ON A TYPE OF SUBGROUPS OF A COMPACT LIE GROUP

YOZÔ MATSUSHIMA

Let G be a connected compact Lie group and H a connected closed subgroup. Then H is an orientable submanifold of G and we may consider H as a cycle in G. In his interesting paper on the topology of group manifolds[1] H. Samelson has proved that, if H is not homologous to 0, then the homology ring[2] of the coset space G/H is isomorphic to the homology ring of a product space of odd dimensional spheres and the homology ring of G is isomorphic to that of the product of the spaces H and G/H. On the other hand, in a recent investigation of fibre bundles[3] T. Kudo has shown that, if the homology ring of the coset space G/H is isomorphic to that of an odd dimensional sphere, then H is not homologous to 0.

In the present paper we shall consider those connected closed subgroups of a connected compact Lie group G such that the homology rings of the coset spaces are isomorphic to that of odd dimensional spheres. We shall first show that the problem to find all such subgroups of G may be reduced to the case where G is a simple group. The determination of such subgroups of the rotation groups of spheres (simple Lie groups of types B and D) is contained essentially in a paper by D. Montgomery and H. Samelson on the transformation groups of spheres.[4] Hence we shall consider here the above problem for simple Lie groups of the other types. The writer is grateful to Mr. M. Kuranishi for his friendly cooperation during the preparation of this paper.

I.

1. All groups considered in the following are compact Lie groups and sub-

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[1] H. Samelson, Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten, Ann. of Math. Vol. **42** (1941); Satz VI. We refer to this paper as [S].

[2] The coefficients of the homology ring are rational numbers.

[3] T. Kudo, On the homological properties of fibre bundles, forthcoming in Journ. of the Institute of Polytechnics, Osaka City University.

[4] D. Montgomery and H. Samelson, Transformation groups of spheres, Ann. of Math. Vol. **44** (1943). We refer to this paper as [M-S].

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groups are always taken as closed.

a) The homology ring[2] of an orientable manifold M is denoted by $R(M)$ and S^n denotes the n-sphere. The homology ring $R(G)$ of a compact Lie group G is isomorphic to $R(S^{m_1} \times \ldots \times S^{m_r})$, where m_i are odd and r is the rank $r(G)$ of G.[5]

b) A connected subgroup H of a compact connected Lie group G is said to be an *S-subgroup*, if $R(G/H)$ is isomorphic to $R(S^m)$, where m is odd. If H is an S-subgroup, then $r(H) = r(G) - 1$.[6]

c) Let $G_1, \ldots, G_k$ be (compact connected) Lie groups and let N be a finite normal subgroup of $\bar{G} = G_1 \times \ldots \times G_k$. We say that the factor group $G = \bar{G}/N$ is *essentially the product* of $G_1, \ldots, G_k$ and we denote $G = G_1 \circ \ldots \circ G_k$. Every compact connected Lie group G is essentially the producted of some simply connected simple groups and a toral group. If G_1 is a connected normal subgroup of a compact connected Lie group G, then there exists a connected normal subgroup G_2 of G such that $G = G_1 \circ G_2$.[7]

d) Let G be a Lie group and H a subgroup and let $W = G/H$. Then we may consider G in a natural way as a transitive transformation group of W. The set of all elements $g \in G$ for which $g(x)$, $x \in W$, are identity transformation of W form a normal subgroup G_0 contained in H. If G_0 is a finite group, then G is said to be *almost effective* on W.

2. We prove now a theorem on the structure of S-subgroups. Let R_1 be the rotation group of 1-sphere and $\tilde{R}_2$ the simply connected covering group of the rotation group R_2 of 2-sphere.

THEOREM I. *Let G be a compact connected Lie group, H an S-subgroup of G, and let G_2 be the maximal connected normal subgroup of G contained in H. Further, let G_1 be a connected normal subgroup of G such that $G = G_1 \circ G_2$. Then $H = H_1 \circ G_2$ and H_1 is an S-subgroups of G_1 and G_1 is simple or essentially the product of two simple groups one of which is R_1 or $\tilde{R}_2$.*

3. To prove Theorem I we need some lemmas.

LEMMA 1. If H is a connected normal subgroup of G, then $H \nsim 0$.

Proof. Let K be a connected normal subgroup of G such that $G = H \circ K$

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[5] H. Hopf, Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, Ann. of Math. Vol. **42** (1941) and H. Hopf, Über den Rang geschlossener Lieschen Gruppen, Commet. Math. Helvet. Vol. **13** (1941).

[6] See, [S], Satz VI. Note that by the results of H. Samelson and T. Kudo H is an S-subgroup if and only if H is not homologous to 0 and $r(H) = r(G) - 1$.

[7] In this case we may consider $G_1 \circ G_2$ as the usual product of two normal subgroups G_1 and G_2.

and let $\bar{G} = H \times K$. Then H is obviously $\nsim 0$ in $\bar{G}$. But the natural homomorphism π of $\bar{G}$ onto G induces the isomorphic mapping of the homology group $B(\bar{G})$ of $\bar{G}$ onto the homology group $B(G)$ of G. From these facts we conclude without difficulty that $H \nsim 0$ in G.

Lemma 2. Let $G = G_1 \circ G_2$, $\bar{G} = G_1 \times G_2$ and let π be the natural homomorphic mapping of $\bar{G}$ onto G. Further let H be a connected subgroup of G and $\bar{H}$ the connected component containing the identity of the group $\pi^{-1}(H)$. H is an S-subgroup of G if and only if $\bar{H}$ is an S-subgroup of $\bar{G}$, and $R(G/H) \cong R(\bar{G}/\bar{H})$.

Proof. π induces the isomorphic mapping of the homology group $B(\bar{G})$ of $\bar{G}$ onto the homology group $B(G)$ of G. The same holds for $B(\bar{H})$ and $B(H)$, since $\pi(\bar{H}) = H$ and π is locally isomorphic. Let $V(\bar{H})$ be the additive subgroup of $B(\bar{H})$ composed of all minimal elements of $B(\bar{H})$[8] and let $\bar{v}_1, \ldots, \bar{v}_k$ be a basis of $V(\bar{H})$. Then $\pi(\bar{v}_1), \ldots, \pi(\bar{v}_k)$ is also a basis of the group $V(H)$ of the minimal elements of $B(H)$.[8] Assume $H \nsim 0$. Then $\pi(\bar{v}_1), \ldots, \pi(\bar{v}_k)$ are linearly independent also when we consider $\pi(\bar{v}_i)$ as homology classes in G.[9] If $\bar{H} \sim 0$, then $\bar{v}_1, \ldots, \bar{v}_k$ would be linearly dependent considered as homology classes in $\bar{G}$, and the same for $\pi(\bar{v}_1), \ldots, \pi(\bar{v}_k)$ considered as the homology class in G. Hence $\bar{H} \nsim 0$. Conversely, if $\bar{H} \nsim 0$, then $H \nsim 0$. Since $r(H) = r(\bar{H})$, $r(G) = r(\bar{G})$ and $r(H) = r(G) - 1$, we have $r(\bar{H}) = r(\bar{G}) - 1$. Hence $R(\bar{G}/\bar{H}) \cong R(S^m)$. Further since $\dim G/H = \dim \bar{G}/\bar{H}$ we have clearly $R(G/H) \cong R(S^m)$.

4. *Proof of Theorem I.*[10] Let $\bar{G} = G_1 \times G_2$, π be the natural homomorphism of $\bar{G}$ onto G and let $\bar{H}$ be the connected component of the group $\pi^{-1}(H)$. Since $\bar{H} \supset G_2$, we have $\bar{H} = H_1 \times G_2$, where H_1 is a connected subgroup of G_1. By Lemma 2 $\bar{H} \nsim 0$ in $\bar{G}$, hence $H_1 \nsim 0$ in G_1 and H_1 is clearly an S-subgroup of G_1 containing no connected normal subgroup of G_1 (different from the group consisting only of identity of G_1). We shall prove that G_1 has the structure stated in Theorem I. For simplicity we write G and H in place of G_1 and H_1. Then H is an S-subgroup of G containing no connected normal subgroup of G. Let $W = G/H$ and $R(W) = R(S^m)$ (m being odd). Then G is almost effective on W. If $G = G_1 \circ G_2$, then we show that one of G_i, say G_1, is transitive on W. For this purpose let $\bar{G} = G_1 \times G_2$, let, as above, $\bar{H}$ be the connected component

8) For the definition and the properties of the minimal element, see H. Hopf, loc. oit. and [S].

9) See, [S], Satz III. Korollar 1.

10) The following proof is similar to the proof of Theorem I b) in [M-S]. But we avoid to use a theorem of Gysin which played an essential role there.

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of $\pi^{-1}(H)$ and $\overline{W} = \overline{G}/\overline{H}$. Then $R(\overline{W}) = R(S^m)$ and $\overline{G}$ is also almost effective on $\overline{W}$. If we can show that G_1 is transitive on $\overline{W}$, then we see easily that G_1 is also transitive on W. Let Γ_i be the image of $\overline{H}$ under the natural homomorphism $g_1 \cdot g_2 \to g_i$ of $\overline{G}$ onto G_i and let $\Gamma = \Gamma_1 \times \Gamma_2$. Γ_i are connected and $\Gamma \supseteqq H$. Let H_i be the intersections $\overline{H} \cap G_i$. H_i are normal subgroups of Γ_i. Then, as in the proof of Theorem I b) in [M-S], the spaces $\Gamma/\overline{H}$, Γ_1/H_1, Γ_2/H_2 are homeomorphic. Consider the space Γ_1/H_1. As H_1 is a normal subgroup of Γ_1, it is a compact connected Lie group. If $\Gamma_1 = H_1$, then $\Gamma_2 = H_2$. If follows that $\overline{H} = H_1 \times H_2$ and $\overline{G}/\overline{H} = G_1/H_1 \times G_2 \times H_2$. But since $R(\overline{G}/\overline{H}) = R(S^m)$, the space $\overline{G}/\overline{H}$ can not be decomposed into a direct product of two manifolds of positive dimensiens. Hence one of the spaces G_i/H_i, for example G_2/H_2, must be a point. This means that $G_2 = H_2$ and hence $\overline{H}$ must contain the normal subgroup G_2, which is impossible. Hence $\Gamma_1 \neq H_1$. Since $r(\overline{H}) = r(\overline{G}) - 1$, we can show that $r(\Gamma_1/H_1) = 1$. Therefore Γ_1/H_1 is homeomorphic with one of the three following manifolds: the 1-sphere S^1, the 3-sphere S^3, and the projective 3-space P^3. We shall show that one of Γ_i is equal to G_i.

i) First let $\Gamma/\overline{H} \approx \Gamma_i/H_i \approx S^3$.

Since S^3 is simply connected, H_i are connected. Clearly $r(\overline{G}) = r(\Gamma)$ and $r(\overline{H}) = r(\overline{G}) - 1$. As H_i are the normal subgroups of $\overline{H}$, $H_i \not\sim 0$ in $\overline{H}$ by Lemma 1. Since $\overline{H} \not\sim 0$ in $\overline{G}$, it follows that $H_i \not\sim 0$ in $\overline{G}$,[11] whence $H_i \not\sim 0$ in G_i. Let $\Delta = H_1 \times H_2$. Then $\Delta \not\sim 0$ in $\overline{G}$ and since $r(H_i) = r(\Gamma_i) - 1, r(\Delta) = r(\overline{H}) - 1$. Hence Δ is an S-subgroup of $\overline{H}$. Then, from the relations $R(\overline{G}) = R(\overline{G}/\overline{H} \times \overline{H})$ and $R(\overline{H}) = R(\overline{H}/\Delta \times \Delta)$,[12] we obtain $R(\overline{G}) = R(\overline{G}/\overline{H} \times \overline{H}/\Delta \times \Delta)$. But since $R(\overline{G}) = R(\overline{G}/\Delta \times \Delta)$, it follows that $R(\overline{G}/\Delta) = R(G_1/H_1 \times G_2/H_2) = R(\overline{G}/\overline{H} \times \overline{H}/\Delta)$. However, $\overline{H}$ and Δ are S-subgroups of $\overline{G}$ and $\overline{H}$ respectively, and therefore $R(\overline{G}/\overline{H}) = R(S^m)$ and $R(\overline{H}/\Delta) = R(S^k)$ (m and k being odd). Hence $R(G_1/H_1 \times G_2/H_2) = R(S^m \times S^k)$. Then it follows that $R(G_1/H_1) = R(S^m)$ and $R(G_2/H_2) = R(S^k)$. On the other hand $R(G_2) = R(H_2 \times G_2/H_2) = R(H_2 \times S^k)$ and $R(\Gamma_2) = R(H_2 \times \Gamma_2/H_2) = R(H_2 \times S^3)$. But since $\dim \Gamma - \dim \Delta = 6$ and $\dim \Gamma - \dim \overline{H} = 3$, we have $\dim \overline{H} - \dim \Delta = 3$. Hence $k = 3$. This shows that $\dim G_2 = \dim \Gamma_2$ and hence $G_2 = \Gamma_2$.

ii) Next let $\Gamma/\overline{H} \approx \Gamma_i/H_i \approx P^3$.

In this case H_i need not be connected. But if H_i^0 are the connected components of H_i, then Γ_i/H_i^0 are the covering spaces of P^3, whence homeomorphic to S^3. So, replacing H_i by H_i^0, if necessary, we obtain $G_2 = \Gamma_2$ by the same argument as in i).

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11) See, [S], Satz III Korollar 3.

12) See, [S], Satz VI.

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iii) Finally let $\Gamma/\overline{H} \approx \Gamma_i/H_i \approx S^1$.

Let H_i^0 be the connected components of H_i. Then Γ_i/H_i^0 are also homeomorphic to S^1. Hence we may assume that H_i are connected. In this case $\overline{H}/\Delta = S^1$ and by the same argument as in i), we have $R(G_1/H_1 \times G_2/H_2) = R(\overline{G}/\overline{H} \times \overline{H}/\Delta) = R(S^m \times S^1)$. Hence $R(G_1/H_1) = R(S^m)$ and $R(G_2/H_2) = R(S^1)$. It follows that $\dim G_2 - \dim H_2 = \dim \Gamma_2 - \dim H_2 = 1$. Hence $G_2 = \Gamma_2$.

Thus we have proved that $G_2 = \Gamma_2$. Then we may show as in [M-S] that G_1 is transitive on W and that $r(G_2) = 1$ i.e. G_2 is R_1 or R_2 or $\widetilde{R}_2$. Theorem I will be proved, if we show that G_1 is simple. Since G_1 is transitive on W, there exists a subgroup H_1 of G_1 such that $W = G_1/H_1$, where $H_1 = G_1 \cap H$. Let H_1^0 be the connected component of H_1. Then H_1^0 is a normal subgroup of H. Hence $H_1^0 \neq 0$ in H. Then, by the same argument as above, $H_1^0 \neq 0$ in G_1. Moreover, we may easily verify that $r(H_1^0) = r(G_1) - 1$, whence H_1^0 is an S-subgroup of G_1 and $R(W_1) = R(S^m)$. Clearly H_1^0 contains no connected normal subgroup of G_1 different from the identity. Suppose that G_1 is not simple and let $G_1 = G' \circ G''$. We use for G_1, H_1^0 and W_1 the same argument we used for G, H and W and find that G' is transitive on W_1 and G'' is R_1 or R_2 or $\widetilde{R}_2$. Then $G = G' \circ (G'' \circ G_2)$. Since the rank of the group $G_3 = G'' \circ G_2$ is 2, it must be transitive on W. By the same argument as above, there exists in G_3 an S-subgroup H_3 such that $R(G_3/H_3) = R(S^m)$ and H_3 contains no connected normal subgroup of G_3. Then G'' or G_2 must be transitive on G_3/H_3. This is impossible if $m > 3$. The cases $m = 1$ and 3 may be treated easily for themselves. Thus Theorem I is proved.

5. By Theorem I the problem to find all S-subgroups of a compact connected group G is reduced to the cases where G is a simple group or a direct product of two simple groups one of which is of rank 1. The latter case may be reduced to the former case. So we consider in the following the S-subgroups of a simple group.

II.

1. If G and G' are locally isomorphic compact simple groups, then, as we may easily see from Lemma 2, the S-subgroups of G and G' correspond to each other. Hence we have only to consider one respresentative from each class of locally isomorphic groups. In particular, if we can show that the S-groups of a simple group G are conjugate to each other, then the same folds for every simple group locally isomorphic to G.

2. First we consider the case $G = R_n$, the rotation group of n-sphere. We denote by Q_{n-k} the subgroup of R_n composed of all elements of R_n which leave

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fixed the unit point on the first k of $n+1$ axis of Euclidean $(n+1)$-space E_{n+1}. Clearly Q_{n-k} is isomorphic to R_{n-k}.

i) $n = 2m-1$.

Then $R_n/Q_{n-1} = S^{2m-1}$ and hence $Q_{n-1} \not\sim 0$ in R_n.[13] By Lemma 7 of [M-S] we see that every S-subgroup of R is conjugate to Q_{n-1} except for a finite number of n's.

ii) $n = 2m$.

In this case $Q_{n-2} \not\sim 0$ in R_n. By the proof of Theorem IV of [M-S] we see that every S-subgroup of R_n is conjugate to Q_{n-2} except for a finite number of n's.

III.

1. Here we consider the group $G = A_n$, the unimodular unitary group in $n+1$ variables. We denote by A_{n-1} the subgroup of $G = A_n$ consisting of all elements of G which leave fixed the unit point on the first of the $n+1$ axis of unitary $(n+1)$-space. Then $G/A_{n-1} = S^{2n+1}$, whence $A_{n-1} \not\sim 0$.[13]

We prove the following

THEOREM II. *Every S-subgroup of $G = A_n$ is conjugate to A_{n-1} for $n \geqq 8$.*[14]

2. Let U be an S-subgroup of G. Then $R(G/U) = R(S^m)$ (m: odd) and $R(G) = R(G/U \times U)$. Hence $R(G) = R(U \times S^m)$. The homology ring of G is

$$R(G) = R(A_n) = R(S^3 \times S^5 \times \ldots \times S^{2n+1}).^{15)}$$

Hence $m = 2k+1$ and

$$R(U) = R(S^3 \times S^5 \times \ldots \times S^{m-2} \times S^{m+2} \times \ldots \times S^{2n+1}). \quad (1)$$

U is simple, for in (1) S^3 appears only once.[16] As we may easily verify the group A_n can not contain the exceptional groups of rank $n-1$. Hence U is a classical simple group. Then (1) is possible for $m < 2n+1$ only when $n = 3$, $m = 5$, and $R(G) = R(S^3 \times S^5 \times S^7)$ and $R(U) = R(S^3 \times S^7)$. Hence if $n > 3$ then $m = 2n+1$ and $R(U) = R(S^3 \times S^5 \times \ldots \times S^{2n-1})$. This shows that U is a simple group of type A_{n-1}.[15]

13) See, [S], Satz IV.

14) The writer can not decide whether Theorem II is also valid for $n < 8$ or not. Since every subgroups of rank 1 is not homologous to 0, Theorem II is not valid for $n = 2$ as we may show by an example. Cf. J. L. Koszul, C. R. Paris **225**, p. 477 (1947), and H. Samelson, C. R, Paris **228**, p. 630 (1949).

15) See, L. Pontrjagin, Homologies in compact Lie groups, Rec. Math. N. S. Vol. **6** (1939) or [S].

16) For, by a theorem of E. Cartan, the 3-dimensional Betti number of any semi-simple group is not equal to 0.

3. Let $\mathfrak{G}_c$ be the Lie algebra of G and $\mathfrak{U}_c$ the subalgebra of $\mathfrak{G}_c$ corresponding to the subgroup U. By taking a suitable conjugate group of U, we may assume that a maximal abelian subalgebra $\mathfrak{h}_c$ of $\mathfrak{U}_c$ is contained in a fixed maximal abelian subalgebra $\mathfrak{H}_c$ of $\mathfrak{G}_c$.[17] We denote by $\mathfrak{G}$ and $\mathfrak{U}$ the Lie algebras obtained from $\mathfrak{G}_c$ and $\mathfrak{U}_c$ respectively by extending the domain of coefficients to the complex number field K. $\mathfrak{H}$ and $\mathfrak{h}$ may be defined analogously. Then as is well known, $\mathfrak{G}$ is the Lie algebra consisting of all matrices of degree $n+1$ with complex numbers as coefficients whose traces are 0. Let e_{ik} $(i, k = 1, \ldots, n+1)$ be the matrix whose (i, k)-element is 1 and others are all 0 and $h_i = e_{ii}$. Then $\mathfrak{G}$ has the following basis:

$$\mathfrak{G} = \mathfrak{H} + \sum_{i,k=1}^{n+1} K e_{ik},\ (i \neq k)\ ;\ \mathfrak{H} = \{\lambda^1 h_1 + \ldots + \lambda^{n+1} h_{n+1}\},\ \sum_{i=1}^{n+1} \lambda^i = 0\ ;$$

$$\left[\sum_{j=1}^{n+1} \lambda^j h_j,\ e_{ik}\right] = (\lambda^i - \lambda^k)\, e_{ik}.$$

The real Lie algebra $\mathfrak{G}_c$ is obtained from $\mathfrak{G}$ by the so called "unitary restriction." *Since* $\mathfrak{U}$ *is of type* A_{n-1}, $\mathfrak{U}$ *has the following basis:*

$$\mathfrak{U} = K\tilde{h}_1' + \ldots + K\tilde{h}_{n-1}' + \sum_{i,k=1}^{n} K u_{ik},\ (i \neq k),$$

where $\tilde{h}_1', \ldots, \tilde{h}_{n+1}'$ is a linearly independent basis of $\mathfrak{h}$ and

$$\left[\sum_{j=1}^{n-1} \mu^j \tilde{h}_j',\ u_{ik}\right] = (\mu^i - \mu^k) u_{ik},\ (i,\ k \leqq n-1)\ ;$$

$$\left[\sum_{j=1}^{n-1} \mu^j \tilde{h}_j',\ u_{in}\right] = (\mu^i + \mu^1 + \ldots + \mu^{n-1}) u_{in},$$

$$\left[\sum_{j=1}^{n-1} \mu^j \tilde{h}_j',\ u_{in}\right] = (-\mu^i - \mu^1 - \ldots - \mu^{n-1}) u_{ni}.$$

Now, let $\tilde{h}_n$ be an element of $\mathfrak{H}$ which is not contained in $\mathfrak{h}$ and let $\tilde{h}_i = \tilde{h}_i' + \tilde{h}_n$ for $i = 1, \ldots, n-1$. Then $\mu^1 \tilde{h}_1' + \ldots + \mu^{n-1} \tilde{h}_{n-1}' = \mu^1 \tilde{h}_1 + \ldots + \mu^n \tilde{h}_n$ and $\mu^1 + \ldots + \mu^n = 0$. Hence $\mathfrak{h}$ is the set of all elements $\lambda^1 \tilde{h}_1 + \ldots + \lambda^n \tilde{h}_n$ such that $\sum_{i=1}^{n} \lambda^i = 0$. We have

$$\left[\sum_{j=1}^{n} \lambda^j \tilde{h}_j,\ u_{ik}\right] = (\lambda^i - \lambda^k)\, u_{ik}$$

[17] For, any toral subgroup of a compact connected Lie group G is conjugate to a subgroup of any maximal toral subgroup G. See, A. Weil, Démonstration topologique d'un théorème fondamental de Cartan. C. R. Paris **200** (1935) ; H. Hopf and H. Samelson, Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen, Commet. Math. Helvet. Vol. **13** (1941).

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for $1 \leqq i, k \leqq n$.

Clearly u_{ik}'s are linear combinations of e_{rs}'s. A u_{ik} is said to be *singular* if it is a linear combination of at least two e_{rs}'s. We shall prove that no u_{ik} is singular for $n \geqq 8$.

4. Let $\tilde{h}_i = \sum_{k=1}^{n+1} \mu_i^k h_k$, where $\sum_{k=1}^{n+1} \mu_i^k = 0$. Since $\tilde{h}_1, \ldots, \tilde{h}_n$ are linearly independent elements of $\mathfrak{H}$, det. $(\mu_i^k)_{i,k=1,\ldots,n} \neq 0$. Let $\mu^i = (\mu_1^i, \ldots, \mu_n^i)$ for $i = 1, \ldots, n+1$. Then the vectors $\mu^1, \ldots, \mu^n$ are linearly independent and $\mu^{n+1} = -\mu^1 - \ldots - \mu^n$. Hence n vectors taken from $\mu^1, \ldots, \mu^{n+1}$ are all linearly independent. Now $\left[\sum_{j=1}^{n} \lambda^j \tilde{h}_j, e_{ik}\right] = \left(\sum_{j=1}^{n} \lambda^j(\mu_j^i - \mu_j^k)\right) e_{ik}$ and $\left[\sum_{j=1}^{n} \lambda^j \tilde{h}_j, e_{rs}\right] = \left(\sum_{j=1}^{n} \lambda^j(\mu_j^r - \mu_j^s)\right) e_{rs}$. If $\sum_{j=1}^{n} \lambda^j(\mu_j^i - \mu_j^k) = \sum_{j=1}^{n} \lambda^j(\mu_j^r - \mu_j^s)$ for every λ^j such that $\sum_{j=1}^{n} \lambda^j = 0$, then, as we may easily verify, $\mu^i - \mu^k = \mu^r - \mu^s + \xi\binom{rs}{ik}$, where $\xi\binom{rs}{ik} = (c, \ldots, c)$. If this relation holds, we define $e_{ik} \equiv e_{rs}$. We may easily see that if e_{ik} and e_{rs} appear in a singular u_{jl} then $e_{ik} \equiv e_{rs}$.

5. First we consider the case where a relation $\mu^i - \mu^k = \xi$ holds for some i, k, where ξ is a vector whose components are all equal, i.e. $\xi = (d, \ldots, d)$.[18] Then $\left[\sum_{j=1}^{n} \lambda^j \tilde{h}_j, e_{ik}\right] = \left(\sum_{j=1}^{n} \lambda^j(\mu_j^i - \mu_j^k)\right) e_{ik} = \left(\sum_{j=1}^{n} \lambda^j d\right) e_{ik} = 0$.

Since $\mathfrak{h}$ is $(n-1)$-dimensional, we may easily see that $\mathfrak{h}$ is the subspace of $\mathfrak{H}$ consisting of all elements $\lambda^1 h_1 + \ldots + \lambda^{n+1} h_{n+1}$ such that $\lambda^i = \lambda^k$. For simplicity, let $i = n$, $k = n+1$. Let $[h, e_{ik}] = \alpha e_{ik}$ and $[h, e_{rs}] = \beta e_{rs}$, where $h \in \mathfrak{h}$. Then $e_{ik} \equiv e_{rs}$, if and only if $\alpha = \beta$ for every $h \in \mathfrak{h}$. Hence in our case, $e_{ik} \equiv e_{rs}$ if and only if $\lambda^i - \lambda^k = \lambda^r - \lambda^s$. As we may easily verify, the followings are possible :

$$e_{n,k} \equiv e_{n+1,k},\ e_{k,n} \equiv e_{k,n+1},\ (k < n),\ e_{n,n+1} \equiv e_{n+1,n}.$$

Since $[h, e_{n,n+1}] = [h, e_{n+1,n}] = 0$ holds for every $h \in \mathfrak{h}$, $e_{n,n+1}$ and $e_{n+1,n}$ can not appear in u's as factors. Hence if some u's are singular, these u's must be the form $u_\alpha = a e_{n,k} + b e_{n+1,k}$, $u_\beta = c e_{k,n} + d e_{k,n+1}$ and the "roots" α and β corresponding to these u's satisfy the relation $\alpha = -\beta$. Hence $[u_\alpha, u_\beta] = [a e_{n,k} + b e_{n+1,k}, c e_{k,n} + d e_{k,n+1}] = ac\,(h_n - h_k) + bd\,(h_{n+1} - h_k) + bc\, e_{n+1,n} + ad\, e_{n,n+1} \in \mathfrak{h}$. Hence $bc = ad = 0$ and $ac = bd$. This is clearly a contradiction and hence all u's are not singular.

6. Now suppose $e_{ik} \equiv e_{rs}$. Then $\mu^i - \mu^k = \mu^r - \mu^s + \xi\binom{rs}{ik}$. If $\xi\binom{rs}{ik} = 0$.

[18] Through in the following we denote by ξ such a vector.

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then $e_{rs} = e_{ik}$ or $e_{rs} = e_{ki}$, since 4 vectors taken from $\mu^1 \ldots \mu^{n+1}$ are linearly independent for $n \geqq 4$ (c.f. 4.) and $i \neq k$ and $r \neq s$. If $e_{ik} \equiv e_{ki}$, then the relation $\mu^i - \mu^k = \xi$ holds and hence all u's are not singular. Thus we may assume $\xi\binom{rs}{ik} \neq 0$. Further if $i = r$ or $k = s$, we also have the relations $\mu^k - \mu^s = \xi$ or $\mu^i - \mu^r = \xi$. Hence we may assume $i \neq r$, $k \neq s$.

7. Suppose that $e_{ik} \equiv e_{rs}$ and $e_{uv} \equiv e_{xy}$ holds. We may assume that $i \neq r$, $k \neq s$, $u \neq x$ and $v \neq y$. Then we have the relations $\mu^i - \mu^k = \mu^r - \mu^s + \xi\binom{rs}{ik}$ and $\mu^u - \mu^v = \mu^x - \mu^y + \xi\binom{xy}{uv}$. We may assume that ξ's are $\neq 0$ and $\xi\binom{rs}{ik} = d\xi\binom{xy}{uv}$, where $d(\neq 0)$ is a complex number. Then we obtain the relation

$$\mu^i - \mu^k - d\mu^u + d\mu^v = \mu^r - \mu^s - d\mu^x + d\mu^y.$$

i) Let $d \neq \pm 1$. Since 8 vectors taken from $\mu^1 \ldots \mu^{n+1}$ are linearly independent for $n \geqq 8$, we must have $\mu^i - \mu^k - \mu^r + \mu^s = 0$, $\mu^u - \mu^v - \mu^x + \mu^y = 0$ and moreover $i = s$, $k = r$, $u = y$ and $v = x$. Hence $e_{ik} \equiv e_{ki}$ and $e_{uv} \equiv e_{vu}$. Then $\mu^i - \mu^k = \xi$ and $\mu^u - \mu^v = \xi$ hold and all u's are not singular (c.f. 5 and 6).

ii) Let $d = \pm 1$. Then $\mu^i \pm \mu^v + \mu^s \pm \mu^x - \mu^k \mp \mu^u - \mu^r \mp \mu^y = 0$. As in i) 8 vectors are linearly independent and since $i \neq k$, $r \neq s$, $u \neq v$, $x \neq y$, $i \neq r$, $k \neq s$, $u \neq x$ and $v \neq y$, we may verify that the following cases are possible:

1) $e_{ik} \equiv e_{rs}$, $e_{ir} \equiv e_{ks}$

2) $e_{ik} \equiv e_{rs}$, $e_{ri} \equiv e_{sk}$

3) $e_{ik} \equiv e_{rs}$, $e_{ki} \equiv e_{sr}$

4) $e_{ik} \equiv e_{ki}$, $e_{uv} \equiv e_{vu}$

5) $e_{ik} \equiv e_{ks}$, $e_{sv} \equiv e_{vi}$

6) $e_{ik} \equiv e_{ks}$, $e_{iv} \equiv e_{vs}$

But in the cases 4), 5) and 6), we obtain the relation of the form $\mu^l - \mu^m = \xi$. Hence we have only to treat 1), 2) and 3).

8. Then it is easily verified that the possible singular u's are as follows:

$$u_\alpha = a_\alpha e_{ik} + b_\alpha e_{rs}, \quad u_{-\alpha} = a_{-\alpha} e_{ki} + b_{-\alpha} e_{sr};$$
$$u_\beta = a_\beta e_{ir} + b_\beta e_{ks}, \quad u_{-\beta} = a_{-\beta} e_{ri} + b_{-\beta} e_{sk}.$$

Let $u_\alpha = u_{pq}$. Then $u_\alpha = [u_{pt}, u_{tq}]$ $(t = 1, \ldots, n)$. Clearly, for some t, u_{pt} and u_{tq} are not singular. Let $u_{pt} = ae_{uv}$ and $u_{tq} = be_{xy}$. Then $u_\alpha = [u_{pt}, u_{tq}] = [a\,e_{uv}, be_{xy}] = ab\,(\delta_{vx} e_{uy} - \delta_{uy} e_{xv})$. Hence it follows that $u = i$, $y = k$, $x = r$, $v = s$ or $u = r$, $y = s$, $x = i$, $v = k$. Then, since $i \neq k$ and $r \neq s$, $\delta_{uy} = \delta_{vx} = 0$. However, they can not bo the case. Thus we have proved that no u is singular.

9. Let $u_\alpha = a_\alpha e_{i_\alpha j_\alpha}$. We want to prove that the set of indices $S = \bigcup_\alpha \{i_\alpha, j_\alpha\}$ is a proper subset of $\{1, 2, \ldots, n+1\}$. Suppose, for this purpose, that $S = \{1, 2, \ldots, n+1\}$. Let $1 \leqq s, t \leqq n+1$. Then there exist α and β such that $s = i_\alpha$ or j_α and $t = i_\beta$ or j_β. Take a u_ε such that $[u_\alpha, u_\varepsilon] \neq 0$ and $[u_\varepsilon, u_\beta] \neq 0$. Then $[e_{i_\alpha j_\alpha}, e_{i_\varepsilon j_\varepsilon}] = \delta_{j_\alpha i_\varepsilon} e_{i_\alpha j_\varepsilon} - \delta_{j_\varepsilon i_\alpha} e_{j_\varepsilon i_\alpha} \neq 0$. Hence $i_\varepsilon = j_\alpha$ or $i_\alpha = j_\varepsilon$. It follows similarly that $i_\varepsilon = j_\beta$ or $j_\varepsilon = i_\beta$. Now there exists an element $h (\neq 0)$ of $\mathfrak{H}$ such that $[h, \mathfrak{U}] = 0$.[19] Then $[h, u_\gamma] = 0$ for every u_γ. Let $h = \lambda^1 h_1 + \ldots + \lambda^{n+1} h_{n+1}$. It follows that $\lambda^{i_\alpha} = \lambda^{j_\alpha}$, $\lambda^{i_\varepsilon} = \lambda^{j_\varepsilon}$ and $\lambda^{i_\beta} = \lambda^{j_\beta}$. From the above relations we obtain $\lambda^{i_\alpha} = \lambda^{j_\alpha} = \lambda^{i_\varepsilon} = \lambda^{j_\varepsilon} = \lambda^{i_\beta} = \lambda^{j_\beta}$. Hence $\lambda^s = \lambda^t$. Since this holds for every pair of s and t, it follows that $\lambda^1 = \lambda^2 = \ldots = \lambda^{n+1}$. Then necessarily $\lambda^1 = \lambda^2 = \ldots = \lambda^{n+1} = 0$, since $\sum_{i=1}^{n+1} \lambda^i = 0$. Hence $h = 0$ and this is impossible. Thus $S \neq \{1, 2, \ldots, n+1\}$. Hence there exists an integer $s (1 \leqq s \leqq n+1)$ such that $s \,\overline{\in}\, S$. Then we see easily that $\mathfrak{U}$ is contained in the Lie algebra $\mathfrak{A} = \mathfrak{h}' + \sum_{\substack{i,k=1 \\ i \neq s, k \neq s}}^{n+1} e_{ik} K$, where $\mathfrak{h}'$ is composed of all elements $\lambda^1 h_1 + \ldots + \lambda^{n+1} h_{n+1}$ of $\mathfrak{H}$ such that $\lambda^s = 0$. But since $\mathfrak{U}$ and $\mathfrak{A}$ have the same dimension, it follows that $\mathfrak{U} = \mathfrak{A}$. Then every matrix of $\mathfrak{U}$ transforms the s-th axis of the $(n+1)$-dimensional complex vector space into 0. Then "unitary ristricted" $\mathfrak{U}_c$ transforms the s-th axis of the unitary $(n+1)$-space into 0. The integrated group U leaves fixed the same axis. Then clearly U is conjugate to A_{n-1}.

IV.

1. We consider now the group $G = C_n$, the unitary simplectic group of $2n$ variables. G consists of all unitary matrices of degree $2n$ which leave the skew-symmetric bilinear form

$$S(x, y) = (x_1 y_1' - x_1' y_1) + (x_2 y_2' - x_2' y_2) + \ldots + (x_n y_n' - x_n' y_n)$$

invariant, where the vector x has the componentes $(x_1, \ldots, x_n, x_1', \ldots, x_n')$.

We denote by C_{n-1} the subgroup of G conisting of all matrices of G which leave fixed the variables x_1 and x_1' i.e. $x_1 \to x_1$, $x_1' \to x_1'$. Then $G/C_{n-1} = S^{4n-1}$ and hence $C_{n-1} \neq 0$.[12]

We prove the following

Theorem III. *Every S-subgroup of $G = C_n$ is conjugate to C_{n-1} for $n > 4$.*

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[19] Take an element h_1 of $\mathfrak{H}$ which is not contained in $\mathfrak{h}$. Since $[h_1, \mathfrak{U}] \subseteqq \mathfrak{U}$, the mapping $u \to [h_1, u]$ $(u \in \mathfrak{U})$ is a derivation of $\mathfrak{U}$. As the derivations of the simple Lie algebra $\mathfrak{U}$ are inner, there exists an element u_0 of $\mathfrak{U}$ such that $[h_1, u] = [u_0, u]$ for every $u \in \mathfrak{U}$. But since $[h_1, u] = 0$ for every $u \in \mathfrak{h}$, u_0 commutes with every element of $\mathfrak{h}$. $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{U}$ and hence $u_0 \in \mathfrak{h}$. Then the element $h_1 - u_0$ satisfies our condition.

2. Let U be an S-subgroup of G. Then $R(G/U) = R(S^m)$ (m: odd) and $R(G) = R(U \times S^m)$. The homology ring of G is

$$R(G) = R(C_n) = R(S^3 \times S^7 \times \dots \times S^{4n-1}).^{15)}$$

Hence $m = 4k - 1$ and

(1) $$R(U) = R(S^3 \times S^7 \times \dots \times S^{4k-5} \times S^{4k+3} \times \dots \times S^{4n-1}).$$

Since S^3 appears in (1) only once, U is simple.[16] As we may easily verify, U can not be the exceptional groups F_4, E_7 and E_8. If $U = E_6$, then we may easily verify that the homology ring of E_6 must be isomorphic with that of C_6 and this is a contradiction.[20] Hence if $n > 3$, U is a classical simple group. Then (1) is impossible if $m < 4n - 1$. Hence $m = 4n - 1$ and $R(U) = R(S^3 \times S^7 \times \dots \times S^{4n-5})$. This shows that U is a simple group of type C_{n-1} or B_{n-1}.[15]

3. Let the Lie algebras $\mathfrak{G}_c$, $\mathfrak{G}$, $\mathfrak{U}_c$, $\mathfrak{U}$, $\mathfrak{H}_c$, $\mathfrak{H}$, $\mathfrak{h}_c$ and $\mathfrak{h}$ be defined as in III, 3. Then $\mathfrak{G}$ has the following basis:

$$\mathfrak{G} = Kh_1 + \dots + Kh_n + \sum_{i=1}^{n} Ke_{\pm 2\lambda^i} + \sum_{i,k=1}^{n} Ke_{\pm\lambda^i\pm\lambda^k}$$

(K: the field of complex numbers),

where $\mathfrak{H} = Kh_1 + \dots + Kh_n$ and $\left[\sum_{j=1}^{n} \lambda^j h_j,\ e_{\pm 2\lambda^i}\right] = (\pm 2\lambda^i)\, e_{\pm 2\lambda^i}$ and $\left[\sum_{j=1}^{n} \lambda_j h_j,\ e_{\pm\lambda^i\pm\lambda^k}\right] = (\pm\lambda^i \pm \lambda^k)\, e_{\pm\lambda^i\pm\lambda^k}$. First let $\mathfrak{U}$ be a simple algebra of type C_{n-1}. Then $\mathfrak{U}$ has the following basis:

$$\mathfrak{U} = K\tilde{h}_1 + \dots + K\tilde{h}_{n-1} + \sum_{i=1}^{n-1} Ku_{\pm 2\lambda^i} + \sum_{i,k=1}^{n-1} Ku_{\pm\lambda^i\pm\lambda^k},$$

where $\mathfrak{h} = K\tilde{h}_1 + \dots + K\tilde{h}_{n-1}$ ($\subset \mathfrak{H}$) and the commutator products of $\tilde{h}$'s and u's are defined anlogously as in $\mathfrak{G}$. If $\mathfrak{U}$ is a simple algebra of type B_{n-1}, $\mathfrak{U}$ has the following basis:

$$\mathfrak{U} = K\tilde{h}_1 + \dots + K\tilde{h}_{n-1} + \sum_{i=1}^{n-1} Ku_{\pm\lambda^i} + \sum_{i,k=1}^{n-1} Ku_{\pm\lambda^i\pm\lambda^k},$$

where $\mathfrak{h} = K\tilde{h}_1 + \dots + K\tilde{h}_{n-1}$ ($\subset \mathfrak{H}$) and $\left[\sum_{i=1}^{n-1} \lambda^j \tilde{h}_j,\ u_{\pm\lambda^i}\right] = (\pm\lambda^i\, u_{\pm\lambda^i})$ and $\left[\sum_{j=1}^{n-1} \lambda^j \tilde{h}_j,\ u_{\pm\lambda^i\pm\lambda^k}\right] = (\pm\lambda^i \pm \lambda^k)\, u_{\pm\lambda^i\pm\lambda^k}$. Clearly u_α's are the linear combinations of e_β's. A u_α is said to be *singular* if it is a linear combination of at least two e_β's. We shall prove that non of them is singular in case $n > 4$.

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[20] See, Yen Chih-Ta, Sur les polynomes de Poincaré des groupes simples exceptionnels, C. R. Paris, **228**, (1949).

4. Let $\tilde{h}_i = \sum_{k=1}^{n} \mu_i^k h_k$. Since $\tilde{h}_i$ are linearly independent the matrix (μ_i^k) $(k = 1, \ldots, n;\ i = 1, \ldots, n-1)$ has rank $n-1$. Let $\mu^k = (\mu_1^k, \ldots, \mu_{n-1}^k)$, $k = 1, \ldots, n$. Then $n-1$ of these n vectors are linearly independent. We may assume that $\mu^1, \mu^2, \ldots, \mu^{n-1}$ are so. Now

$$\left[\sum_{j=1}^{n-1} \lambda^j \tilde{h}_j,\ e_{\varepsilon\lambda^i+\delta\lambda^k}\right] = \left(\sum_{j=1}^{n-1} \lambda^j (\varepsilon\mu_j^i + \delta\mu_j^k)\right) e_{\varepsilon\lambda^i+\delta\lambda^k},$$

where $\varepsilon = \pm 1$, $\delta = \pm 1$ for $i \neq k$ and for $i = k$, $\varepsilon = \delta = \pm 1$, if $\mathfrak{U}$ is of type C_{n-1}, and $\varepsilon = 0$, $\delta = \pm 1$, if $\mathfrak{U}$ is of type B_{n-1}.

If $\sum_{j=1}^{n-1} \lambda^j (\varepsilon\mu_j^i + \delta\mu_j^k) = \sum_{j=1}^{n-1} \lambda^j (\varepsilon'\mu_j^r + \delta'\mu_j^s)$ holds for every λ^j, then the relation $\varepsilon\mu^i + \delta\mu^k = \varepsilon'\mu^r + \delta'\mu^s$ holds. In this case we write $e_{\varepsilon\lambda^i+\delta\lambda^k} \equiv e_{\varepsilon'\lambda^r+\delta'\lambda^s}$. If e_α and e_β appear in a singular u_γ, then $e_\alpha \equiv e_\beta$.

5. Let $\mu^n = 0$. Then the possible relations are $e_{\varepsilon\lambda^i+\lambda^n} \equiv e_{\varepsilon\lambda^i-\lambda^n}$ $(\varepsilon = \pm 1;\ 1 \leqq i \leqq n)$. Hence the possible singular u's are the forms

$$u_{\alpha_i} = a_i e_{\lambda^i+\lambda^n} + b_i e_{\lambda^i-\lambda^n},\quad u_{-\alpha_i} = c_i e_{-\lambda^i-\lambda^n} + d_i e_{-\lambda^i+\lambda^n}.$$

Then $[u_{\alpha_i}, u_{-\alpha_i}] = h + b_i c_i N_i e_{-2\lambda^n} + a_i d_i N_i' e_{2\lambda^n} \in \mathfrak{h}$, where $h \in \mathfrak{h}$ and $[e_{\lambda^i-\lambda^n}, e_{-\lambda^i-\lambda^n}] = N_i e_{-2\lambda^n}$ and $[e_{\lambda^i+\lambda^n}, e_{-\lambda^i+\lambda^n}] = N_i' e_{2\lambda^n}$ and $N_i \neq 0$, $N_i' \neq 0$. Hence it follows $b_i c_i = a_i d_i = 0$. This shows that u_{α_i} and $u_{-\alpha_i}$ can not be singular.

6. Now let $\mu^n \neq 0$. Then $\mu^n = \sum_{i=1}^{n-1} a_i \mu^i$. Let m be the number of indices i such that $a_i \neq 0$. If $m \geqq 4$, then no u_α is singular. For, if $e_{\varepsilon\lambda^i+\delta\lambda^k} \equiv e_{\varepsilon'\lambda^r+\delta'\lambda^s}$, then $\varepsilon\mu^i + \delta\mu^k = \varepsilon'\mu^r + \delta'\mu^s$ and at least one of these vectors must be μ^n. Then μ^n is a linear combination of at most 3 vectors and this is impossible. Hence we may assume $m \leqq 3$.

i) First let $m = 1$. For simplicity let $\mu^n = a_1\mu^1$. Let $\alpha = \varepsilon\lambda^i + \delta\lambda^k$. We denote for simplicity the indices i, k as the indices of α and e_α. Let $u_\alpha = \sum_i a_{\beta_i} e_{\beta_i}$ be singular. We show that the indices of β_i's are 1 and n. Suppose that β_1 has an index i different from 1 and n and let $\beta_1 = \varepsilon_1\lambda^i + \varepsilon_2\lambda^j$. Further let $\beta_2 = \eta_1\lambda^k + \eta_2\lambda^l$. Then since $e_{\beta_1} \equiv e_{\beta_2}$, $\varepsilon_1\mu^i + \varepsilon_2\mu^j = \eta_1\mu^k + \eta_2\mu^l$. If $i, j, k, l < n$, this is impossible. Since $i \neq 1, n$, α) first let $j = n$. Then $\varepsilon_1\mu^i + \varepsilon_2 a_1\mu^1 = \eta_1\mu^k + \eta_2\mu^l$. Since $i \neq 1$, it follows that $i = k$, $\varepsilon_1 = \eta_1$ and $1 = l$, $\varepsilon_2 a_1 = \eta_2$. Then $\beta_1 = \varepsilon_1\lambda^i + \varepsilon_2\lambda^n$ and $\beta_2 = \varepsilon_1\lambda^i + \varepsilon_2 a_1\lambda^1$. β) Let $k = n$. Then $\beta_1 = \varepsilon_1\lambda^i + \varepsilon_2\lambda^1$ and $\beta_2 = \varepsilon_1\lambda^i + \varepsilon_2 a_1\lambda^n$. γ) Let $l = n$. Then $\beta_1 = \varepsilon_1\lambda^i + \varepsilon_2\lambda^1$ and $\beta_2 = \varepsilon_1\lambda^i + \varepsilon_2 a_1\lambda^n$. In either case β_2 is determined by β_1 uniquely, whence e_{β_3} can not appear in u_α, and $u_{-\alpha}$ must be the forms $u_\alpha = ae_{\lambda^i+\varepsilon_2\lambda^1} + be_{\lambda^i+\eta_2\lambda^n}$, $u_{-\alpha} = ce_{-\lambda^i-\varepsilon_2\lambda^1} + de_{-\lambda^i-\eta_2\lambda^n}$. Then $[u_\alpha, u_{-\alpha}] \in \mathfrak{h}$ and we may prove in the same way as in 5 that $bc = 0$ and $ad = 0$. Then u_α and $u_{-\alpha}$ are not singular. Hence the indices of β_i are 1 and n. But there are only 8 e_β's having indices 1 and 2, i.e. $e_{\pm 2\lambda^1}$, $e_{\pm\lambda^n}$, $e_{\pm\lambda^1\pm\lambda^n}$. Hence the number of possible singular u's is at most 4.

ii) Next let $m = 2$. For simplicity let $\mu^n = a_1\mu^1 + a_2\mu^2$. If $u_\alpha = \sum_i a_{\beta_i}e_{\beta_i}$ is singular, then we may prove as in i) that the indices of β_i's form subset of $\{1, 2, n\}$. But in each singular u_α at least one e_{β_i} must appear, one of whose indices is n. The number of such e_β's are 10, i.e. $e_{\pm\lambda^n}$, $e_{\pm\lambda^1\pm\lambda^n}$ and $e_{\pm\lambda^2\pm\lambda^n}$. But it is impossible that, for example, $e_{\lambda^1+\varepsilon\lambda^n}$ and $e_{-\lambda^1+\varepsilon\lambda^n}$ both appear in the singular u's. For, then the relations $e_{\varepsilon_1\lambda^1+\varepsilon_2\lambda^2+\varepsilon_3\lambda^n} \equiv e_{\lambda^1+\varepsilon\lambda^n}$ and $e_{\delta_1\lambda^1+\delta_2\lambda^2+\delta_3\lambda^n} \equiv e_{-\lambda^1+\varepsilon\lambda^n}$ hold, where ε_i, $\delta_i = \pm 1$ or 0 and at least one of ε_i and δ_i is 0 respectively. These lead to a contradiction as we may easily verify. From these facts we see that the number of the possible singular u's is at most 6.

iii) Finally let $m = 3$ and let, for simplicity, $\mu^n = a_1\mu^1 + a_2\mu^2 + a_3\mu^3$. Then as above the singular u's are the linear combinations of e_{β_i}'s whose indices form subsets of $\{1, 2, 3, n\}$ and in each u_α at least one e_{β_i} with index n must appear. The number of such e_{β_i}'s is 14, i.e. $e_{\pm 2\lambda^n}$, $e_{\pm\lambda^1\pm\lambda^n}$, $e_{\pm\lambda^2\pm\lambda^n}$ and $e_{\pm\lambda^3\pm\lambda^n}$. But $e_{\pm 2\lambda^n}$ can not appear in the singular u's, for if did μ^n would be a linear combination of two μ's. As in ii) it is also impossible that, for example, $e_{\lambda^1+\varepsilon\lambda^n}$ and $e_{-\lambda^1+\varepsilon\lambda^n}$ both appear in the singular u's. Hence the number of possible singular u's is at most 6. Thus we have shown that the number of singular elements is at most 6. We see also from the above consideration that if the number s of singular elements is 6, then for every singular u_β, $u_{-\beta}$ is also singular. Further we see that if $s = 5$, then there exist singular u_{β_1} and u_{β_2} such that $u_{-\beta_1}$ and $u_{-\beta_2}$ are also singular. These hold equally for $\mathfrak{U}$ of type B_{n-1} and of type C_{n-1}.

7. Next we prove that if $n > 4$, no u is singular.

i) Let $\mathfrak{U}$ be the type B_{n-1}. Suppose that $u_{\varepsilon\lambda^i+\delta\lambda^k}$ $(i \neq k,\ \varepsilon = \pm 1,\ \delta = \pm 1)$ is singular. Then since $[u_{\varepsilon\lambda^i}, u_{\delta\lambda^k}] = N_{i,k}\, u_{\varepsilon\lambda^i+\delta\lambda^k}$, $u_{\varepsilon\lambda^i}$ or $u_{\delta\lambda^k}$ must be singular. Let $u_{\varepsilon\lambda^i}$ be singular. Then since $u_{\varepsilon\lambda^i} = N_t[u_{\varepsilon\lambda^i+\lambda^t}, u_{-\lambda^t}]$, $(t = 1, \ldots, n-1,\ t \neq i)$, $u_{\varepsilon\lambda^i+\lambda^t}$ or $u_{-\lambda^t}$ is singular for every t. Hence we get a set of $n-2$ singular elements. Further from the relations $u_{\varepsilon\lambda^i} = N_t'[u_{\varepsilon\lambda^i-\lambda^t}, u_{\lambda^t}]$ $(t = 1, \ldots, n-1,\ t \neq i)$, we get also a set of $n-2$ singular elements and these two sets have no common elements. $u_{\lambda\varepsilon^i}$ is also a singular element different from these $2(n-2)$ singular element. Hence there are at least $2(n-2)+1$ singular elements. If the number s of singular elements is $\leqq 4$, then $2(n-2)+1 \leqq 4$ and hence $n \leqq 3$. If $s = 5$ we see easily that there exists at least one singular u_β different from the above $2(n-2)+1$ elements and hence also $n \leqq 3$. If $s = 6$, there exist at least two singular elements different from the above $2(n-2)+1$ element and so also $n \leqq 3$. Hence we may assume that all $u_\alpha(\alpha = \pm\lambda^i \pm\lambda^k,\ i, k = 1, \ldots, n-1)$ are not singular. If one of $u_{\pm\lambda^i}$, say u_{λ^1}, is not singular,

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then since $[u_{\lambda^1}, u_{-\lambda^1\pm\lambda^i}] = N_i u_{\pm\lambda^i}$ $(i = 2, \ldots, n = 1)$, $u_{\pm\lambda^i}$ $(i = 2, \ldots, n-1)$ are also not singular. Then since $[u_{-\lambda^1+\lambda^i}, u_{-\lambda^i}] = N_i' u_{-\lambda^1}$, $u_{-\lambda^1}$ is also not singular and hence all u's are not singular. If all of $u_{\pm\lambda^i}$ $(i = 1, \ldots, n-1)$ are singular, there exist $2(n-1)$ singular element. Hence $2(n-1) \leqq 6$, i.e. $n \leqq 4$.[21] Thus if $n > 4$ u's are not singular.

ii) If $\mathfrak{U}$ is of type C_{n-1}, we may prove analogously that none of u is singular in case $n > 3$.

8. We prove now that $\mathfrak{U}$ is not of type B_{n-1}. Let $u_{\lambda^i} = a_i e_{\alpha_i}$ and $u_{-\lambda^i} = b_i e_{-\alpha_i}$ $(i = 1, 2, \ldots, n-1)$. Then since $[u_{\varepsilon\lambda^i}, u_{\delta\lambda^j}] \neq 0$ $(\varepsilon, \delta = \pm 1)$, we see that the set of $2(n-1)$ "roots" $\{\pm\alpha_i\}$ of $\mathfrak{G}$ has the property that $\pm\alpha_i \pm\alpha_j$ are also the roots of $\mathfrak{G}$. Let one of α_i be the form $\pm 2\lambda^j$; for example let $\alpha_1 = 2\lambda^j$. Further let $\alpha_2 = \varepsilon\lambda^i + \delta\lambda^k$, then since $\alpha_1 + \alpha_2$ is a root, it follows that $\alpha_2 = -\lambda^j + \delta\lambda^k$. But then $-\alpha_1 + \alpha_2 = -2\lambda^j - \lambda^j + \delta\lambda^k$ is not a root of $\mathfrak{G}$. Hence α_i are of the forms $\pm\lambda^k \pm\lambda^j$. Let $\alpha_1 = \varepsilon\lambda^i + \delta\lambda^k$ and $\alpha_2 = \varepsilon_1\lambda^j + \delta_1\lambda^l$. Since $\alpha_1 + \alpha_2 = \varepsilon\lambda^i + \delta\lambda^k + \varepsilon_1\lambda^j + \delta_1\lambda^l$ is a root, it follows that $\alpha_2 = -\varepsilon\lambda^i + \eta\lambda^s$ or $\alpha_2 = -\delta\lambda^k + \eta\lambda^s$, where $s = j$ or l. But since $\alpha_1 - \alpha_2$ is also a root it follows that $\alpha_2 = -\varepsilon\lambda^i + \delta\lambda^k$ or $\alpha_2 = \varepsilon\lambda^i - \delta\lambda^k$. Thus α_2 and $-\alpha_2$ are determined uniquely by α_1. Hence if $n > 3$, there is no such a set of roots $\{\pm\alpha_i\}$ of $\mathfrak{G}$. Therefore $\mathfrak{U}$ is not of type B_{n-1}.

9. By 8. we know that $\mathfrak{U}$ is of type C_{n-1}. Now let $u_{2\lambda^i} = a_i e_{\alpha_i}$ and $u_{-2\lambda^i} = b_i e_{-\alpha_i}$ $(i = 1, 2, \ldots, n-1)$. Since $[u_{\varepsilon 2\lambda^i}, u_{\delta 2\lambda^j}] = 0$, the set of $2(n-1)$ roots $\{\pm\alpha_i\}$ of $\mathfrak{G}$ has the property that $\pm\alpha_i \pm\alpha_j$ are not the roots of $\mathfrak{G}$ at all. Suppose that $\alpha_j = \varepsilon_j 2\lambda^{i_j}$ for $1 \leqq j \leqq k$ and $\alpha_s = \delta_s\lambda^{j_s} + \eta_s\lambda^{l_s}$ $(j_s \neq l_s)$ for $k+1 \leqq s \leqq n-1$. Then the sets of indices $\{i_1\}, \ldots, \{i_k\}, \{j_{k+1}, l_{k+1}\}, \ldots, \{j_{n-1}, l_{n-1}\}$ have no common indices. Hence the number of these indices is $k + 2(n-k-1) = 2(n-1) - k$. Thus $2(n-1) - k \leqq n$ and hence $k \geqq n-2$. Suppose that $k = n-2$ and let, for simplicity, $\{i_1, \ldots, i_{n-1}\} = \{1, 2, \ldots, n-2\}$, $\{j_{n-1}, l_{n-1}\} = \{n-1, n\}$ and $u_{2\lambda^n} = a_{n-1} e_{\varepsilon\lambda^{n-1}+\delta\lambda^n}$. Now there exists an element $h = \lambda^1 h_1 + \ldots + \lambda^n h_n \in \mathfrak{H}$ $(h \neq 0)$ such that $[h, u] = 0$ for all $u \in \mathfrak{U}$.[19] Since $[h, u_{2\lambda^i}] = 0$, it follows that $\lambda^1 = \lambda^2 = \ldots = \lambda^{n-2} = 0$ and $\varepsilon\lambda^{n-1} + \delta\lambda^n = 0$. Let $u_{\lambda^{n-1}+\lambda^i} = ce_{\beta_1}$ and $u_{\lambda^{n-1}-\lambda^i} = de_{\beta_2}$. As $[h, e_{\beta_k}] = 0$ $(k = 1, 2)$ and $\beta_k \neq \pm(\varepsilon\lambda^{n-1} + \delta\lambda^n)$, the indices of β_k's are $< n-1$. But $[u_{\lambda^{n-1}+\lambda^i}, u_{\lambda^{n-1}-\lambda^i}] = N_i u_{2\lambda^{n-1}}$ and this is a contradiction. Hence $k = n-1$. Suppose $\{i_1, \ldots, i_{n-1}\} = \{1, 2, \ldots, n-1\}$. Then we easily see that $u_{\varepsilon\lambda^i+\delta\lambda^k} = a_{ik} e_{\varepsilon_1\lambda^j+\delta_1\lambda^l}$ $(1 \leqq i, k \leqq n-1)$ and $1 \leqq j, l \leqq n-1$. Now let $\mathfrak{A} = Kh_1 + \ldots + Kh_{n+1} + \sum_{i=1}^{n-1} Ke_{\pm 2\lambda^i} + \sum_{i,k=1}^{n-1} Ke_{\pm\lambda^i\pm\lambda^k}$. Then $\mathfrak{A}$ is a subalgebra of $\mathfrak{G}$ of type C_{n-1} containing $\mathfrak{U}$. Hence $\mathfrak{A} = \mathfrak{U}$. Then each matrix of $\mathfrak{U}$

21) If $n = 4$, there may be 6 singular elements.

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transforms the variables x_n and x_n' into 0. The same holds for every matrix of $\mathfrak{U}_c$. Hence each matrix of the integrated group U leaves fixed the variables x_n and x_n' (i.e. $x_n \to x_n$, $x_n' \to x_n'$). Then clearly U is conjugate to C_{n-1}. Thus Theorem III is proved.

V.

Now let G be the exceptional group of rank > 2. The Poincaré polynomials of the exceptional groups are as follows:[20]

G_2: $(1+x^3)\ (1+x^{11})$;
F_4: $(1+x^3)\ (1+x^{11})\ (1+x^{15})\ (1+x^{23})$;
E_6: $(1+x^3)\ (1+x^9)\ (1+x^{11})\ (1+x^{15})\ (1+x^{17})\ (1+x^{23})$;
E_7: $(1+x^3)\ (1+x^{11})\ (1+x^{15})\ (1+x^{19})\ (1+x^{23})\ (1+x^{27})\ (1+x^{35})$;
E_8: $(1+x^3)\ (1+x^{15})\ (1+x^{23})\ (1+x^{27})\ (1+x^{35})\ (1+x^{39})\ (1+x^{47})\ (1+x^{59})$.

From this table, we may easily see the following

THEOREM IV. *Exceptional simple groups of rank > 2 have no S-subgroups.*

Mathematical Institute,
Nagoya University.

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