)$. Since G is arc-transitive on X, by Proposition 2.16, G_v either is a 2-group or has order dividing $2^4 \cdot 3^6$. It follows that $|G| \mid 2^5 \cdot 3^6 \cdot p^n$ or $|G| = 2^{m+1} \cdot p^n$ for some integer m. Let N be a minimal normal subgroup of G.

Suppose that N is nonsolvable. Then $p > 3$ and $|G| \mid 2^5 \cdot 3^6 \cdot p^n$ because a $\{2, p\}$-group is solvable by a theorem of Burnside [25, Theorem 8.5.3]. It follows that X is $(G, 2)$-arc-transitive. Furthermore, $3 \mid |N_v|$ for any $v \in V(X)$, because $p > 3$, and the 2-arc-transitivity of G implies that N_v acts transitively on $X(v)$ because $N_v \trianglelefteq G_v$. By Proposition 2.15, N has at most two orbits on $V(X)$. Hence, p^n divides $|N|$. Since N is minimal, it is a product of isomorphic nonabelian simple groups. Since $|N| \mid 2^5 \cdot 3^6 \cdot p^n$, by [16, pp. 12–14], each direct factor of N is one of the following:

$$A_5, A_6, \mathrm{PSL}(2, 7), \mathrm{PSL}(2, 8), \mathrm{PSL}(2, 17), \mathrm{PSL}(3, 3), \mathrm{PSU}(3, 3) \text{ and } \mathrm{PSU}(4, 2). \tag{3.1}$$

An inspection of the orders of such groups gives $n = 2$ and $N \cong A_5 \times A_5$. If N is transitive on $V(X)$, then X must be $(N, 2)$-transitive. Clearly, a direct factor T of N

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has at least $p = 5$ orbits on $V(X)$. This forces T to be semiregular on $V(X)$ which is impossible because $|V(X)| = 2p^2$. Thus, N has exactly two orbits on $V(X)$. Then $N_v \cong A_4 \times A_4$. This is also impossible by Proposition 2.14. Thus, N is solvable. □

THEOREM 3.2. *Let p be a prime and s a positive integer. Let X be a connected tetravalent s-transitive graph of order $2p^n$. Then $s = 1, 2, 3$ or 4. Assume also that $s \geq 2$. Then X is a normal cover of Y, where s and Y are given in the following table.*

s	Y
2	$K_{4,4}$, Q_4, K_5, $K_{5,5} - 5K_2$, $B'(PG(2, 2))$ or $X^3_{2p^3}$
3	$K_{4,4}$
4	$B(PG(2, 3))$

PROOF. By Proposition 2.12, the theorem is true for $n = 1$. Thus, one may assume that $n > 1$ and $s > 1$. Let $A = \mathrm{Aut}(X)$. For $p = 2$, by Proposition 2.14, either $s = 2$ and X is a normal cover of the complete bipartite graph $K_{4,4}$ or the 4-cube Q_4, or $s = 3$ and X is a normal cover of $K_{4,4}$. Thus, assume $p > 2$.

Let $M \trianglelefteq A$ be maximal subject to M having at least three orbits on $V(X)$. Since X is 2-arc-transitive, by Proposition 2.15, the quotient graph X_M of X relative to the orbit set of M is a tetravalent $(A/M, s)$-transitive graph of order $2p^r$ or p^t with $1 \leq r, t \leq n$. Furthermore, X is a normal cover of X_M. To complete the proof, it suffices to show that either $s = 2$ and $X_M \cong K_5$, $K_{5,5} - 5K_2$, $B'(PG(2, 2))$ or $X^3_{2p^3}$, or $s = 4$ and $X_M \cong B(PG(2, 3))$. Let T/M be a minimal normal subgroup of A/M. Then T/M has at most two orbits on $V(X_M)$.

Assume that X_M has order $2p^r$. Let $r > 1$. By Lemma 3.1, $T/M \cong \mathbb{Z}_p^r$. Since $s > 1$, by Proposition 2.10, $X_M \cong X^3_{2p^3}$ and by Lemma 2.9, one has $s = 2$. Now let $r = 1$. By Proposition 2.12, $X_M \cong K_{5,5} - 5K_2$, $B'(PG(2, 2))$ or $B(PG(2, 3))$. Clearly, if $X_M \cong K_{5,5} - 5K_2$ or $B'(PG(2, 2))$, then X_M is 2-transitive and hence X is 2-transitive. If $X_M \cong B(PG(2, 3))$, then $p = 13$ and X_M is 4-transitive, implying that X is at most 4-transitive. Note that $A/M \leq \mathrm{Aut}(X_M)$. For a vertex $\overline{v} \in V(X_M)$, let $(A/M)_{\overline{v}}$ and $\mathrm{Aut}(X_M)_{\overline{v}}$ be the stabilizers of $\overline{v}$ in A/M and $\mathrm{Aut}(X_M)$, respectively. Then $(A/M)_{\overline{v}} \leq \mathrm{Aut}(X_M)_{\overline{v}}$ and $\mathrm{Aut}(X_M)_{\overline{v}} \cong \mathbb{Z}_3^2 \rtimes \mathrm{GL}(2, 3)$ by Proposition 2.16. Suppose that $(A/M)_{\overline{v}} \neq \mathrm{Aut}(X_M)_{\overline{v}}$. Then X_M is $(A/M, s)$-transitive for $s = 2$ or 3. Again by Proposition 2.16, $(A/M)_{\overline{v}}$ has a subgroup isomorphic to A_4. However, $\mathbb{Z}_3^2 \rtimes \mathrm{GL}(2, 3)$ has no subgroups isomorphic to A_4, a contradiction. Thus, $(A/M)_{\overline{v}} = \mathrm{Aut}(X_M)_{\overline{v}}$, and hence X is 4-transitive.

Now assume that X_M has order p^t. By Proposition 2.14, X_M is a normal cover of the complete graph K_5. This means that $p = 5$, and there exists a normal 5-subgroup, say $\overline{H}$, in $\mathrm{Aut}(X_M)$ such that the quotient graph $(X_M)_{\overline{H}}$ of X_M relative to the orbit set of $\overline{H}$ is K_5. Further, $\mathrm{Aut}(X_M)/\overline{H} \leq \mathrm{Aut}(K_5) \cong S_5$. Noting that $\mathrm{Aut}(X_M)/\overline{H}$ is

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TABLE 1. Tetravalent s-transitive graphs of order $2p^2$.

X	s-transitive	Aut(X)	Comments
$K_{4,4}$	3-transitive	$(S_4 \times S_4) \rtimes \mathbb{Z}_2$	Example 2.1, $p = 2$
$C_{p^2}[2K_1]$	1-transitive	$\mathbb{Z}_2^{p^2} \rtimes D_{2p^2}$	Example 2.1, $p > 2$
$X^0_{2p^2}$	1-regular	$\mathbb{Z}_{2p^2} \rtimes \mathbb{Z}_4$	Example 2.2, $p \equiv 1 \bmod 4$
$X^1_{2p^2}$	1-transitive	$(\mathbb{Z}_2 \times \mathbb{Z}_p^2) \rtimes D_8$	Example 2.4, $p > 2$
$X^2_{2p^2}$	1-regular	$G(2p^2) \rtimes \mathbb{Z}_4$	Example 2.7, $p \equiv 1 \bmod 4$

2-arc-transitive on K_5, one has $\mathrm{Aut}(X_M)/\overline{H} \cong A_5$ or S_5. This tells us that $\overline{H}$ is the largest normal 5-subgroup of $\mathrm{Aut}(X_M)$. It is easily seen that $A/M \leq \mathrm{Aut}(X_M)$ and that every Sylow 5-subgroup of A/M is also a Sylow 5-subgroup of $\mathrm{Aut}(X_M)$. Therefore, $\overline{H}$ is also a normal subgroup of A/M. Set $H/M = \overline{H}$. Then $H \trianglelefteq A$, and it is easy to see that H has five orbits on $V(X)$. The maximality of M forces that $H = M$. Thus, $t = 1$ and $X_M \cong K_5$. In particular, $s = 2$. □

Let G be a nonabelian simple group and Z an abelian group. We call an extension E of Z by G a *central extension* of G if $Z \leq Z(E)$. If E is perfect, that is, the derived group $E' = E$, we call E a *covering group* of G. Schur proved that for every simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M. This group M is called the *full covering group* of G, and the center of M is called the *Schur multiplier* of G, denoted by $\mathrm{Mult}(G)$. For more information for the Schur multiplier, see, for example, [17, Ch. 5].

THEOREM 3.3. *Let p be a prime and X a connected tetravalent graph of order $2p^2$. Then X is s-transitive for some positive integer s if and only if it is isomorphic to one of the graphs in Table 1. Furthermore, all graphs in Table 1 are pairwise nonisomorphic.*

PROOF. By Examples 2.1–2.7, all graphs in Table 1 are pairwise nonisomorphic tetravalent symmetric graphs. Let X be a tetravalent s-transitive graph of order $2p^2$ for an integer s. To finish the proof, it suffices to show that X is one of the graphs listed in Table 1. If $p \leq 3$ then $|X| = 8$ or 18, and by [22], up to isomorphism there are three connected tetravalent symmetric graph of order 8 or 18. It follows that $X \cong K_{4,4}$, $C_9[2K_1]$ or $X^1_{2\cdot 3^2}$. In what follows, we assume that $p > 3$. Set $A = \mathrm{Aut}(X)$ and let $v \in V(X)$. We first prove a claim.

Claim. If A has a nontrivial normal 2-subgroup, say M, then X is isomorphic to $C_{p^2}[2K_1]$, $X^0_{2p^2}$ or $X^1_{2p^2}$.

Consider the quotient graph X_M of X relative to the orbit set of M, and let K be the kernel of A acting on $V(X_M)$. Since $p > 3$, every orbit of M has length 2, and

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hence $|X_M| = p^2$. By the symmetry of X, every orbit of M contains no edges, and by Proposition 2.15, X_M is of valency 2 or 4. If X_M has valency 2, then $X \cong C_{p^2}[2K_1]$. If X_M has valency 4, by Proposition 2.15, K is semiregular, implying that $K = M \cong \mathbb{Z}_2$. Let P be a Sylow p-subgroup of A. Then $|P| = p^2$ and hence P is abelian. Since X_M is a tetravalent graph of order p^2 and $p > 2$, PM/M must be regular on $V(X_M)$. It follows that X_M is a Cayley graph on PM/M. By [1, Corollary 1.3], every connected tetravalent Cayley graph on an odd order abelian group G is normal except for $G = \mathbb{Z}_5$. It follows that X_M, as a Cayley graph on PM/M, is normal, and hence $PM/M \trianglelefteq A/M$, namely, $PM \trianglelefteq A$. It is easily seen that PM is transitive on $V(X)$. Since $|PM| = 2p^2$, PM is also regular on $V(X)$, implying that X is a normal Cayley graph on PM. Since M is a normal subgroup of order 2, M is in the center of A, implying that PM is abelian. By Propositions 2.3 and 2.5, $X \cong X^0_{2p^2}$ or $X^1_{2p^2}$, as claimed.

Now take a minimal normal subgroup, say N, in A. By Lemma 3.1, N is solvable. Since $p > 3$ and $|V(X)| = 2p^2$, N is an elementary abelian 2-group or p-group. By the claim, we may assume further that N is a p-group. Take a nontrivial maximal normal p-subgroup, say M, of A. Clearly, since $p > 3$, one has $|M| \mid p^2$. If $M \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then M has exactly two orbits on $V(X)$, and by Proposition 2.10, $X \cong X^1_{2p^2}$ or $X^2_{2p^2}$. Assume now that M is cyclic. Set $C = C_A(M)$. Then $M \leq C$ and $A/C \leq \mathrm{Aut}(M) \cong \mathbb{Z}_{p(p-1)}$.

Suppose that $C = M$. Then $M \cong \mathbb{Z}_{p^2}$. Clearly, M acts semiregularly on $V(X)$ with two orbits. Let $R(M)$ and $L(M)$ be the two orbits of M. Since M acts regularly on $R(M)$ and $L(M)$, one may assume that $R(M) = \{R(g) \mid g \in M\}$ and $L(M) = \{L(g) \mid g \in M\}$, and that the actions of M on $R(M)$ and $L(M)$ are just by right multiplication, that is, $R(h)^g = R(hg)$ and $L(h)^g = L(hg)$ for any $h, g \in M$. By the symmetry of X, there is no edge in $R(M)$ or $L(M)$, implying that X is bipartite. Let the neighbors of $R(1)$ be $L(g_1)$, $L(g_2)$, $L(g_3)$ and $L(g_4)$, where $g_1, g_2, g_3, g_4 \in M$. Since M is abelian, for any $g \in M$, the neighbors of $R(g)$ are $L(gg_1)$, $L(gg_2)$, $L(gg_3)$ and $L(gg_4)$, and furthermore, the neighbors of $L(g)$ are $R(gg_1^{-1})$, $R(gg_2^{-1})$, $R(gg_3^{-1})$ and $R(gg_4^{-1})$. The map α defined by $R(g) \mapsto L(g^{-1})$, $L(g) \mapsto R(g^{-1})$ for any $g \in G$, is an automorphism of X of order 2. Let $B = \langle M, \alpha \rangle$. Then B is transitive on $V(X)$, and since M is normal in A, $B = M \rtimes \langle \alpha \rangle$ has order $2p^2$, and hence it acts regularly on $V(X)$. Thus, $A = BA_v$, where A_v is the stabilizer of $v \in V(X)$ in A. Since $|M| = p^2$, $A_v M/M$ has even order and since B/M has order 2, A/M is not cyclic. This leads to a contradiction.

Thus, $M < C$. Let T/M be a minimal normal subgroup of A/M contained in C/M. Suppose that T/M is nonsolvable. Then $M \cong \mathbb{Z}_p$ and X is 2-arc-transitive. It follows that the quotient graph X_M of X relative to the orbit set of M is a tetravalent 2-arc-transitive graph of order $2p$. From Proposition 2.12 it can be easily deduced that $T/M \cong A_5$, PSL(2, 7) or PSL(3, 3). Let T' be the derived subgroup of T. Since T/M is simple, one has $T'M/M = T/M$, that is, $T = T'M$. If $M \leq T'$, then $T' = T$, and T is a covering group of one of the groups A_5, PSL(2, 7) and PSL(3, 3). However, by [6], $\mathrm{Mult}(A_5) \cong \mathrm{Mult}(\mathrm{PSL}(2, 7)) \cong \mathbb{Z}_2$ and $\mathrm{Mult}(\mathrm{PSL}(3, 3)) = 1$. This

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forces $p \leq 2$, a contradiction. Thus, $M \not\leq T'$. Since $M \cong \mathbb{Z}_p$, $T' \cap M = 1$ and $T = T' \times M$. Clearly, T' is nonabelian simple. Then T' is characteristic in T, and hence it is normal in A because $T \trianglelefteq A$, contrary to Lemma 3.1. Thus, T/M is solvable. Then T/M must be a 2-group and hence $T = M \times Q$, where Q is a Sylow 2-subgroup of T. Then Q is characteristic in T, and since $T \trianglelefteq A$, one has $Q \trianglelefteq A$. By the claim, $X \cong C_{p^2}[2K_1]$, $X^0_{2p^2}$ or $X^1_{2p^2}$. □

COROLLARY 3.4. *Let p be an odd prime. Every connected tetravalent symmetric graph of order $2p^2$ is a 1-transitive Cayley graph.*

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