ON THE CONSTRUCTION OF SEQUENCE SPACES THAT HAVE SCHAUDER BASES

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1. Introduction. It is known that every Banach space which possesses a Schauder basis is essentially a space of sequences (**6**, Section 11.4). The primary objectives of this paper are: (1) to illustrate the close connection between sectionally bounded BK spaces and Banach spaces which have a Schauder basis, and (2) to consider some results in these theories in such a way as to render them easy and natural. In order to reach the largest number of readers we shall use (**6**) as the sole basis of our discussion. References to other authors are made in order to direct the reader to the original source of a theorem or to a related discussion.

A BK space is a Banach space of sequences of real or complex numbers, $S = \{(x_i)\}$, on which the coordinate functionals are continuous; that is, $(x_i^\nu) \to (x_i)$ in S implies $x_i^\nu \to x_i$ for each i. The topology that makes a given space, S, of sequences a BK space is unique (**6**, Section 11.3, Corollary 1). Thus if the norms $|| \ ||$ and $|| \ ||_0$ on S make S a BK space, they are equivalent, i.e. there are k and $K > 0$ such that $k||(x_i)|| \leqslant ||(x_i)||_0 \leqslant K\,||(x_i)||$ for each (x_i) in S. If $\mathbf{x} = (x_i)$, $P_n\,\mathbf{x}$ denotes the nth section of $\mathbf{x}$, $(x_1, x_2, \ldots, x_n, 0, \ldots)$; $\mathbf{e}_j = (\delta_{ij})_{i=1}^\infty = (0, 0, \ldots, 0, 1, 0, \ldots)$ where 1 is in the jth place. The BK spaces which we shall discuss are always assumed to contain all finite sequences. A BK space is sectionally bounded if $\sup_n ||P_n\,\mathbf{x}|| < \infty$ for each $\mathbf{x} \in S$ (**4**, p. 58).

A Schauder basis for a Banach space $(X, || \ ||)$ is a sequence $Z = \{\mathbf{z}_1, \mathbf{z}_2, \ldots\}$ such that each member $\mathbf{z}$ of X has a unique expansion

$$\mathbf{z} = \sum_{i=1}^{\infty} t_i\,\mathbf{z}_i$$

where t_i, $i = 1, 2, \ldots,$ are scalars. The norm defined by

$$||\mathbf{x}||_0 = \sup_n \left\| \sum_{i=1}^{n} t_i\,\mathbf{z}_i \right\|$$

is equivalent to $|| \ ||$, and the functionals $f_i(\mathbf{z}) = t_i$ are continuous (**6**, Section 11.4, Theorem 1). Hence, the space S_z of all sequences (t_i) for which

$$\sum_{i=1}^{\infty} t_i\,\mathbf{z}_i$$

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converges in X, given the norm

$$||(t_i)|| = \left\|\sum_{i=1}^{\infty} t_i \mathbf{z}_i\right\|,$$

is a BK space, and it is a sectionally bounded BK space with the equivalent norm

$$||(t_i)||_0 = \sup_n \left\|\sum_{i=1}^{n} t_i \mathbf{z}_i\right\|.$$

Under the mapping

$$(t_i) \leftrightarrow \sum_{i=1}^{\infty} t_i \mathbf{z}_i,$$

S with the norm $|| \ ||_0$ is topologically isomorphic to X. Under this mapping $\mathbf{e}_1, \mathbf{e}_2, \ldots$ in S_z correspond to the basis Z, so they form a basis for S_z. Thus as far as isomorphic properties are concerned, we may restrict our study of bases to the case of $\mathbf{e}_1, \mathbf{e}_2, \ldots$ a basis for a BK space. In addition, we shall assume that the basis is bounded away from 0 and ∞, i.e.

$$0 < \inf_n ||\mathbf{e}_n|| \leqslant sup_n ||\mathbf{e}_n|| < \infty.$$

This is not a severe restriction since a basis can always be normalized and the resulting sequence is also a basis.

2. The proper sequential norm. In this section we shall discuss a means of constructing sectionally bounded BK spaces in which $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is bounded away from 0 and ∞. Let s represent the space of all sequences with addition and scalar multiplication defined coordinatewise.

2.1. *Definition.* A proper sequential norm (p.s.n.) is a function, N, from s into R^* which satisfies the following conditions:

(1) N is a norm, i.e.
  (a) $N(\mathbf{x} + \mathbf{y}) \leqslant N(\mathbf{x}) + N(\mathbf{y})$,
  (b) $N(a\mathbf{x}) = |a| \, N(\mathbf{x})$ for each scalar a,
  (c) $N(\mathbf{x}) \geqslant 0$, $N(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$,
(2) $0 < \inf_n N(\mathbf{e}_n) \leqslant \sup_n N(\mathbf{e}_n) < \infty$.
(3) $N(\mathbf{x}) = \sup_n N(P_n \mathbf{x})$.

Let $K_N = \sup_n N(\mathbf{e}_n)$ and $k_N = \inf_n N(\mathbf{e}_n)$. Since

$$2N(\mathbf{x}) \geqslant N(P_n \mathbf{x}) + N(P_{n-1} \mathbf{x}) \geqslant N(P_n \mathbf{x} - P_{n-1} \mathbf{x}) = N(x_n \mathbf{e}_n) \geqslant k_N \, |\mathbf{x}_n|$$

and

$$N(x) \leqslant N(x_1 \mathbf{e}_1) + N(x_2, \mathbf{e}_2) + \ldots \leqslant |x_1| \, K_N + |x_2| \, K_N + \ldots,$$

we have

$$\tfrac{1}{2} k_N \sup_n |x_n| \leqslant N(x) \leqslant K_N \sup_n \sum_{i=1}^{n} |x_i|. \tag{4}$$

2.2. THEOREM. *For a* p.s.n., N, *the set* S_N *of all* x *for which* $N(\mathbf{x}) < \infty$, *is a Banach space with norm* N, *and* $\mathbf{e}_1, \mathbf{e}_2, \ldots$ *form a basis for their closed linear span in* S_N.

Proof. Condition (1) of 2.1 shows that S_N is a linear space and N is a norm in S_N. If $\{\mathbf{x}^{(n)}\}$ is a Cauchy sequence in S_N, $x_i = \lim_n x_i^{(n)}$ exists for each i because of (4). Write $\mathbf{x} = (x_i)$; then $\lim_n N(P_k(\mathbf{x} - \mathbf{x}^{(n)})) = 0$ for each k by the second inequality in (4). If H is such that p and $q \geqslant H$ imply

$$N(\mathbf{x}^{(p)} - \mathbf{x}^{(q)}) < \epsilon,$$

then for each k, $N(P_k(\mathbf{x}^{(p)} - \mathbf{x}^{(q)}) < \epsilon$, so $N(P_k(\mathbf{x}^{(p)} - \mathbf{x})) \leqslant \epsilon$ for $p \geqslant H$. Thus $N(\mathbf{x}^{(p)} - \mathbf{x}) \leqslant \epsilon$ for $p \geqslant H$ which implies $N(\mathbf{x}) < \infty$ and

$$\lim_n N(\mathbf{x} - \mathbf{x}^{(n)}) = 0.$$

The elements $\mathbf{e}_1, \mathbf{e}_2, \ldots$ are a basis for their closed linear span; in other words, basic; because for each pair of integers $p > q > 0$ and arbitrary scalars $a_1, a_2, \ldots, a_p$;

$$N\left(\sum_{i=1}^{q} a_i \mathbf{e}_i\right) \leqslant N\left(\sum_{i=1}^{p} a_i \mathbf{e}_i\right)$$

(**6**, Section 11.4, Theorem 5).

Inequality (4) shows that S_N is a BK space. The subspace of S_N for which $\{\mathbf{e}_i\}$ is a basis will be designated $S_N{}^0$. If $(S, \| \ \|)$ is any BK space having $\mathbf{e}_1, \mathbf{e}_2, \ldots$ for a basis which is bounded away from 0 and ∞, define

$$N(x) = \sup_n \left\| \sum_{i=1}^{n} x_i e_i \right\|.$$

Then N is a p.s.n. which is equivalent to $\| \ \|$ on S, so S is a closed subspace of S_N which implies $S = S_N{}^0$. Therefore, every Banach space which has a basis is topologically isomorphic to a space $S_N{}^0$ for N, a p.s.n.

Examples of p.s.n.'s are: $N(\mathbf{x}) = \sup |x_i|$ for which $S_N = m$, the space of bounded sequences and $S_N{}^0 = c_0$, the space of sequences which converge to 0;

$$N(x) = \sup_n \sum_{i=1}^{n} |x_i|$$

for which $S_N = S_N{}^0 = l_1$, the space of absolutely convergent series; and the l_p norms,

$$N(x) = \sup_n \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $1 \leqslant p < \infty$; see (**6**, p. 289).

2.3. THEOREM. *Let* $S(, \| \ \|)$ *be a sectionally bounded* BK *space in which* $\mathbf{e}_1, \mathbf{e}_2, \ldots$ *are bounded away from* 0 *and* ∞. *Define* $N(\mathbf{x})$ *to be*

$$\sup_n \left\| \sum_{i=1}^{n} x_i \mathbf{e}_i \right\|;$$

then N *is a* p.s.n. *and* S *is a closed subspace of* S_N *which contains* $S_N{}^0$.

Proof. Direct calculation shows that N is a p.s.n. Since S is sectionally bounded, $N(\mathbf{x}) < \infty$ for each x in S, thus $S \subseteq S_N$. But S and S_N are BK spaces, so by (**6**, Section 11.3, Corollary 1) the topology of S is stronger than that of S_N. This implies that S is a closed subspace of S_N because $N(\mathbf{x}) \geqslant \|\mathbf{x}\|$ for $\mathbf{x}$ in S. Also $\mathbf{e}_1, \mathbf{e}_2, \ldots$ are in S, so $S_N{}^0 \subseteq S$.

A sequence $\mathbf{x} = (x_i)$ can be partitioned into consecutive blocks; thus:

$$\mathbf{x} = \sum_{j=1}^{\infty} \mathbf{x}_j, \qquad \mathbf{x}_j = \sum_{i=p(j-1)}^{p(j)-1} x_i \mathbf{e}_i$$

where $1 = p(0) < p(1) < \ldots$ is an arbitrary increasing sequence of indices. The convergence of

$$\sum_{j=1}^{\infty} \mathbf{x}_j$$

refers to coordinatewise convergence, i.e. the usual topology of s. The following idea is reminiscent of that of the block basis (**1**, p. 152; **3**).

2.4. LEMMA. *Let N be a* p.s.n. *and* $\mathbf{t}$ *be any sequence partitioned into blocks,*

$$\mathbf{t} = \sum_{j=1}^{\infty} \mathbf{t}_j, \qquad \mathbf{t}_j = \sum_{i=p(j-1)}^{p(j)-1} t_i \mathbf{e}_i,$$

such that $0 < \inf_j N(\mathbf{t}_j) \leqslant \sup_j N(\mathbf{t}_j) < \infty$. *The function defined by*

$$M(\mathbf{x}) = N\left(\sum_{i=1}^{\infty} x_i \mathbf{t}_i\right)$$

is a p.s.n., *and S_M and $S_M{}^0$ are isometric to closed subspaces of S_N and $S_N{}^0$ respectively.*

Proof. That M is a p.s.n. can be verified directly from the defining conditions. The correspondence between $\mathbf{x}$ in S_M and

$$\sum_{i=1}^{\infty} x_i \mathbf{t}_i$$

in S_N is an isometry between S_M and a closed subspace of S_N. If

$$\sum_{i=1}^{\infty} x_i \mathbf{t}_i$$

is in $S_N{}^0$, then $\mathbf{x}$ is in $S_M{}^0$ since

$$M(\mathbf{x} - P_n \mathbf{x}) = N\left(\sum_{i=1}^{\infty} x_i \mathbf{t}_i - \sum_{i=1}^{p(n)} x_i \mathbf{t}_i\right)$$

which converges to 0 as $n \to \infty$. On the other hand, if $\mathbf{x}$ is in $S_M{}^0$,

$$\sum_{i=1}^{\infty} x_i \mathbf{t}_i$$

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converges in S_N, but $\mathbf{t}_i$ is in $S_N{}^0$ for each i, so

$$\sum_{i=1}^{\infty} x_i \mathbf{t}_i$$

is in $S_N{}^0$.

2.5. *Definition.* A p.s.n. M such as described in the previous lemma will be called the p.s.n. subordinate to N (with respect to $\mathbf{t}_1, \mathbf{t}_2, \ldots$).

For example, let $N(\mathbf{x}) = \sup_n |x_n|$, and let

$$\mathbf{t}_1 = \mathbf{e}_1, \qquad \mathbf{t}_2 = \mathbf{e}_2 + \mathbf{e}_3, \qquad \mathbf{t}_3 = \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6, \ldots .$$

Then the subordinate p.s.n. with respect to $\mathbf{t}_1, \mathbf{t}_2, \ldots$ is $M(\mathbf{x})$

$$M(\mathbf{x}) = N(x_1, x_2, x_2, x_3, x_3, x_3, \ldots) = \sup_n |x_n|.$$

Thus $S_M = S_N = m$ and S_M is isometric to the subspace of S_N of all x for which $x_2 = x_3$, $x_4 = x_5 = x_6$, etc.

3. The conjugate space of $S_N{}^0$.

3.1. *Definition.* The conjugate p.s.n. of N, a p.s.n., is the function from s into R^* given by

$$N'(\mathbf{y}) = \sup\left(\sup_n \left|\sum_{i=1}^{n} x_i y_i\right| ; N(x) \leqslant 1\right).$$

The verification that N' is a p.s.n. is a straightforward application of its definition and properties of the supremum. However, this task is largely unnecessary in view of the following theorem.

3.2. THEOREM. *The conjugate space of $S_N{}^0$, $(S_N{}^0)^*$, is isometric to $S_{N'}$ under the correspondence of f in $(S_N{}^0)^*$ to $\mathbf{y}$ in $S_{N'}$ where $y_i = f(\mathbf{e}_i)$ for each i. Also*

$$f(\mathbf{x}) = \sum_{i=1}^{\infty} x_i y_i$$

for each $\mathbf{x}$ in $S_N{}^0$.

Proof. Given f in $(S_N{}^0)^*$ and

$$\mathbf{x} = \sum_{i=1}^{\infty} x_i \mathbf{e}_i \text{ in } S_N{}^0, \qquad f(\mathbf{x}) = \sum_{i=1}^{\infty} x_i y_i$$

by the continuity of f. If $N(\mathbf{x}) \leqslant 1$,

$$\left|\sum_{i=1}^{n} x_i y_i\right| = |f(P_n \mathbf{x})| \leqslant ||f||\, N(P_n \mathbf{x}) \leqslant ||f||$$

where $||\ ||$ is the norm in $(S_N{}^0)^*$. Thus $y \in S_N$ and $N(\mathbf{y}) \leqslant ||f||$.

Assume $N'(\mathbf{y}) < \infty$. For $\mathbf{x}$ in $S_N{}^0$ and each n,

$$\left|\sum_{i=1}^{n} x_i y_i\right| = N(\mathbf{x}) \left|\sum_{i=1}^{n} (x_i/N(\mathbf{x}))y_i\right| \leqslant N(\mathbf{x})N'(\mathbf{y}).$$

Hence the functional

$$f_n(\mathbf{x}) = \sum_{i=1}^{n} x_i y_i$$

is continuous on $S_N{}^0$ and $||f_n|| \leqslant N'(\mathbf{y})$. Since $\lim_n f_n(\mathbf{e}_i) = y_i$ for each i, and $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is fundamental in $S_N{}^0$, the Banach Steinhaus Theorem (**6**, Section 7.6, Theorem 3) implies that $f(\mathbf{y}) = \lim_n f_n(\mathbf{y})$ is a continuous linear functional defined on $S_N{}^0$ and $||f|| \leqslant N'(\mathbf{y})$.

For $\mathbf{x}$ in $S_N{}^0$ and $\mathbf{y}$ in $S_{N'}$, define

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} x_i y_i.$$

Theorem 3.2 implies that $S_{N'}$ can be identified with $(S_N{}^0)^*$ and that the duality between these two spaces given by $(\mathbf{x}, \mathbf{y})$ can be identified with that between $S_N{}^0$ and $(S_N{}^0)^*$ given by $f(\mathbf{x})$. For convenience, we describe this by saying that $S_{N'}$ represents $(S_N{}^0)^*$.

3.3. COROLLARY. *For N, a* p.s.n., *$S_{N'}$ is* (1) *the space of all sequences* $\mathbf{y}$ *such that*

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for each $\mathbf{x}$ *in* $S_N{}^0$ *and* (2) *the space of all sequences* $\mathbf{y}$ *for which*

$$\sup_n \left| \sum_{i=1}^{n} x_i y_i \right| < \infty$$

for each $\mathbf{x}$ *in* S_N.

Proof. If $\mathbf{y}$ is such that

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for each $\mathbf{x}$ in $S_N{}^0$, define the linear functional f on $S_N{}^0$ by

$$f(\mathbf{x}) = \sum_{i=1}^{\infty} x_i y_i.$$

Since $S_N{}^0$ is a BK space, f is continuous (**6**, Section 11.3, Corollary 5). Thus $\mathbf{y}$ is in $S_{N'}$ and $N'(\mathbf{y}) = ||f||$ by Theorem 3.2. In the course of proving the same theorem, we showed that if $\mathbf{y} \in S_{N'}$,

$$\sum_{i=1}^{\infty} x_i y_i$$

converges for each $\mathbf{x} \in S_N{}^0$; cf. (**6**, Section 11.3, Problem 1).

If $\mathbf{x} \in S_N$ and $\mathbf{y} \in S_{N'}$,

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leqslant N(\mathbf{x}) N(\mathbf{y})$$

for each n; hence

$$\sup_n \left| \sum_{i=1}^{\infty} x_i y_i \right| < \infty .$$

On the other hand, if $\mathbf{y}$ is such that

$$\sup_n \left| \sum_{i=1}^{\infty} x_i y_i \right| < \infty$$

for each $\mathbf{x}$ in S_N, define the seminorm

$$\phi_n(\mathbf{x}) = \left| \sum_{i=1}^{n} x_i y_i \right| \text{ on } S_N.$$

Then the collection $\{\phi_n\}$ is pointwise bounded on $S_{N'}$, hence uniformly bounded (**6**, Section 7.6, Theorem 1). Thus

$$N'(\mathbf{y}) = \sup\left\{\sup_n \left| \sum_{i=1}^{n} x_i y_i \right| : N(x) \leqslant 1\right\}$$

$$= \sup\{\sup_n \ \phi_n(\mathbf{x}) : N(\mathbf{x}) \leqslant 1\} = \sup_n ||\phi_n|| < \infty .$$

3.4. Lemma. *For each sequence* $\mathbf{x}$, $(N')'(\mathbf{x}) = N(\mathbf{x})$.

Proof. The statement will follow from (3) of 2.1 if we prove that

$$(N')'(P_k \mathbf{x}) = N(P_k \mathbf{x}) \qquad \text{for } k = 1, 2, \ldots .$$

When $N'(\mathbf{y}) \leqslant 1$, $N'(P_n \mathbf{y}) \leqslant 1$, so

$$(N')'(P_k \mathbf{x}) = \sup\left\{ \left| \sum_{i=1}^{k} x_i y_i : N'(\mathbf{y}) \leqslant 1\right\} = \sup\{f(P_k \mathbf{x}) : ||f|| \leqslant 1\} = N(P_k \mathbf{x}).$$

The last statement is a consequence of (**6**, Section 4.4, Theorem 2 and Corollary 3).

From Lemma 3.4 and Theorem 2.2 we obtain:

3.5. Theorem. (a) $S_{N'}$ *represents* $(S_N{}^0)^*$, (b) S_N *represents* $(S_{N'}{}^0)^*$.

3.6. Theorem. *For p.s.n.'s M and N the following statements are equivalent:* (a) $S_M{}^0 = S_N{}^0$, (b) $S_M = S_N$, (c) $S_{M'} = S_{N'}$, (d) $S_{M'}{}^0 = S_{N'}{}^0$.

Proof. (a) $\Rightarrow$ (b). If $S_M{}^0 = S_N{}^0$, M and N are equivalent norms on $S_M{}^0$; so for each $\mathbf{x}$, $\sup_n M(P_n \mathbf{x}) < \infty$ if and only if $\sup_n N(P_n x) < \infty$. Thus by (2.1–3) $S_M = S_N$.

(b) $\Rightarrow$ (c) follows from Corollary 3.3.

(c) $\Rightarrow$ (d). Since M and N are equivalent norms on S_M, $\lim_n N(\mathbf{x} - \mathrm{P}_n \mathbf{x}) = 0$, if and only if $\lim_n M(\mathbf{x} - P_n \mathbf{x}) = 0$.

That (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) follows from Lemma 3.4 and the preceding.

4. The balanced and the symmetric proper sequential norms.

4.1. *Definition.* A p.s.n. N is balanced if $N(\mathbf{x}) = \sup\{N(a_i x_i) : |a_i| \leqslant 1\}$ for each sequence $\mathbf{x}$.

4.2. LEMMA. *Suppose M is a* p.s.n. *with the property that $M(\mathbf{x}) < \infty$ implies $M(a_i x_i) < \infty$ for each $(a_i) \in m$. Then there is a balanced* p.s.n., *N for which $S_M = S_N$.*

Proof. Given $\mathbf{x}$ in S_M, define the linear operator T_x from m into S_M by $T_x \mathbf{a} = (a_i x_i)$. Since both m and S_M are BK spaces, T_x is continuous (**6**, Section 11.3, Corollary 5). If

$$N(\mathbf{x}) = \sup\{M(a_i x_i) : |a_i| \leqslant 1\},$$

$N(\mathbf{x})$ is the norm of T_x in the uniform topology of operators between m and S_M. Direct calculations will prove that N is a p.s.n. and the previous discussion shows that $S_N \supseteq S_M$. But $N(\mathbf{x}) \geqslant M(\mathbf{x})$ for each $\mathbf{x}$, so $S_M \subseteq S_N$.

4.3. LEMMA. *If M is a balanced* p.s.n., *so is M'.*

Proof. Suppose $\mathbf{y}$ is a sequence and $|a_i| \leqslant 1$ for each i. Since $M(a_i x_i) \leqslant 1$ whenever $M(\mathbf{x}) \leqslant 1$,

$$M'(a_i y_i) = \sup\left\{\sup_n \left|\sum_{i=1}^{n} a_i y_i x_i\right| : M(\mathbf{x}) \leqslant 1\right\} \leqslant \sup\left\{\sup_n \left|\sum_{i=1}^{n} x_i y_i\right| : M(\mathbf{x}) \leqslant 1\right\} = M'(\mathbf{y}).$$

4.4. *Definition.* A Schauder basis X of a Banach space X is unconditional if for each

$$\mathbf{x} = \sum_{i=1}^{\infty} a_i \mathbf{x}_i \text{ in } X$$

the convergence of the series to $\mathbf{x}$ is unconditional. That is, given $\epsilon > 0$ there is a finite set of integers F such that if G is a finite set of integers containing F,

$$\left\|\sum_{i \in G} a_i \mathbf{x}_i - \mathbf{x}\right\| \leqslant \epsilon$$

(**2**, p. 73; **3**, p. 518).

4.5. THEOREM. *If $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is an unconditional basis for a* BK *space T, there is a balanced p.s.n. N such that $T = S_N{}^0$.*

Proof. By the discussion following Theorem 2.2, there is a p.s.n. M for which $S_M{}^0 = T$. If $\mathbf{x}$ is in $S_M{}^0$ and $\mathbf{y}$ is in $S_{M'}$,

$$\sum_{i=1}^{\infty} x_i y_i$$

converges unconditionally to $(\mathbf{x}, \mathbf{y})$ since

$$\sum_{i=1}^{\infty} x_i \mathbf{e}_i$$

converges unconditionally to $\mathbf{x}$. This fact plus Corollary 3.3 imply

$$S_{M'} = \left\{\mathbf{y}: \sum_{i=1}^{\infty} x_i y_i \text{ converges absolutely for each } \mathbf{x} \text{ in } S_M{}^0\right\}.$$

Hence, if $\mathbf{y} \in S_{M'}$ so is $(a_i y_j)$ for (a_i) in m. By Lemma 4.2 there is a balanced p.s.n. N_1 such that $S_{N_1} = S_{M'}$. Let $N = N_1'$; then N is balanced according to Lemma 4.3 and $S_N{}^0 = S_M{}^0$ by Theorem 3.6.

The preceding theorem implies that every Banach space having an unconditional basis can be realized as a space of the type $S_N{}^0$ for N a balanced p.s.n. Conversely, if N is a balanced p.s.n., it is not hard to show that $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is an unconditional basis of $S_N{}^0$.

4.6. *Definition.* A p.s.n. N is symmetric if $N(\mathbf{x}) = N(x_{\pi(1)}, x_{\pi(2)}, \ldots)$ for each permutation π on the integers.

We shall not study the properties of the symmetric p.s.n. in detail here. If N is symmetric, $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a symmetric basis for $S_N{}^0$; that is, if

$$\sum_{i=1}^{\infty} a_i \mathbf{e}_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty} a_{\pi(i)} \mathbf{e}_i$$

converges for each permutation π on the integers **(5)**. Conversely, every Banach space with a symmetric basis can be realized as a space $S_N{}^0$ where N is a symmetric p.s.n.

It can be shown that every symmetric p.s.n. is balanced. Symmetric p.s.n.'s are the l_p norms and the supremum norm. A p.s.n. that is balanced but not symmetric is

$$N(\mathbf{x}) = \sup_n \sum_{i=1}^{n} |x_{2i}| + \sup_n |x_{2n-1}|.$$

5. Applications. The subject matter of this section is primarily a development of results found in **(3)** in a manner that emphasizes the duality of sequence spaces. By examining the relevant definitions, it can be seen that condition (a) of Theorem 5.1 below is equivalent to having $\mathbf{e}_1, \mathbf{e}_2, \ldots$ a boundedly complete basis for $S_N{}^0$ **(3**, Theorem 1, Condition (a)**)** and condition (b) is equivalent to having $\mathbf{e}_1, \mathbf{e}_2, \ldots$ a shrinking basis for $S_N{}^0$ **(3**, Theorem 1, Condition (b)**)**; cf. **(2**, p. 69, Definition 3; **6**, Section 11.4, Theorem 7**)**.

5.1. THEOREM. *The space $S_N{}^0$ is reflexive if and only if* (a) $S_N{}^0 = S_N$ *and* (b) $S_{N'}{}^0 = S_{N'}$.

Proof. If (a) and (b) hold, then $S_N{}^0$ is reflexive by Theorem 3.2.

If $S_N{}^0$ is reflexive and $\mathbf{x} \in S_N$, then $\mathbf{x}$ represents a continuous linear functional on $S_{N'}{}^0$. Let F be a continuous extension of this functional to all of $S_{N'}$ (**6**, Section 4.4, Corollary 1). Since $S_N{}^0$ is reflexive and $S_{N'}$ represents the conjugate space of $S_N{}^0$, F corresponds to a sequence $\mathbf{z}$ in $S_N{}^0$. But

$$(\mathbf{z}, \mathbf{e}_i) = f(\mathbf{e}_i) = (\mathbf{x}, \mathbf{e}_i)$$

for each i, which implies $\mathbf{x} \in S_N{}^0$, so (a) holds. To obtain (b) observe that $S_N{}^0$ reflexive implies $S_{N'}$, and, hence, $S_{N'}{}^0$ reflexive (**6**, Section 7.2, problem 19).

We omit the proofs of the following two lemmas which are easy.

5.2. LEMMA. *If N is a balanced p.s.n. and* $(1, 1, 1, \ldots)$ *is in S_N, then $S_N = m$.*

5.3. LEMMA. *A* p.s.n. *that is subordinate to a balanced* p.s.n. *is balanced.*

The next theorem corresponds to (**3**, Lemmas 1 and 2).

5.4. THEOREM. *Suppose N is a balanced* p.s.n. *and $S_N{}^0 \neq S_N$, then*

(a) *There is a closed subspace of S_N topologically isomorphic to m.*

(b) *There is a closed subspace of $S_N{}^0$ topologically isomorphic to c_0.*

(c) *There is a closed subspace of $S_N{}^0$ topologically isomorphic to l_1.*

Proof. Let $\mathbf{t}$ be in S_N but not $S_N{}^0$. Then

$$\left\{\sum_{i=1}^{n} t_i \mathbf{e}_i\right\}$$

is not a Cauchy sequence, so there is a number $\eta > 0$ and integers

$$1 = p(0) < p(1) < \ldots$$

such that

$$N\left(\sum_{i=p(k-1)}^{p(k)-1} t_i \mathbf{e}_i\right) > \eta.$$

Let

$$\mathbf{t}_k = \sum_{i=p(k-1)}^{p(k)-1} t_i \mathbf{e}_i$$

and let M be the p.s.n. subordinate to N with respect to $\mathbf{t}_1, \mathbf{t}_2, \ldots$. Then M is a balanced p.s.n. by Lemma 5.4 and $S_M = m$ by Lemma 5.2 because $M(1, 1, \ldots) = N(\mathbf{t}) < \infty$. By Lemma 2.4, $S_M = m$ is isometric to a closed subspace of S_N and $S_M{}^0 = c_0$ to a closed subspace of $S_N{}^0$. This establishes (a) and (b).

Let

$$\mathbf{s}_k = \sum_{i=p(k-1)}^{p(k)-1} s_i \mathbf{e}_i$$

be a sequence in $S_{N'}$, such that $N_{'}(\mathbf{s}_k) \leqslant 1$ and $(\mathbf{t}_k, \mathbf{s}_k) > \eta$ for each k. The existence of such a sequence follows from the fact that $S_{N'}$ is the conjugate space of $S_N{}^0$ (**6**, Section 4.4, Corollary 3). If

$$\sum_{k=1}^{\infty} y_k \, \mathbf{s}_k$$

converges, so do

$$\sum_{k=1}^{\infty} \sum_{i=p(k-1)}^{p(k)-1} y_k \, s_i \, \mathbf{e}_1 \quad \text{and} \quad \sum_{k=1}^{\infty} \sum_{i=p(k-1)}^{p(k)-1} |y_k| \, s_i \, \mathbf{e}_i$$

since N is balanced. Therefore,

$$\sum_{k=1}^{\infty} \sum_{i=p(k-1)}^{p(k)-1} |y_k| \, s_i \, t_i$$

converges, which implies

$$\sum_{k=1}^{\infty} |y_k|$$

converges. Consequently, the closed linear span of $\mathbf{s}_1, \mathbf{s}_2, \ldots$ in $S_{N'}{}^0$ is topologically isomorphic to l_1.

5.5. Theorem. *For N a balanced* p.s.n., *$S_N{}^0$ is reflexive if it has no subspace isomorphic with l_1 or c_0, or if $S_N{}^0$ and $S_{N'}$ have no subspace isomorphic to l_1.*

Proof. If $S_N{}^0$ is not reflexive, then either $S_N{}^0 \neq S_N$ or $S_{N'} \neq S_{N'}{}^0$. If $S_N{}^0 \neq S_N$, then (b) and (c) of 5.4 show that there is a closed subspace of $S_N{}^0$ topologically isomorphic to c_0 and a closed subspace of $S_{N'}{}^0$ topologically isomorphic to l_1. If $S_N{}'^0 \neq S_{N'}$, apply (b) and (c) of 5.5 to $S_{N'}{}^0$ and $S_{N''}{}^0 = S_N{}^0$.

The method of proving the second statement is similar.

We shall now discuss the example in (**3**) of a non-reflexive Banach space isomorphic to its second conjugate. Define

$$J(\mathbf{x}) = \sup\left[\sum_{i=1}^{n} (x_{p(2i-1)} - x_{p(2i)})^2 + (x_{p(2n+1)})^2\right]^{\frac{1}{2}}$$

where the supremum is taken over all positive integers n and finite increasing sequences of integers $p(1), p(2), \ldots, p(2n+1)$. It is clear that J satisfies the conditions defining a p.s.n. except possibly for (1)-(a), the triangular inequality. For the reader's convenience we repeat James's proof that it does (**3**, p. 524). Given $\epsilon > 0$, there is an increasing sequence of integers

$$p(1), p(2), \ldots, p(2n+1)$$

for which

$$J(\mathbf{x}+\mathbf{y}) \leqslant \epsilon + \left[\sum_{i=1}^{n} (x_{p(2i-1)} + y_{p(2i-1)} - x_{p(2i)} - y_{p(2i)})^2 + (x_{p(2n+1)} + y_{p(2n+1)})^2\right]^{\frac{1}{2}} \leqslant J(\mathbf{x}) + J(\mathbf{y}) + \epsilon.$$

5.6. LEMMA. (a) $S_J \subseteq c$, (b) $S_J{}^0 = S_J \cap c_0$, (c) $S_{J'} = S_{J'}{}^0$.

Proof. (a) Suppose $\lim_n x_n$ does not exist; then there is $\epsilon > 0$ and a sequence $p(1) < p(2) < \ldots$ such that $|x_{p(2i-1)} - x_{p(2i)}| > \epsilon$ for each i. Thus

$$J(\mathbf{x}) \geqslant \left[\sum_{i=1}^{n} (x_{p(2i-1)} - x_{p(2i)})^2 + (x_{p(2n+1)})^2 \right]^{\frac{1}{2}} \geqslant n^{\frac{1}{2}}\epsilon$$

for each n, which implies $\mathbf{x} \notin S_J$.

(b) Let $\mathbf{x} \in S_J \cap c_0$. Given $\epsilon > 0$, there must be K so large that whenever $K < p(1) < p(2) < \ldots < p(2n)$,

$$\sum_{i=1}^{n} (x_{p(2i-1)} - x_{p(2i)})^2 < \epsilon^2/2$$

and such that $n \geqslant K$ implies $|x_n| < \epsilon/\sqrt{2}$. If $k > K$ and

$$k \leqslant p(1) < p(2) < \ldots < p(2n+1),$$

then

$$\sum_{i=1}^{n} (x_{p(2i-1)} - x_{p(2i)})^2 + (x_{p(2n+1)})^2 < \epsilon^2/2 + \epsilon^2/2 = \epsilon^2.$$

Therefore, $J(\mathbf{x} - P_k \mathbf{x}) \leqslant \epsilon$ for $k \geqslant K$.

(c) The argument used here is essentially that of James (**3**, p. 524). Suppose there is $\mathbf{y} \in S_N$, for which $\lim_k J'(\mathbf{y} - P_k \mathbf{y})\,;0$ is false. Then there is $\epsilon > 0$ and integers $1 = k(1) \leqslant k(2) \leqslant \ldots$ such that

$$J\left(\sum_{j=k(i)}^{k(i+1)-1} y_j \mathbf{e}_j\right) > \epsilon \qquad \text{for } i = 1, 2, \ldots .$$

Since $S_{J'}$ represents the conjugate space of $S_J{}^0$, there is a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ such that

$$\mathbf{x}_i = \sum_{j=k(i)}^{k(i+1)-1} x_j \mathbf{e}_j, \quad J(\mathbf{x}_i) \leqslant 1, \quad \text{and} \quad \left(\mathbf{x}_i, \sum_{j=k(i)}^{k(i+1)-1} y_j \mathbf{e}_j\right) > \epsilon \qquad \text{for each } i.$$

Let $\mathbf{z}$ be the sequence with $z_i = (1/n)x_i$ for $k(n) \leqslant i \leqslant k(m+1)$. Note that $|z_i| < 1/n$ for any given n when i becomes sufficiently large so $\mathbf{z} \in c_0$. For $p(1) < p(2) < \ldots < p(2n+1)$ consider the sum

$$\sum_{i=1}^{n} (z_{p(2i-1)} - z_{p(2i)})^2 + (z_{p(2n+1)})^2.$$

For each i either (i) $z_{p(2i-1)}$ and $z_{p(2i)}$ are coordinates of the same $\mathbf{x}_n/n$ or (ii) they are coordinates of different elements, $\mathbf{x}_n/n$ and $\mathbf{x}_{n+r}/(n+r)$. The sum of the squares of the differences of the numbers of type (ii) is less than or equal to

$$\sum_{n=1}^{\infty} [1/n + 1/(n+1)]^2 < 4\sum_{n=1}^{\infty} (1/n)^2$$

while the remaining terms have sum less than or equal to

$$\sum_{n=1}^{\infty} (J(\mathbf{z}_n)/n)^2 \leqslant 1/n^2.$$

Thus the sum in question is

$$\leqslant 5 \sum_{n=1}^{\infty} 1/n^2;$$

so $J(\mathbf{z}) < \infty$, which implies $\mathbf{z} \in S_N{}^0$. But

$$(\mathbf{z}, \mathbf{y}) \geqslant \epsilon \left(\sum_{n=1}^{p} 1/n \right)$$

for each p, which is a contradiction.

Since $\mathbf{e}_0 = (1, 1, \ldots)$ is in S_J but not $S_J{}^0$, $S_J{}^0$ is not reflexive. However, by (a) and (b) of Lemma 5.6 every member $\mathbf{x}$ of S_J can be written uniquely

$$\mathbf{x} = x_0\,\mathbf{e}_0 + \sum_{i=1}^{\infty} (x_i - x_0)\mathbf{e}_i, \qquad x_0 = \lim_i x_i.$$

Define T from S_J into $S_J{}^0$ by $T\mathbf{x} = \mathbf{y}$ where $y_1 = x_0$, $y_i = x_{i-1} - x_0$ for $i > 1$. It can be verified directly that $J(\mathbf{y}) < \infty$ and that $\mathbf{y} \in c_0$; hence $\mathbf{y}$ is in $S_J{}^0$ Since the coefficient functionals of a basis are continuous, T is a closed, thus a continuous, operator, It is also easy to show that T is onto and one to one. Since S_J represents the conjugate space of $S_J{}^0$ which is equal to $S_{J'}$, $S_J{}^0$ is isomorphic to its second conjugate.

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