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Period relations for Rankin–Selberg convolutions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$

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Abstract

We formulate and prove the archimedean period relations for Rankin–Selberg convolutions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$. As a consequence, we prove the period relations for critical values of the Rankin–Selberg L-functions for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ over arbitrary number fields.

Contents

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1. Introduction

The cases of $\mathrm{GL}(n)\times\mathrm{GL}(n-1)$ and $\mathrm{GL}(n)\times\mathrm{GL}(n)$ are fundamental in the general Rankin–Selberg theory, and many problems for general Rankin–Selberg convolutions are reduced to these two cases. The goal of this article is to give an unconditional proof of the period relations for critical values of Rankin–Selberg L-functions for $\mathrm{GL}(n)\times\mathrm{GL}(n-1)$ over arbitrary number fields, which is a long-standing problem and has been studied by many authors (see § 1.2 for some relevant works). In the framework of Langlands program, it is compatible with the celebrated conjecture of Deligne [Del79] on the rationality of critical values of L-functions attached to pure motives. More general conjectures concerning period relations for critical values of Rankin–Selberg L-functions are formulated by Blasius in [Bla97].

1.1 Whittaker periods

Let k be a number field, and write $\mathbb{A}$ for the adele ring of k. Denote by k_v the completion of k at a place v. Write

$$\mathrm{k}_\infty:=\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{R}=\prod_{v\mid\infty}\mathrm{k}_v\hookrightarrow\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C}=\prod_{\iota\in\mathcal{E}_{\mathrm{k}}}\mathbb{C},$$

where $\mathcal{E}_{\mathrm{k}}$ is the set of field embeddings $\iota:\mathrm{k}\hookrightarrow\mathbb{C}$.

Let Π be an irreducible subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$ $(n\geqslant 1)$. Here $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$ denotes the space of all smooth automorphic forms on $\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A})$, which is a smooth representation of $\mathrm{GL}_n(\mathbb{A})$ (see [LS19, § 3.2] and [GZ24]). Assume that Π is cuspidal or (more generally) tamely isobaric as defined in (63). It should be mentioned that allowing Π to be isobaric is an old idea going back to Schmidt, Mahnkopf, and Grobner (see [Sch93, Mah05, Gro18]). Suppose that Π is regular algebraic in the sense of Clozel (see [Clo90]). By [Clo90, § 3], up to isomorphism there is a unique irreducible algebraic representation F_μ of $\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})$, say of highest weight $\mu=\{\mu^\iota\}_{\iota\in\mathcal{E}_{\mathrm{k}}}\in(\mathbb{Z}^n)^{\mathcal{E}_{\mathrm{k}}}$, such that the total continuous cohomology

$$\mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathrm{k}_\infty)^0;F_\mu^\vee\otimes\Pi_\infty)\neq\{0\}. \tag{1}$$

Here $\Pi_\infty:=\widehat{\otimes}_{v\mid\infty}\Pi_v$ is the infinite part of Π, a superscript '$\vee$' over a representation indicates the contragradient representation, and a superscript '0' over a Lie group indicates the identity connected component of the Lie group. Moreover, μ is pure in the sense that there exists $w_\mu\in\mathbb{Z}$ such that

$$\mu_1^\iota+\mu_n^{\bar\iota}=\mu_2^\iota+\mu_{n-1}^{\bar\iota}=\cdots=\mu_n^\iota+\mu_1^{\bar\iota}=w_\mu$$

for all $\iota\in\mathcal{E}_{\mathrm{k}}$. Here we write $\mu^\iota=(\mu_1^\iota,\dots,\mu_n^\iota)$, and $\bar\iota$ is the composition of

$$\mathrm{k}\xrightarrow{\iota}\mathbb{C}\xrightarrow{\text{complex conjugation}}\mathbb{C}.$$

The representation F_μ is called the coefficient system of Π.

Let $\Pi_f:=\otimes'_{v\nmid\infty}\Pi_v$ be the finite part of Π. The rationality field $\mathbb{Q}(\Pi)$ of Π is the fixed field of the group of field automorphisms $\sigma\in\mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma(\Pi_f)=\Pi_f$. This is a number field contained in $\mathbb{C}$. By [Clo90, Theorem 3.13] and [Gro18, Lemma 1.2], for every $\sigma\in\mathrm{Aut}(\mathbb{C})$, there exists a unique irreducible subrepresentation ${}^\sigma\Pi$ of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$ that is tamely isobaric and regular algebraic, and whose finite part $({}^\sigma\Pi)_f$ is isomorphic to ${}^\sigma(\Pi_f)$. See § 6.2 for more details.

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The cohomology space in (1) is naturally a representation of the group

$$\pi_0(\mathrm{GL}_n(\mathrm{k}_\infty)) := \mathrm{GL}_n(\mathrm{k}_\infty)/\mathrm{GL}_n(\mathrm{k}_\infty)^0.$$

By using the determinant homomorphism, the latter group is identified with

$$\pi_0(\mathrm{k}_\infty^\times) := \mathrm{k}_\infty^\times/(\mathrm{k}_\infty^\times)^0 = \{\pm 1\}^{\mathcal{E}_\mathrm{k}^\mathbb{R}},$$

where $\mathcal{E}_\mathrm{k}^\mathbb{R}$ denotes the set of real places of k, which is identified with a subset of $\mathcal{E}_\mathrm{k}$. The group of characters of $\pi_0(\mathrm{k}_\infty^\times)$ is denoted by $\widehat{\pi_0(\mathrm{k}_\infty^\times)}$, which is obviously identified with the group of quadratic characters of $\mathrm{k}_\infty^\times$.

For every archimedean local field $\mathbb{K}$, put

$$b_{n,\mathbb{K}} := \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor, & \text{if } \mathbb{K} \cong \mathbb{R}; \\ \dfrac{n(n-1)}{2}, & \text{if } \mathbb{K} \cong \mathbb{C}. \end{cases}$$

Write

$$b_{n,\infty} := \sum_{v|\infty} b_{n,\mathrm{k}_v}.$$

Let ε_{Π_∞} denote the central character of $F_\mu^\vee \otimes \Pi_\infty$. Note that ε_{Π_∞} is a quadratic character of $\mathrm{k}_\infty^\times$, and is trivial when n is even. By [Clo90, Lemma 3.14],

$$\mathrm{H}_{\mathrm{ct}}^i(\mathrm{GL}_n(\mathrm{k}_\infty)^0; F_\mu^\vee \otimes \Pi_\infty) = \{0\}, \quad \text{if } i < b_{n,\infty},$$

and as a representation of $\pi_0(\mathrm{GL}_n(\mathrm{k}_\infty))$,

$$\mathrm{H}_{\mathrm{ct}}^{b_{n,\infty}}(\mathrm{GL}_n(\mathrm{k})^0; F_\mu^\vee \otimes \Pi_\infty) \cong \begin{cases} \bigoplus_{\varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)}} \varepsilon, & \text{if } n \text{ is even}; \\ \varepsilon_{\Pi_\infty}, & \text{if } n \text{ is odd}. \end{cases} \tag{2}$$

We are particularly interested in the bottom degree cohomology space (2).

For every $\varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)}$ that occurs in the bottom degree cohomology space (2), by comparing the Betti and de Rham cohomologies of the (tower of) locally symmetric spaces attached to $\mathrm{GL}_n(\mathbb{A})$, Raghuram and Shahidi define a nonzero complex number, to be called the Whittaker period for Π and ε (see [RS08b, Definition/Proposition 3.3]). The basic idea of this period construction goes back to Hida, Harder, Mahnkopf, and Schmidt. These Whittaker periods play an important role in the arithmetic study of special values of Rankin–Selberg L-functions. However, the definition of Whittaker period in [RS08b] is not canonical since it depends on an arbitrarily fixed generator of the ε-eigenspace of (2). In § 6, based on the non-vanishing hypothesis that is proved in [Sun17], we will canonically define Raghuram–Shahidi's Whittaker period by fixing a canonical generator of the concerning ε-eigenspace.

With a slight variation, we define the Whittaker period $\Omega_\varepsilon(\Pi)$ for every $\varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)}$ that occurs in

$$\mathcal{H}(\Pi_\infty) := \mathrm{H}_{\mathrm{ct}}^{b_{n,\infty}}(\mathrm{GL}_n(\mathrm{k})^0; F_\mu^\vee \otimes \Pi_\infty) \otimes \widetilde{\mathfrak{O}}_{n,\infty},$$

where $\widetilde{\mathfrak{O}}_{n,\infty}$ is a certain one-dimensional complex vector space defined by orientations (see (58)), which is naturally a representation of $\pi_0(\mathrm{k}_\infty^\times)$ that is isomorphic to $\mathrm{sgn}_\infty^{(n-1)(n-2)/2}$. Here sgn_∞ is the quadratic character of $\mathrm{k}_\infty^\times$ that is nontrivial on $\mathrm{k}_v^\times$ for every real place v of k. Note that the isomorphism class of the representation $\mathcal{H}(({}^\sigma\Pi)_\infty)$ of $\pi_0(\mathrm{k}_\infty^\times)$ is independent of $\sigma \in \mathrm{Aut}(\mathbb{C})$ (see Remark 6.3).

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In fact, by fixing a generator of a certain one-dimensional $\mathbb{Q}(\Pi)$-vector space, we will simultaneously define a family

$$\{\Omega_\varepsilon(\Pi')\}_{\Pi'\in\{\,^\sigma\Pi\,:\,\sigma\in\mathrm{Aut}(\mathbb{C})\}}$$

of Whittaker periods, which are nonzero complex numbers. Moreover, the family is unique up to scalar multiplication by $\mathbb{Q}(\Pi)^\times$ in the following sense (see Lemma 6.6): suppose that another generator yields another family $\{\Omega'_\varepsilon(\Pi')\}_{\Pi'\in\{\,^\sigma\Pi\,:\,\sigma\in\mathrm{Aut}(\mathbb{C})\}}$ of Whittaker periods. Then for all $\Pi_1, \Pi_2 \in \{\,^\sigma\Pi : \sigma\in\mathrm{Aut}(\mathbb{C})\}$ and all $\sigma\in\mathrm{Aut}(\mathbb{C})$ such that $^\sigma\Pi_1 = \Pi_2$,

$$\sigma\left(\frac{\Omega'_\varepsilon(\Pi_1)}{\Omega_\varepsilon(\Pi_1)}\right) = \frac{\Omega'_\varepsilon(\Pi_2)}{\Omega_\varepsilon(\Pi_2)}.$$

In particular, like Deligne's periods for pure motives, the Whittaker period $\Omega_\varepsilon(\Pi)$ is uniquely defined up to scalar multiplication by $\mathbb{Q}(\Pi)^\times$. See § 6.3 for details. When $n = 1$, the Whittaker period $\Omega_\varepsilon(\Pi)\in\mathbb{Q}(\Pi)^\times$.

Remark 1.1. By comparing Deligne's conjecture and the global period relation (Theorem 1.2 of this article), Hara and Namikawa [HN24, Theorem 1.1] supply a conjectural description of our Whittaker period $\Omega_\varepsilon(\Pi)$ in terms of Deligne's periods and Yoshida's fundamental periods (see [Yos01]). They also partially prove Theorem 1.2 in [HN24, Theorem 6.11] under the assumption that [HN24, Conjecture 6.8] holds true. We remark that the archimedean period relation (Theorem 3.2) that is proved in this article implies their Conjecture 6.8.

1.2 Period relations

Suppose that $n \geqslant 2$ and Π is cuspidal. Let Σ be an irreducible subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_{n-1}(\mathrm{k})\backslash\mathrm{GL}_{n-1}(\mathbb{A}))$ that is tamely isobaric and regular algebraic. Assume that the coefficient systems F_μ and F_ν of Π and Σ, respectively, are balanced, that is, there is an integer j such that

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k}\otimes_\mathbb{Q}\mathbb{C})}(F_\mu^\vee\otimes F_\nu^\vee, \otimes_{\iota\in\mathcal{E}_\mathrm{k}}\mathrm{det}^j) \neq \{0\}.$$

We call such an integers j a balanced place (for F_μ and F_ν). These balanced places j are in bijection with the critical places $\frac{1}{2}+j$ of $\Pi\times\Sigma$ (see § 7.2). As before, F_ν has highest weight $\nu = \{\nu^\iota\}_{\iota\in\mathcal{E}_\mathrm{k}} \in (\mathbb{Z}^{n-1})^{\mathcal{E}_\mathrm{k}}$, and $\nu^\iota = (\nu_1^\iota,\ldots,\nu_{n-1}^\iota)$.

Let $\frac{1}{2}+j$ be a critical place of $\Pi\times\Sigma$. Put

$$\Omega_{\mu,\nu,j} := \mathrm{i}^{\,j(n(n-1)/2)[\mathrm{k}:\mathbb{Q}]+\sum_{\iota\in\mathcal{E}_\mathrm{k}}\sum_{i=1}^{n-1}(n-i)(\mu_i^\iota+\nu_i^\iota)} \quad (\mathrm{i} := \sqrt{-1}).$$

Let $\chi : \mathrm{k}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a finite-order Hecke character. We are concerned with the rationality of the critical value $\mathrm{L}(\frac{1}{2}+j, \Pi\times\Sigma\times\chi)$, when both the critical place $\frac{1}{2}+j$ and the finite-order Hecke character χ vary. Here $\mathrm{L}(s, \Pi\times\Sigma\times\chi)$ denotes the completed Rankin–Selberg L-function. Define the composition field

$$\mathbb{Q}(\Pi,\Sigma,\chi) := \mathbb{Q}(\Pi)\mathbb{Q}(\Sigma)\mathbb{Q}(\chi)\subset\mathbb{C}.$$

Similar to Π_∞, we have the archimedean parts Σ_∞ and χ_∞ of Σ and χ, respectively.

The main result of this article is the following global period relation.

Theorem 1.2. *Let the notation and assumptions be as above. Then*

$$\frac{\mathrm{L}(\frac{1}{2}+j, \Pi\times\Sigma\times\chi)}{\Omega_{\mu,\nu,j}\cdot\mathcal{G}(\chi_\Sigma)\cdot\mathcal{G}(\chi)^{n(n-1)/2}\cdot\Omega_{\varepsilon_n}(\Pi)\cdot\Omega_{\varepsilon_{n-1}}(\Sigma)} \in \mathbb{Q}(\Pi,\Sigma,\chi), \qquad (3)$$

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where χ_Σ *is the central character of* Σ, *'$\mathcal{G}$' indicates the Gauss sum (see* (81) *and* (82)*), and* $\varepsilon_n, \varepsilon_{n-1}$ *are the quadratic characters of* $\mathrm{k}_\infty^\times$ *given by*

$$(\varepsilon_n, \varepsilon_{n-1}) := \begin{cases} \big(\varepsilon_{\Sigma_\infty}\cdot \mathrm{sgn}_\infty^{(n-2)(n-3)/2+j}\cdot\chi_\infty,\, \varepsilon_{\Sigma_\infty}\cdot\mathrm{sgn}_\infty^{(n-2)(n-3)/2}\big), & \text{if } n \text{ is even;}\\ \big(\varepsilon_{\Pi_\infty}\cdot\mathrm{sgn}_\infty^{(n-1)(n-2)/2},\, \varepsilon_{\Pi_\infty}\cdot\mathrm{sgn}_\infty^{(n-1)(n-2)/2+j}\cdot\chi_\infty\big), & \text{if } n \text{ is odd.}\end{cases}$$

Moreover, the quotient (3) *is* $\mathrm{Aut}(\mathbb{C})$*-equivariant in the sense that*

$$\sigma\left(\frac{\mathrm{L}(\frac{1}{2}+j, \Pi\times\Sigma\times\chi)}{\Omega_{\mu,\nu,j}\cdot\mathcal{G}(\chi_\Sigma)\cdot\mathcal{G}(\chi)^{n(n-1)/2}\cdot\Omega_{\varepsilon_n}(\Pi)\cdot\Omega_{\varepsilon_{n-1}}(\Sigma)}\right) = \frac{\mathrm{L}(\frac{1}{2}+j, {}^\sigma\Pi\times{}^\sigma\Sigma\times{}^\sigma\chi)}{\Omega_{\mu,\nu,j}\cdot\mathcal{G}(\chi_{{}^\sigma\Sigma})\cdot\mathcal{G}({}^\sigma\chi)^{n(n-1)/2}\cdot\Omega_{\varepsilon_n}({}^\sigma\Pi)\cdot\Omega_{\varepsilon_{n-1}}({}^\sigma\Sigma)} \tag{4}$$

for every $\sigma\in\mathrm{Aut}(\mathbb{C})$.

The proof of Theorem 1.2 crucially depends on three local results that are responsible for the occurrence of the denominator in (3). More precisely:

- the definition of the canonical Whittaker periods $\Omega_{\varepsilon_n}(\Pi)$ and $\Omega_{\varepsilon_{n-1}}(\Sigma)$ relies on the non-vanishing hypothesis that was proposed by Kazhdan and Mazur in 1970s and proved by Sun in 2017 [Sun17];
- the appearance of the term $\mathcal{G}(\chi_\Sigma)\cdot\mathcal{G}(\chi)^{n(n-1)/2}$ is a consequence of the non-archimedean period relation (Proposition 5.1), which is essentially due to Harder [Har83, § III] for $n=2$ and Mahnkopf [Mah05, § 3.4] and Raghuram [Rag10, § 3.3] in general;
- the explicit calculation of $\Omega_{\mu,\nu,j}$ is a consequence of the archimedean period relation (Theorem 3.2); the key contribution of this article is a proof of the archimedean period relation, based on the preparatory work in [LLSS23]; the proof is much more involved than that of the non-archimedean period relation.

In what follows we comment on some previous works concerning Theorem 1.2. The first result was obtained by Shimura in 1959 [Shi59, § 9]. He proved that for certain nonzero complex numbers $\{\Omega_\epsilon\}_{\epsilon\in\{\pm1\}}$,

$$\frac{\mathrm{L}(k,\Delta)}{(2\pi\mathrm{i})^k\cdot\Omega_{(-1)^k}}\in\mathbb{Q}\quad\text{for all } k=1,2,\ldots,11. \tag{5}$$

Here Δ is Ramanujan's cusp form of weight 12 and level 1 given by

$$\Delta(z) = q\prod_{n=1}^\infty(1-q^n)^{24} = \sum_{n=1}^\infty \tau(n)q^n \quad (q := e^{2\pi\mathrm{i}\cdot z}),$$

and the (incomplete) L-function $\mathrm{L}(s,\Delta)$ is given by

$$\mathrm{L}(s,\Delta) = \sum_{n=1}^\infty\frac{\tau(n)}{n^s}\quad\text{(when the real part of } s \text{ is sufficiently large).}$$

When $n=2$, $\mathrm{k}=\mathbb{Q}$, χ and Σ are trivial, and Π is the automorphic representation associated with Δ, Theorem 1.2 is a reformulation of the relation (5).

After the aforementioned pioneering work of Shimura, a series of results towards Theorem 1.2 for $n=2$ were obtained by Manin [Man72, Man73, Man76], Shimura [Shi76, Shi77, Shi78], and Harder [Har83]. Theorem 1.2 for $n=2$ was finally proved in full generality by Hida in 1994 [Hid94, Theorem I].

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For general n, the representation-theoretic problems behind Theorem 1.2 are much more difficult than the case of $n = 2$. The non-archimedean period relation is responsible for the rationality of $\mathrm{L}(\frac{1}{2} + j, \Pi \times \Sigma \times \chi)$ when the finite-order Hecke character χ varies, and the archimedean period relation is responsible for the rationality of $\mathrm{L}(\frac{1}{2} + j, \Pi \times \Sigma \times \chi)$ when the critical place $\frac{1}{2} + j$ varies. The non-archimedean period relation is much easier to prove than the archimedean period relation. Partly because of this reason, more complete results on the rationality of $\mathrm{L}(\frac{1}{2} + j, \Pi \times \Sigma \times \chi)$ have been obtained for fixed j and varying χ, in a series of works including [Sch93, KMS00, Mah05, KS13, Rag10, Rag16, GH16, Gro18]. See also the survey paper [HL17] for more relevant works.

However, it is also crucial to understand the rationality of $\mathrm{L}(\frac{1}{2} + j, \Pi \times \Sigma \times \chi)$ when χ is fixed and j varies, as in Shimura's result (5). For example, as explained in the introduction of [HN21], this is essentially important for the Kummer congruence (also called Manin congruence) in the construction of p-adic Rankin–Selberg L-functions (see [Jan24]). Only some partial or conditional results (for varying j) have been obtained in this direction (see [Jan19, HR20, GL21, HN21, Rag22]).

We have some more specific comments that compare Theorem 1.2 with the existing results in the literature.

– The number field k is assumed to be $\mathbb{Q}$ in [KMS00, KS13, Mah05, Rag10]. It is assumed to be imaginary quadratic or CM in [GH16, Gro18, GL21], with extra assumptions on Π and Σ. In [GL21, Theorem A] the rationality for varying j is obtained under the hypotheses that certain central L-values are non-vanishing, which themselves remain a variety of difficult open problems.
– Under a hypothesis that is more or less equivalent to the archimedean period relation, a less precise version of Theorem 1.2 is proved in [Jan19] for general k. For $n = 3$, $\mathrm{k} = \mathbb{Q}$ and $\chi = 1$, based on the explicit calculation of certain Rankin–Selberg zeta integrals in [HIM22], Theorem 1.2 is proved in [HN21].
– Roughly speaking, Theorem 1.2 asserts that the transcendency of the critical L-values is captured by the Whittaker periods. Harder and Raghuram prove in [HR20, Theorem 7.21] that the transcendency of the ratio of two successive critical L-values is captured by the 'relative period' (which is in fact the ratio of two Whittaker periods). They use Langlands–Shahidi method, and their result is proved for more general Rankin–Selberg L-functions and for k totally real. This is extended to the case that k is totally imaginary in [Rag22]. The results in the case of $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ are immediate consequences of Theorem 1.2.
– We say that Π is of symplectic type if the L-function $\mathrm{L}(s, \Pi, \wedge^2 \otimes \eta)$ has a pole at $s = 1$ for some character η of $\mathrm{k}^\times \backslash \mathbb{A}^\times$. When this is the case, a rationality result for the standard automorphic L-function $\mathrm{L}(s, \Pi \otimes \chi)$ similar to (3) is proved by Jiang *et al.* in [JST19]. The reciprocity law, namely (4), is not proved in [JST19] for those L-functions.

In this article, we complete the story by giving an unconditional proof of Theorem 1.2, which is over arbitrary number fields. As we mentioned earlier, the key ingredient is the archimedean period relation whose proof is very much involved.

Last but not least, it is clear that the period relations (Theorem 1.2) have further applications towards the arithmetic study of other L-functions and Deligne's conjecture (see [Mah05, RS08a, Rag10, Rag16, Che22b, Che23, Che22a, HN24]), and they are also indispensable for the study of p-adic L-functions (see [Man73, Man76, Sch88, Sch93, Sch01, KMS00] and [Jan11, Jan15, Jan16, Jan19, Jan24]). In an ongoing work, the main results of this article and [LLSS23] will be used to construct nearly ordinary Rankin–Selberg p-adic L-functions under a general framework.

The article is organized as follows. In § 2 we translate general cohomological representations to the cohomological representation with trivial coefficient system. This is the main idea used in the proof of the archimedean period relations (Theorem 3.2), which is formulated in § 3 and proved in § 4. To this end, we recall the main result of [LLSS23] and use it to compare the Rankin–Selberg integrals with the integrals over a certain open orbit. In § 5 we reformulate the non-archimedean period relations (Proposition 5.1) and provide a proof of it for completeness. In § 6 we define the Whittaker periods of irreducible smooth automorphic representations that are tamely isobaric and regular algebraic, and study their properties under Galois twist. We formulate the global modular symbols and modular symbols at infinity, and explain their relationship in § 7, which amounts to the unfolding of global Rankin–Selberg integrals as in [JS81b]. Finally, the global period relation Theorem 1.2 is proved in § 7 based on the results established in earlier sections.

2. Cohomological representations and their translations

In this section we introduce some generalities for cohomological representations, and give an explicit construction of the translation from the cohomological representations with trivial coefficient system to general ones.

2.1 Cohomological representations

Let $\mathbb{K}$ be an archimedean local field. Thus, it is a topological field that is topologically isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Its complexification

$$\mathbb{K} \otimes_{\mathbb{R}} \mathbb{C} = \prod_{\iota \in \mathcal{E}_{\mathbb{K}}} \mathbb{C},$$

where $\mathcal{E}_{\mathbb{K}}$ denotes the set of all continuous field embeddings $\iota : \mathbb{K} \to \mathbb{C}$. Note that $\mathcal{E}_{\mathbb{R}}$ consists of the inclusion map, and $\mathcal{E}_{\mathbb{C}}$ consists of the identity map and the complex conjugation.

Fix an integer $n \geqslant 1$, and fix a weight

$$\mu^{\iota} = (\mu_1^{\iota} \geqslant \mu_2^{\iota} \geqslant \cdots \geqslant \mu_n^{\iota}) \in \mathbb{Z}^n$$

for every $\iota \in \mathcal{E}_{\mathbb{K}}$. Write $\mu := \{\mu^{\iota}\}_{\iota \in \mathcal{E}_{\mathbb{K}}}$, and denote by F_μ the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{K} \otimes_{\mathbb{R}} \mathbb{C}) = \prod_{\iota \in \mathcal{E}_{\mathbb{K}}} \mathrm{GL}_n(\mathbb{C})$ of highest weight μ. Recall that all algebraic representations of algebraic groups are assumed to be finite-dimensional.

We say that μ is pure if

$$\mu_1^{\iota} + \mu_n^{\bar{\iota}} = \mu_2^{\iota} + \mu_{n-1}^{\bar{\iota}} = \cdots = \mu_n^{\iota} + \mu_1^{\bar{\iota}},$$

for every $\iota \in \mathcal{E}_{\mathbb{K}}$, where $\bar{\iota}$ denotes the composition of ι with the complex conjugation. We suppose that μ is pure. Denote by $\Omega(\mu)$ the set of isomorphism classes of irreducible Casselman–Wallach representations π_μ of $\mathrm{GL}_n(\mathbb{K})$ such that:

- π_μ is generic, and essentially unitarizable in the sense that $\pi_\mu \otimes \chi'$ is unitarizable for some character χ' of $\mathrm{GL}_n(\mathbb{K})$; and
- the total continuous cohomology

$$\mathrm{H}^*_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0; F_\mu^{\vee} \otimes \pi_\mu) \neq \{0\}.$$

We remark that no such π_μ exists when μ is not pure (see [Clo90, Lemma 4.9]).

By [Clo90, § 3],

$$\#(\Omega(\mu)) = \begin{cases} 2, & \text{if } \mathbb{K} \cong \mathbb{R} \text{ and } n \text{ is odd;} \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

Write $\mathrm{sgn}_{\mathbb{K}^\times} : \mathbb{K}^\times \to \mathbb{C}^\times$ for the quadratic character that is nontrivial if and only if $\mathbb{K} \cong \mathbb{R}$, and define the sign character

$$\mathrm{sgn} := \mathrm{sgn}_{\mathbb{K}^\times} \circ \det$$

of a general linear group $\mathrm{GL}_n(\mathbb{K})$. Then in the first case of (6) the two members of $\Omega(\mu)$ are twists of each other by the sign character, and in the second case of (6) the only representation in $\Omega(\mu)$ is isomorphic to its own twist by the sign character. Recall that by [Clo90, Lemma 3.14],

$$\mathrm{H}^i_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0; F_\mu^\vee \otimes \pi_\mu) = \{0\}, \quad \text{if } i < b_{n,\mathbb{K}},$$

and

$$\mathrm{H}^{b_{n,\mathbb{K}}}_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0; F_\mu^\vee \otimes \pi_\mu) \cong \begin{cases} 1_{\mathbb{K}^\times} \oplus \mathrm{sgn}_{\mathbb{K}^\times}, & \text{if } \mathbb{K} \cong \mathbb{R} \text{ and } n \text{ is even,} \\ \varepsilon_{\pi_\mu}, & \text{otherwise,} \end{cases}$$

as representations of $\pi_0(\mathbb{K}^\times)$, where $1_{\mathbb{K}^\times}$ denotes the trivial character of $\mathbb{K}^\times$, and ε_{π_μ} denotes the central character of $F_\mu^\vee \otimes \pi_\mu$. Here and henceforth we make the identification

$$\pi_0(\mathrm{GL}_n(\mathbb{K})) = \pi_0(\mathbb{K}^\times) \quad (\pi_0 \text{ indicates the set of connected components})$$

through the determinant map $\mathrm{GL}_n(\mathbb{K}) \to \mathbb{K}^\times$. Note that ε_{π_μ} is equal to either $1_{\mathbb{K}^\times}$ or $\mathrm{sgn}_{\mathbb{K}^\times}$ for $\mathbb{K} \cong \mathbb{R}$ and n odd, and is trivial otherwise.

For every commutative ring R, let $\mathrm{B}_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ consisting of all the upper triangular matrices, and let $\mathrm{N}_n(R)$ be the subgroup of matrices in $\mathrm{B}_n(R)$ whose diagonal entries are 1. Likewise let $\bar{\mathrm{B}}_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ consisting of all the lower triangular matrices, and let $\bar{\mathrm{N}}_n(R)$ be the subgroup of matrices in $\bar{\mathrm{B}}_n(R)$ whose diagonal entries are 1. Let $\mathrm{T}_n(R)$ be the subgroup of diagonal matrices in $\mathrm{GL}_n(R)$.

Note that the invariant spaces $(F_\mu)^{\mathrm{N}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}$ and $(F_\mu^\vee)^{\bar{\mathrm{N}}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}$ are one-dimensional. We shall fix a generator $v_\mu \in (F_\mu)^{\mathrm{N}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}$ and a generator $v_\mu^\vee \in (F_\mu^\vee)^{\bar{\mathrm{N}}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}$ such that their pairing

$$\langle v_\mu, v_\mu^\vee \rangle = 1. \tag{7}$$

To be more concrete, by the Borel–Weil–Bott theorem [Bot57] we can realize F_μ as the algebraic induction

$$F_\mu = {}^{\mathrm{alg}}\mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}^{\mathrm{GL}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})} \chi_\mu, \tag{8}$$

which consists of all algebraic functions $f : \mathrm{GL}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C}) \to \mathbb{C}$ such that

$$f(\bar{b}g) = \chi_\mu(\bar{b}) f(g) \quad \text{for all } \bar{b} \in \bar{\mathrm{B}}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C}) \text{ and } g \in \mathrm{GL}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C}).$$

Here $\chi_\mu = \otimes_{\iota \in \mathcal{E}_\mathbb{K}} \chi_{\mu^\iota}$ denotes the algebraic character of $\mathrm{T}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C})$ corresponding to the weight $\mu \in (\mathbb{Z}^n)^{\mathcal{E}_\mathbb{K}}$, to be viewed as an algebraic character of $\bar{\mathrm{B}}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C})$ as usual. Then we realize v_μ as the $\mathrm{N}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C})$-invariant algebraic function f in F_μ such that

$$f(1_n) = 1 \quad (1_n \text{ denotes the identity element of } \mathrm{GL}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C})).$$

Similarly, we realize $F_\mu^\vee$ as the algebraic induction

$$F_\mu^\vee = {}^{\mathrm{alg}}\mathrm{Ind}_{\mathrm{B}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})}^{\mathrm{GL}_n(\mathbb{K}\otimes_\mathbb{R}\mathbb{C})} \chi_{-\mu},$$

and realize $v_\mu^\vee$ as the $\bar{\mathrm{N}}_n(\mathbb{K} \otimes_\mathbb{R} \mathbb{C})$-invariant algebraic function $f^\vee$ in $F_\mu^\vee$ such that $f^\vee(1_n) = 1$. The invariant pairing $\langle\, , \rangle : F_\mu \times F_\mu^\vee \to \mathbb{C}$ is determined by the equality (7). Note that as a linear functional on $F_\mu^\vee$, v_μ equals the evaluation map at 1_n. Similarly, $v_\mu^\vee$ equals the evaluation map at 1_n as a linear functional on F_μ.

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Fix a unitary character

$$\psi_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}^\times, \quad x \mapsto e^{2\pi \mathrm{i} x}, \tag{9}$$

which induces a unitary character

$$\psi_{\mathbb{K}} : \mathbb{K} \to \mathbb{C}^\times, \quad x \mapsto \psi_{\mathbb{R}}\Bigg(\sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \iota(x)\Bigg). \tag{10}$$

This further induces a unitary character

$$\psi_{n,\mathbb{K}} : \mathrm{N}_n(\mathbb{K}) \to \mathbb{C}^\times, \quad [x_{i,j}]_{1 \leqslant i,j \leqslant n} \mapsto \psi_{\mathbb{K}}\Bigg((-1)^n \cdot \sum_{i=1}^{n-1} x_{i,i+1}\Bigg). \tag{11}$$

By abuse of notation, we will still use $\psi_{n,\mathbb{K}}$ to denote the space $\mathbb{C}$ carrying the representation of $\mathrm{N}_n(\mathbb{K})$ corresponding to the character $\psi_{n,\mathbb{K}}$. Similar notation will be freely used for other characters. Let $\pi_\mu \in \Omega(\mu)$. Recall that the space $\mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(\pi_\mu, \psi_{n,\mathbb{K}})$ is one-dimensional. Fix a generator

$$\lambda_\mu \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(\pi_\mu, \psi_{n,\mathbb{K}}), \tag{12}$$

to be called the Whittaker functional on π_μ.

Write $0_{n,\mathbb{K}}$ for the zero element of $(\mathbb{Z}^n)^{\mathcal{E}_{\mathbb{K}}}$. Then $F_{0_{n,\mathbb{K}}}$ is the trivial representation. Specifying the above argument to the case when $\mu = 0_{n,\mathbb{K}}$, we take a representation $\pi_{0_{n,\mathbb{K}}} \in \Omega(0_{n,\mathbb{K}})$, together with the Whittaker functional $\lambda_{0_{n,\mathbb{K}}} \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(\pi_{0_{n,\mathbb{K}}}, \psi_{n,\mathbb{K}}) \setminus \{0\}$.

Throughout this article, we assume that the representation $\pi_{0_{n,\mathbb{K}}} \in \Omega(0_{n,\mathbb{K}})$ is chosen such that $\pi_{0_{n,\mathbb{K}}}$ and $F_\mu^\vee \otimes \pi_\mu$ have the same central character, to be denoted by $\varepsilon_{n,\mathbb{K}}$.

2.2 Explicit translations

We will prove the following result in this subsection.

Proposition 2.1. *There is a unique element $\jmath_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(\pi_{0_{n,\mathbb{K}}}, F_\mu^\vee \otimes \pi_\mu)$ such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_{0_{n,\mathbb{K}}} & \xrightarrow{\jmath_\mu} & F_\mu^\vee \otimes \pi_\mu \\ {\scriptstyle \lambda_{0_{n,\mathbb{K}}}}\downarrow & & \downarrow{\scriptstyle v_\mu \otimes \lambda_\mu} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

Moreover, $\jmath_\mu$ induces a linear isomorphism

$$\jmath_\mu : \mathrm{H}^i_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0; \pi_{0_{n,\mathbb{K}}}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0; F_\mu^\vee \otimes \pi_\mu)$$

of representations of $\pi_0(\mathbb{K}^\times)$ for each $i \in \mathbb{Z}$.

It is known from the Vogan–Zuckerman theory of cohomological representations (see Proposition 1.2 and § 5 of [VZ84]) that

$$\dim \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(\pi_{0_{n,\mathbb{K}}}, F_\mu^\vee \otimes \pi_\mu) = 1. \tag{13}$$

We first recall the realization of π_μ and introduce a certain principal series representation I_μ of $\mathrm{GL}_n(\mathbb{K})$. Define a character

$$\rho_n := \otimes_{i=1}^n |\cdot|_{\mathbb{K}}^{(n+1)/2-i} \quad (|\cdot|_{\mathbb{K}} \text{ denotes the normalized absolute value})$$

of $\mathrm{T}_n(\mathbb{K})$. For $\iota \in \mathcal{E}_{\mathbb{K}}$, define the half-integers

$$\tilde{\mu}_i^\iota := \mu_i^\iota + \frac{n+1}{2} - i, \quad i = 1, \dots, n. \tag{14}$$

Then $\{(\tilde{\mu}_1^\iota, \tilde{\mu}_2^\iota, \dots, \tilde{\mu}_n^\iota)\}_{\iota \in \mathcal{E}_{\mathbb{K}}}$ is the infinitesimal character of the algebraic representation F_μ.

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For $\mathbb{K} \cong \mathbb{R}$, $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z} \setminus \{0\}$, denote by $D_{a,b}$ the essentially square-integrable irreducible Casselman–Wallach representation of $\mathrm{GL}_2(\mathbb{K})$ with infinitesimal character (a, b). Note that such a representation is unique up to isomorphism. In this article we use the notation $\widehat{\otimes}$ to denote the completed projective tensor product of locally convex topological vector spaces (see [Trè67, Definition 43.5]).

If n is even, then

$$\pi_\mu \cong \mathrm{Ind}_{\bar{\mathrm{P}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} \big(D_{\tilde{\mu}_1^\iota, \tilde{\mu}_n^\iota} \widehat{\otimes} \cdots \widehat{\otimes} D_{\tilde{\mu}_{n/2}^\iota, \tilde{\mu}_{n/2+1}^\iota}\big) \quad \text{(normalized smooth induction)},$$

where $\bar{\mathrm{P}}_n$ is the lower triangular parabolic subgroup of type $(2, \dots, 2)$. More precisely, the above normalized smooth induction consists of all smooth maps $f : \mathrm{GL}_n(\mathbb{K}) \to D_{\tilde{\mu}_1^\iota, \tilde{\mu}_n^\iota} \widehat{\otimes} \cdots \widehat{\otimes} D_{\tilde{\mu}_{n/2}^\iota, \tilde{\mu}_{n/2+1}^\iota}$ such that

$$f(\bar{p}g) = \delta_{\bar{\mathrm{P}}_n}^{1/2}(\bar{p}) \cdot (\bar{p} \cdot (f(g))) \quad \text{for all } \bar{p} \in \bar{\mathrm{P}}_n \text{ and } g \in \mathrm{GL}_n(\mathbb{K}), \tag{15}$$

where $\delta_{\bar{\mathrm{P}}_n}$ denotes the modular character of $\bar{\mathrm{P}}_n$. Thereafter Ind always denotes the normalized smooth induction which is similarly defined as above.

If n is odd, then

$$\pi_\mu \cong \mathrm{Ind}_{\bar{\mathrm{P}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} \big(D_{\tilde{\mu}_1^\iota, \tilde{\mu}_n^\iota} \widehat{\otimes} \cdots \widehat{\otimes} D_{\tilde{\mu}_{(n-1)/2}^\iota, \tilde{\mu}_{(n+3)/2}^\iota} \otimes (\cdot)^{\tilde{\mu}_{(n+1)/2}^\iota} \varepsilon_{n,\mathbb{K}}\big),$$

where $\bar{\mathrm{P}}_n$ is the lower triangular parabolic subgroup of type $(2, \dots, 2, 1)$, and we recall that $\varepsilon_{n,\mathbb{K}} = 1_{\mathbb{K}^\times}$ or $\mathrm{sgn}_{\mathbb{K}^\times}$ is the common central character of $F_\mu^\vee \otimes \pi_\mu$ and $\pi_{0_{n,\mathbb{K}}}$.

For $\mathbb{K} \cong \mathbb{C}$, we have that

$$\pi_\mu \cong \mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} \big(\iota^{\tilde{\mu}_1^\iota} \bar{\iota}^{\tilde{\mu}_n^{\bar{\iota}}} \otimes \cdots \otimes \iota^{\tilde{\mu}_n^\iota} \bar{\iota}^{\tilde{\mu}_1^{\bar{\iota}}}\big),$$

where for $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$, $\iota^a \bar{\iota}^b$ denotes the character

$$\iota^a \bar{\iota}^b : \mathbb{K}^\times \to \mathbb{C}^\times, \quad z \mapsto \iota(z)^{a-b} (\iota(z)\bar{\iota}(z))^b.$$

In both the real and complex cases, we define the principal series representation

$$I_\mu := \mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} (\chi_\mu \cdot \rho_n \cdot (\varepsilon_{n,\mathbb{K}} \circ \det)),$$

so that I_μ and π_μ have the same central character.

LEMMA 2.2. *The principal series representation I_μ has a unique irreducible quotient as well as a unique generic irreducible subquotient. Moreover, the irreducible quotient is generic and isomorphic to π_μ.*

Proof. By [Jac09, Lemma 2.5], $I_\mu^\vee$ has a unique irreducible subrepresentation, which is also the unique generic irreducible subquotient. This implies the first statement of the lemma.

For $\mathbb{K} \cong \mathbb{R}$, by the well-known realization of essentially square-integrable representations of $\mathrm{GL}_2(\mathbb{R})$ as quotients of principal series representations, π_μ is a quotient of

$$\mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} (w(\chi_\mu \cdot \rho_n) \cdot (\varepsilon_{n,\mathbb{K}} \circ \det))$$

for a certain $w \in W_n$. Here W_n is the subgroup of permutation matrices in $\mathrm{GL}_n(\mathbb{Z})$ which is identified with the Weyl group and acts on $\mathrm{T}_n(\mathbb{K})$ by conjugation, and thus it acts on the set of characters of $\mathrm{T}_n(\mathbb{K})$. The above representation and I_μ have the same irreducible constituents. Hence, π_μ is isomorphic to the unique generic irreducible subquotient of I_μ, which is in fact a quotient as we mentioned at the beginning of the proof.

For $\mathbb{K} \cong \mathbb{C}$, the lemma follows easily from [JL70, Theorem 6.2] (for the case of $\mathrm{GL}_2(\mathbb{C})$) and parabolic induction in stages. □

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The group $\mathrm{N}_n(\mathbb{K})$ is equipped with the Haar measure

$$\mathrm{d}u := \prod_{1\leqslant i<j\leqslant n} \mathrm{d}u_{i,j}, \quad u = [u_{i,j}]_{1\leqslant i,j\leqslant n} \in \mathrm{N}_n(\mathbb{K}), \tag{16}$$

where $\mathrm{d}u_{i,j}$ is the self-dual Haar measure on $\mathbb{K}$ with respect to $\psi_{\mathbb{K}}$. By [Wal92, Theorem 15.4.1],

$$\dim \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(I_\mu, \psi_{n,\mathbb{K}}) = 1 \tag{17}$$

and there is a unique $\lambda'_\mu \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(I_\mu, \psi_{n,\mathbb{K}})$ such that

$$\lambda'_\mu(f) = \int_{\mathrm{N}_n(\mathbb{K})} f(u)\overline{\psi_{n,\mathbb{K}}}(u)\,\mathrm{d}u \tag{18}$$

for all $f \in I_\mu$ such that $f|_{\mathrm{N}_n(\mathbb{K})} \in \mathcal{S}(\mathrm{N}_n(\mathbb{K}))$. Here and henceforth, for a Nash manifold X, denote by $\mathcal{S}(X)$ the space of Schwartz functions on X (see [DC91, AG08]).

As usual, an element $u \otimes f \in F_\mu^\vee \otimes I_\mu$ is identified with the function

$$\mathrm{GL}_n(\mathbb{K}) \to F_\mu^\vee, \quad g \mapsto f(g)\cdot u.$$

Then $F_\mu^\vee \otimes I_\mu$ is identified with the space of $F_\mu^\vee$-valued smooth functions φ on $\mathrm{GL}_n(\mathbb{K})$ satisfying that

$$\varphi(bx) = \big(((\varepsilon_{n,\mathbb{K}} \circ \det)\cdot \chi_\mu)(b)\big)\cdot \varphi(x), \quad \text{for all } b \in \bar{\mathrm{B}}_n(\mathbb{K}),\ x \in \mathrm{GL}_n(\mathbb{K}),$$

on which $\mathrm{GL}_n(\mathbb{K})$ acts by

$$(g.\varphi)(x) := g.(\varphi(xg)), \quad \text{where } g,\, x \in \mathrm{GL}_n(\mathbb{K}).$$

Define a $\mathrm{GL}_n(\mathbb{K})$-homomorphism

$$\begin{array}{rcl} \imath_\mu : I_{0_{n,\mathbb{K}}} & \to & F_\mu^\vee \otimes I_\mu, \\ f & \mapsto & \big(g \mapsto f(g)\cdot(g^{-1}.v_\mu^\vee)\big). \end{array} \tag{19}$$

LEMMA 2.3. *The map $\imath_\mu$ satisfies that*

$$(v_\mu \otimes \lambda'_\mu)\circ \imath_\mu = \lambda'_{0_{n,\mathbb{K}}}. \tag{20}$$

Proof. Recall that $v_\mu \in F_\mu$ is $\mathrm{N}_n(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{C})$-invariant, and $\langle v_\mu, v_\mu^\vee\rangle = 1$. For $f \in I_{0_{n,\mathbb{K}}}$ with $f|_{\mathrm{N}_n(\mathbb{K})} \in \mathcal{S}(\mathrm{N}_n(\mathbb{K}))$, we have that

$$\begin{aligned} \big((v_\mu \otimes \lambda'_\mu)\circ \imath_\mu\big)(f) &= \int_{\mathrm{N}_n(\mathbb{K})} \langle v_\mu, \imath_\mu(f)(u)\rangle \overline{\psi_{n,\mathbb{K}}}(u)\,\mathrm{d}u \\ &= \int_{\mathrm{N}_n(\mathbb{K})} f(u)\overline{\psi_{n,\mathbb{K}}}(u)\,\mathrm{d}u \\ &= \lambda'_{0_{n,\mathbb{K}}}(f). \end{aligned}$$

This proves (20), in view of [Wal92, Theorem 15.4.1]. □

By Lemma 2.2 and (17), there is a unique $p_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(I_\mu, \pi_\mu)$ such that

$$\lambda_\mu \circ p_\mu = \lambda'_\mu. \tag{21}$$

Let $J_\mu := \mathrm{Ker}(p_\mu)$, which is the largest subrepresentation of I_μ such that

$$\mathrm{Dim}\, J_\mu < \mathrm{Dim}\, I_\mu.$$

Here and below, Dim indicates the Gelfand–Kirillov dimension of a Casselman–Wallach representation of $\mathrm{GL}_n(\mathbb{K})$. Likewise, we have $J_{0_{n,\mathbb{K}}} := \mathrm{Ker}(p_{0_{n,\mathbb{K}}}) \subset I_{0_{n,\mathbb{K}}}$.

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Lemma 2.4. *It holds that*

$$\imath_\mu(J_{0_{n,\mathbb{K}}}) \subset F_\mu^\vee \otimes J_\mu.$$

Proof. It suffices to show that

$$\tilde{\imath}_\mu(F_\mu \otimes J_{0_{n,\mathbb{K}}}) \subset J_\mu,$$

where $\tilde{\imath}_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(F_\mu \otimes I_{0_{n,\mathbb{K}}}, I_\mu)$ is the linear map induced by ι_μ. This follows from the fact that (see [Vog78, Lemma 2.2])

$$\mathrm{Dim}\,(F_\mu \otimes J_{0_{n,\mathbb{K}}}) = \mathrm{Dim}\,J_{0_{n,\mathbb{K}}}. \qquad \square$$

By Lemma 2.4, there is a unique $\jmath_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(\pi_{0_{n,\mathbb{K}}}, F_\mu^\vee \otimes \pi_\mu)$ such that the following diagram commutes.

$$\begin{array}{ccc} F_\mu^\vee \otimes I_\mu & \xrightarrow{\mathrm{id}\otimes p_\mu} & F_\mu^\vee \otimes \pi_\mu \\ {\scriptstyle \imath_\mu}\uparrow & & \uparrow{\scriptstyle \jmath_\mu} \\ I_{0_{n,\mathbb{K}}} & \xrightarrow{p_{0_{n,\mathbb{K}}}} & \pi_{0_{n,\mathbb{K}}} \end{array} \qquad \text{(id indicates the identity map)} \tag{22}$$

By (20) and (21),

$$\begin{aligned}(v_\mu \otimes \lambda_\mu) \circ \jmath_\mu \circ p_{0_{n,\mathbb{K}}} &= (v_\mu \otimes \lambda_\mu) \circ (\mathrm{id}_{F_\mu^\vee} \otimes p_\mu) \circ \imath_\mu \\ &= (v_\mu \otimes \lambda'_\mu) \circ \imath_\mu \\ &= \lambda'_{0_{n,\mathbb{K}}} \\ &= \lambda_{0_{n,\mathbb{K}}} \circ p_{0_{n,\mathbb{K}}},\end{aligned}$$

which implies that

$$(v_\mu \otimes \lambda_\mu) \circ \jmath_\mu = \lambda_{0_{n,\mathbb{K}}}.$$

This proves the existence part of Proposition 2.1. The uniqueness follows from (13). The last statement of the proposition follows from [VZ84, § 5].

3. Archimedean period relations

In this section we explain the statement of the archimedean period relation (Theorem 3.2), whose proof will be given in the next section.

3.1 Some cohomology spaces

For simplicity, write

$$\mathrm{H}_\mu := \mathrm{H}_{\mathrm{ct}}^{b_{n,\mathbb{K}}}(\mathrm{GL}_n(\mathbb{K})^0; F_\mu^\vee \otimes \pi_\mu),$$

which is of dimension 1 or 2. As in Proposition 2.1, we have a linear isomorphism

$$\jmath_\mu : \mathrm{H}_{0_{n,\mathbb{K}}} \xrightarrow{\sim} \mathrm{H}_\mu$$

of representations of $\pi_0(\mathbb{K}^\times)$.

Fix a maximal compact subgroup

$$K_{n,\mathbb{K}} := \begin{cases} \mathrm{O}(n), & \text{if } \mathbb{K} \cong \mathbb{R}; \\ \mathrm{U}(n), & \text{if } \mathbb{K} \cong \mathbb{C} \end{cases} \tag{23}$$

of $\mathrm{GL}_n(\mathbb{K})$. The determinant homomorphism yields identifications

$$\pi_0(\mathrm{GL}_n(\mathbb{K})) = \pi_0(K_{n,\mathbb{K}}) = \pi_0(\mathbb{K}^\times).$$

We use the corresponding lowercase Gothic letter to denote the Lie algebra of a Lie group. For example, the Lie algebra of $K_{n,\mathbb{K}}$ will be denoted by $\mathfrak{k}_{n,\mathbb{K}}$. Put

$$d_{n,\mathbb{K}} := b_{n+1,\mathbb{K}} + b_{n,\mathbb{K}} = \dim_{\mathbb{R}}(\mathfrak{gl}_n(\mathbb{K})/\mathfrak{k}_{n,\mathbb{K}}) = \begin{cases} \dfrac{n(n+1)}{2}, & \text{if } \mathbb{K} \cong \mathbb{R}; \\ n^2, & \text{if } \mathbb{K} \cong \mathbb{C}. \end{cases}$$

Define a one-dimensional real vector space

$$\omega_{n,\mathbb{K}}(\mathbb{R}) := \wedge^{d_{n,\mathbb{K}}}(\mathfrak{gl}_n(\mathbb{K})/\mathfrak{k}_{n,\mathbb{K}}).$$

Put

$$\omega_{n,\mathbb{K}} := \omega_{n,\mathbb{K}}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C},$$

which is naturally a representation of $\pi_0(\mathbb{K}^\times)$ that is isomorphic to $\mathrm{sgn}_{\mathbb{K}^\times}^{n-1}$. Here and henceforth, we also view $1_{\mathbb{K}^\times}$ and $\mathrm{sgn}_{\mathbb{K}^\times}$ as representations of $\pi_0(\mathbb{K}^\times)$. Then there is an identification

$$\mathrm{H}_{\mathrm{ct}}^{d_{n,\mathbb{K}}}(\mathrm{GL}_n(\mathbb{K})^0; \omega_{n,\mathbb{K}}) = 1_{\mathbb{K}^\times} \tag{24}$$

of representations of $\pi_0(\mathbb{K}^\times)$.

Write $\omega_{n,\mathbb{K}}^+$ and $\omega_{n,\mathbb{K}}^-$ for the two connected components of $\omega_{n,\mathbb{K}}(\mathbb{R}) \setminus \{0\}$, which are viewed as left invariant orientations on $\mathrm{GL}_n(\mathbb{K})/K_{n,\mathbb{K}}^0$. The complex orientation space of $\omega_{n,\mathbb{K}}(\mathbb{R})$ is defined to be the one-dimensional space

$$\mathfrak{O}_{n,\mathbb{K}} := \frac{\mathbb{C} \cdot \omega_{n,\mathbb{K}}^+ \oplus \mathbb{C} \cdot \omega_{n,\mathbb{K}}^-}{\{a(\omega_{n,\mathbb{K}}^+ + \omega_{n,\mathbb{K}}^-) \,:\, a \in \mathbb{C}\}}. \tag{25}$$

Then $\pi_0(\mathbb{K}^\times) = \pi_0(K_{n,\mathbb{K}})$ acts on $\mathfrak{O}_{n,\mathbb{K}}$ by $\mathrm{sgn}_{\mathbb{K}^\times}^{n-1}$, through the right translation on $\mathrm{GL}_n(\mathbb{K})/K_{n,\mathbb{K}}^0$. We identify $\omega_{n,\mathbb{K}}^* \otimes \mathfrak{O}_{n,\mathbb{K}}$ with the space of invariant measures on $\mathrm{GL}_n(\mathbb{K})/K_{n,\mathbb{K}}^0$ in the obvious way. Here and as usual, a superscript $*$ over a vector space indicates the dual space. Denote by $\mathfrak{M}_{n,\mathbb{K}}$ the one-dimensional space of invariant measures on $\mathrm{GL}_n(\mathbb{K})$. By push-forward of measures through the map $\mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})/K_{n,\mathbb{K}}^0$, we have an identification

$$\mathfrak{M}_{n,\mathbb{K}} = \omega_{n,\mathbb{K}}^* \otimes \mathfrak{O}_{n,\mathbb{K}}.$$

In view of this and (24), we have that

$$\mathrm{H}_{\mathrm{ct}}^{d_{n,\mathbb{K}}}(\mathrm{GL}_n(\mathbb{K})^0; \mathfrak{M}_{n,\mathbb{K}}^*) \otimes \mathfrak{O}_{n,\mathbb{K}} = \mathbb{C}.$$

3.2 Archimedean modular symbols and archimedean period relations

Now we assume that $n \geqslant 2$. Let $\nu \in (\mathbb{Z}^{n-1})^{\mathcal{E}_{\mathbb{K}}}$ be a highest weight and assume that it is pure. Then, as before, we have representations F_ν and π_ν of $\mathrm{GL}_{n-1}(\mathbb{K} \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathrm{GL}_{n-1}(\mathbb{K})$, respectively, an element $v_\nu \in F_\nu$, an element $v_\nu^\vee \in F_\nu^\vee$, and a Whittaker functional λ_ν on π_ν. The representation $\pi_{0_{n-1,\mathbb{K}}}$ is determined by π_ν as before, and we have a linear isomorphism

$$\jmath_\nu : \mathrm{H}_{0_{n-1,\mathbb{K}}} \xrightarrow{\sim} \mathrm{H}_\nu$$

of representations of $\pi_0(\mathbb{K}^\times)$.

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As usual, we have an embedding

$$\imath : \mathrm{GL}_{n-1}(R) \hookrightarrow \mathrm{GL}_n(R), \quad g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, \tag{26}$$

where R is an arbitrary commutative ring. We also view $\mathrm{GL}_{n-1}(R)$ as a subgroup of $\mathrm{GL}_n(R) \times \mathrm{GL}_{n-1}(R)$ via the diagonal embedding

$$\mathrm{GL}_{n-1}(R) \hookrightarrow \mathrm{GL}_n(R) \times \mathrm{GL}_{n-1}(R), \quad g \mapsto \left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, g \right). \tag{27}$$

For all $k, l \in \mathbb{N}$, denote by $R^{k \times l}$ the set of $k \times l$ matrices with entries in R.

Recall the pure weight $\mu \in (\mathbb{Z}^n)^{\mathcal{E}_{\mathbb{K}}}$ from § 2.1. Put $\xi := (\mu, \nu)$. Write $F_\xi := F_\mu \otimes F_\nu$. Assume that ξ is *balanced* in the sense that there is an integer j such that

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{C})}(F_\xi^\vee, \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \mathrm{det}^j) \neq 0. \tag{28}$$

For each $k \in \mathbb{N}$, write

$$w_k := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ & \cdots & \cdots & \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathrm{GL}_k(\mathbb{Z}).$$

Following [LLSS23], define a series $\{z_k \in \mathrm{GL}_k(\mathbb{Z})\}_{k \in \mathbb{N}}$ of matrices inductively by

$$z_0 := \varnothing \text{ (the unique element of } \mathrm{GL}_0(\mathbb{Z})), \quad z_1 := [1],$$

and

$$z_k := \begin{bmatrix} w_{k-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^t z_{k-2}^{-1} & 0 \\ 0 & 1_2 \end{bmatrix} \begin{bmatrix} {}^t z_{k-1} w_{k-1} z_{k-1} & {}^t e_{k-1} \\ 0 & 1 \end{bmatrix}, \quad \text{for all } k \geqslant 2. \tag{29}$$

Here, and as usual, a left superscript t over a matrix indicates the transpose, 1_2 stands for the 2×2 identity matrix, and $e_{k-1} := [0, \ldots, 0, 1] \in \mathbb{Z}^{1 \times (k-1)}$.

Let $j \in \mathbb{Z}$ be as in (28). The following proposition follows from the fact that

$$\big(\bar{\mathrm{B}}_n(\mathbb{C}) \times \bar{\mathrm{B}}_{n-1}(\mathbb{C})\big) \cdot (z_n, z_{n-1}) \cdot \mathrm{GL}_{n-1}(\mathbb{C})$$

is Zariski open in $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C})$ (see [LLSS23, Lemma 1.1]).

Proposition 3.1. *There is a unique element*

$$\phi_{\xi,j} \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{C})}(F_\xi^\vee, \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \mathrm{det}^j)$$

such that

$$\phi_{\xi,j}((z_n^{-1}.v_\mu^\vee) \otimes (z_{n-1}^{-1}.v_\nu^\vee)) = 1.$$

Fix a quadratic character $\chi_{\mathbb{K}}$ of $\mathbb{K}^\times$. Define characters

$$\chi_{\mathbb{K},t} := \chi_{\mathbb{K}} \cdot |\cdot|_{\mathbb{K}}^t, \quad \chi_{\mathbb{K}}^{(j)} := \chi_{\mathbb{K}} \cdot \mathrm{sgn}_{\mathbb{K}^\times}^j \quad (t \in \mathbb{C}), \tag{30}$$

and, more generally,

$$\chi_{\mathbb{K},t}^{(j)} := \chi_{\mathbb{K}} \cdot |\cdot|_{\mathbb{K}}^t \cdot \mathrm{sgn}_{\mathbb{K}^\times}^j,$$

of the group $\mathbb{K}^\times$. When no confusion arises, for every commutative ring R, every character of $R^\times$ is identified with a character of $\mathrm{GL}_{n-1}(R)$ via the pullback through the determinant homomorphism. In particular, $\chi_{\mathbb{K},t}^{(j)}$ is also viewed as a character of $\mathrm{GL}_{n-1}(\mathbb{K})$.

Put

$$\pi_\xi := \pi_\mu \widehat{\otimes} \pi_\nu.$$

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We have the normalized Rankin–Selberg integral (see [Jac09])

$$\mathrm{Z}^\circ_\xi(\cdot, s, \chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(\pi_\xi \otimes \mathfrak{M}_{n-1,\mathbb{K}},\, \chi_{\mathbb{K},-s+1/2}\big) = \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(\pi_\xi, \chi_{\mathbb{K},-s+1/2} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}}\big),$$

such that

$$\mathrm{Z}^\circ_\xi(f \otimes f' \otimes m, s, \chi_{\mathbb{K}}) := \frac{1}{\mathrm{L}(s, \pi_\mu \times \pi_\nu)} \cdot \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} \lambda_\mu\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \cdot f \right) \lambda_\nu(g \cdot f') \cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s-1/2}\, \mathrm{d}\bar{m}(g) \tag{31}$$

for all $f \in \pi_\mu$, $f' \in \pi_\nu$, $m \in \mathfrak{M}_{n-1,\mathbb{K}}$, and $s \in \mathbb{C}$ with the real part $\mathrm{Re}(s)$ sufficiently large (it extends to all $s \in \mathbb{C}$ by holomorphic continuation). Here and henceforth, $\bar{m}$ is the quotient measure on $\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})$ induced by m. Recall that a Haar measure on $\mathrm{N}_{n-1}(\mathbb{K})$ has been fixed as in (16).

Put

$$\mathrm{H}_{\chi_{\mathbb{K}}^{(j)}} := \mathrm{H}^0_{\mathrm{ct}}(\mathrm{GL}_{n-1}(\mathbb{K})^0; \chi_{\mathbb{K}}^{(j)}).$$

Let $\phi_{\xi,j}$ be as in Proposition 3.1. Then we have a $\mathrm{GL}_{n-1}(\mathbb{K})$-equivariant continuous linear map

$$\phi_{\xi,j} \otimes \mathrm{Z}^\circ_\xi(\cdot, \tfrac{1}{2} + j, \chi_{\mathbb{K}}) : F_\xi^\vee \otimes \pi_\xi \to (\otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \det{}^j) \otimes (\chi_{\mathbb{K},-j} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}}) = \chi_{\mathbb{K}}^{(j)} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}}. \tag{32}$$

By restriction of cohomology, this induces a linear map

$$\begin{aligned} \wp_{\xi,\chi_{\mathbb{K}},j} &: \mathrm{H}_\mu \otimes \mathrm{H}_\nu \otimes \mathrm{H}_{\chi_{\mathbb{K}}^{(j)}} \otimes \mathfrak{O}_{n-1,\mathbb{K}} \\ &= \mathrm{H}^{d_{n-1,\mathbb{K}}}_{\mathrm{ct}}(\mathrm{GL}_n(\mathbb{K})^0 \times \mathrm{GL}_{n-1}(\mathbb{K})^0; F_\xi^\vee \otimes \pi_\xi \otimes \chi_{\mathbb{K}}^{(j)}) \otimes \mathfrak{O}_{n-1,\mathbb{K}} \\ &\to \mathrm{H}^{d_{n-1,\mathbb{K}}}_{\mathrm{ct}}(\mathrm{GL}_{n-1}(\mathbb{K})^0; \mathfrak{M}^*_{n-1,\mathbb{K}}) \otimes \mathfrak{O}_{n-1,\mathbb{K}} = \mathbb{C}. \end{aligned}$$

We call this map the *archimedean modular symbol*, which is nonzero by the non-vanishing hypothesis that is proved in [Sun17].

Specifying the above argument to the case when $\xi = \xi_0 := (0_{n,\mathbb{K}}, 0_{n-1,\mathbb{K}})$ and $j = 0$, we get a linear map (with $\chi_{\mathbb{K}}$ replaced by $\chi_{\mathbb{K}}^{(j)}$)

$$\wp_{\xi_0,\chi_{\mathbb{K}}^{(j)},0} : \mathrm{H}_{0_{n,\mathbb{K}}} \otimes \mathrm{H}_{0_{n-1,\mathbb{K}}} \otimes \mathrm{H}_{\chi_{\mathbb{K}}^{(j)}} \otimes \mathfrak{O}_{n-1,\mathbb{K}} \to \mathbb{C}. \tag{33}$$

The archimedean period relation is the following theorem.

THEOREM 3.2. *Let the notation and assumptions be as above. Let*

$$\Omega'_{\mu,\nu,j} := \mathrm{i}^{j(n(n-1)/2)[\mathbb{K}:\mathbb{R}]} \cdot c'_\mu \cdot c_\nu \cdot \varepsilon_{\mu,\nu}, \tag{34}$$

where

$$c'_\mu := \prod_{i=1}^{n-1} ((-1)^n \mathrm{i})^{(n-i)\sum_{\iota \in \mathcal{E}_{\mathbb{K}}} \mu_i^\iota},$$

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$$c_\nu := \prod_{i=1}^{n-1} ((-1)^n \mathrm{i})^{(n-i)\sum_{\iota\in\mathcal{E}_{\mathbb{K}}} \nu_i^\iota}, \quad \textit{and}$$
$$\varepsilon_{\mu,\nu} := \prod_{i>k,\, i+k\leqslant n} (-1)^{\sum_{\iota\in\mathcal{E}_{\mathbb{K}}}(\mu_i^\iota+\nu_k^\iota)}.$$

Then the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{H}_\mu \otimes \mathrm{H}_\nu \otimes \mathrm{H}_{\chi_{\mathbb{K}}^{(j)}} \otimes \mathfrak{O}_{n-1,\mathbb{K}} & \xrightarrow{\Omega'_{\mu,\nu,j}\cdot \wp_{\xi,\chi_{\mathbb{K}},j}} & \mathbb{C} \\
\jmath_\mu\otimes\jmath_\nu\otimes\mathrm{id}\otimes\mathrm{id}\uparrow & & \Vert \\
\mathrm{H}_{0_{n,\mathbb{K}}} \otimes \mathrm{H}_{0_{n-1,\mathbb{K}}} \otimes \mathrm{H}_{\chi_{\mathbb{K}}^{(j)}} \otimes \mathfrak{O}_{n-1,\mathbb{K}} & \xrightarrow{\wp_{\xi_0,\chi_{\mathbb{K}}^{(j)},0}} & \mathbb{C}
\end{array}$$

Remark 3.3. The Rankin–Selberg integrals for minimal K-type vectors of principal series representations of $\mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_{n-1}(\mathbb{K})$ have been explicitly calculated by Ishii and Miyazaki in [IM22]. The Rankin–Selberg integrals for minimal K-type vectors of irreducible generalized principal series representations of $\mathrm{GL}_3(\mathbb{K}) \times \mathrm{GL}_2(\mathbb{K})$ have been calculated explicitly by Hirano *et al.* in [HIM22]. It should be also possible to prove Theorem 3.2 when $n \leqslant 3$ or $\mathbb{K} \cong \mathbb{C}$, by using these results and the method in [Sun17].

4. Proof of archimedean period relations

In this section we prove the archimedean period relations (Theorem 3.2). Retain the notation of the last section.

Put

$$\jmath_\xi := \jmath_\mu \otimes \jmath_\nu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})\times\mathrm{GL}_{n-1}(\mathbb{K})}(\pi_{\xi_0}, F_\xi^\vee \otimes \pi_\xi). \tag{35}$$

We will prove the following result, which implies Theorem 3.2 by specifying s to $\frac{1}{2}$.

Theorem 4.1. *The diagram*

$$\begin{array}{ccc}
F_\xi^\vee \otimes \pi_\xi = (F_\mu^\vee \otimes \pi_\mu)\widehat{\otimes}(F_\nu^\vee \otimes \pi_\nu) & \xrightarrow{\Omega'_{\mu,\nu,j}\cdot\phi_{\xi,j}\otimes \mathrm{Z}_\xi^\circ(\cdot,s+j,\chi_{\mathbb{K}})} & \chi^{(j)}_{\mathbb{K},\frac{1}{2}-s} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}} \\
\jmath_\xi \uparrow & & \Vert \\
\pi_{\xi_0} = \pi_{0_{n,\mathbb{K}}} \widehat{\otimes} \pi_{0_{n-1,\mathbb{K}}} & \xrightarrow{\mathrm{Z}^\circ_{\xi_0}(\cdot,s,\chi_{\mathbb{K}}^{(j)})} & \chi^{(j)}_{\mathbb{K},\frac{1}{2}-s} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}}
\end{array}$$

commutes for all $s \in \mathbb{C}$, where $\Omega'_{\mu,\nu,j}$ is given by (34) as in Theorem 3.2.

Here $\phi_{\xi,j} \otimes \mathrm{Z}_\xi^\circ(\cdot, s+j, \chi_{\mathbb{K}})$ is defined in the way similar to (32).

4.1 Reduction to principal series representations

Recall that in § 2.2 we have defined a principal series representation I_μ with a Whittaker functional $\lambda'_\mu \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(I_\mu, \psi_{n,\mathbb{K}})$, and a unique $p_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(I_\mu, \pi_\mu)$ such that

$$\lambda_\mu \circ p_\mu = \lambda'_\mu.$$

We have also defined $\imath_\mu \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{K})}(I_{0_{n,\mathbb{K}}}, F_\mu^\vee \otimes I_\mu)$ such that

$$(v_\mu \otimes \lambda'_\mu) \circ \imath_\mu = \lambda'_{0_{n,\mathbb{K}}}$$

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and that the diagram (22) commutes. We have similar data for ν. Put

$$I_\xi := I_\mu \widehat{\otimes} I_\nu, \quad p_\xi := p_\mu \otimes p_\nu, \quad \imath_\xi := \imath_\mu \otimes \imath_\nu.$$

Define the normalized Rankin–Selberg integral

$$\begin{aligned} \mathrm{Z}_\xi^\diamond(\cdot, s, \chi_{\mathbb{K}}) &= \frac{1}{\mathrm{L}(s, \pi_\mu \times \pi_\nu)} \mathrm{Z}_\xi(\cdot, s, \chi_{\mathbb{K}}) \\ &\in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(I_\xi \otimes \mathfrak{M}_{n-1,\mathbb{K}},\ \chi_{\mathbb{K},-s+1/2}\big) \end{aligned}$$

as the composition

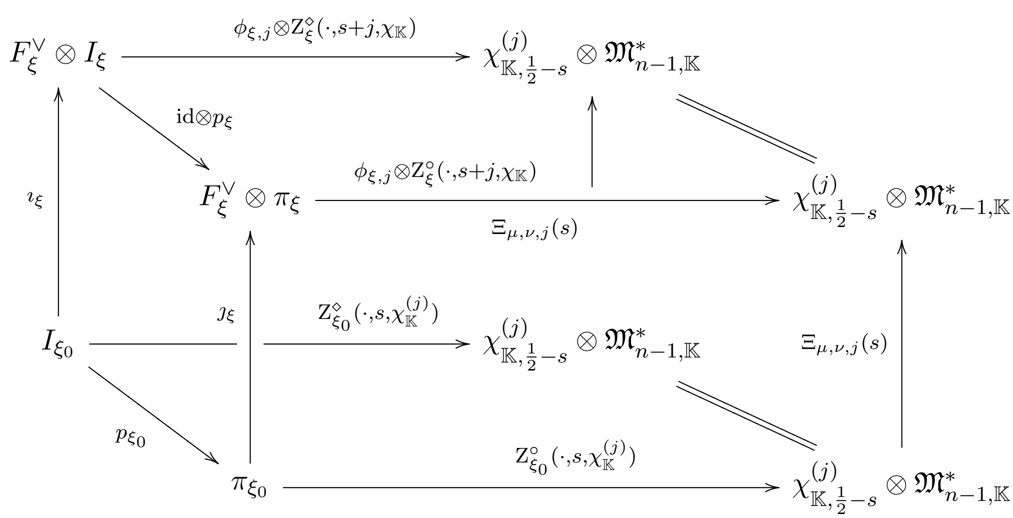
$$I_\xi \otimes \mathfrak{M}_{n-1,\mathbb{K}} \xrightarrow{p_\xi \otimes \mathrm{id}} \pi_\xi \otimes \mathfrak{M}_{n-1,\mathbb{K}} \xrightarrow{\mathrm{Z}_\xi^\circ(\cdot,s,\chi_{\mathbb{K}})} \chi_{\mathbb{K},-s+1/2}.$$

Then

$$\begin{aligned} \mathrm{Z}_\xi^\diamond(f \otimes f' \otimes m, s, \chi_{\mathbb{K}}) = &\frac{1}{\mathrm{L}(s, \pi_\mu \times \pi_\nu)} \\ &\cdot \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} \lambda'_\mu\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} . f \right) \lambda'_\nu(g.f') \\ &\cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s-1/2}\, \mathrm{d}\bar{m}(g), \end{aligned}$$

for $f \in I_\mu$, $f' \in I_\nu$, $m \in \mathfrak{M}_{n-1,\mathbb{K}}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large.

In view of all the above, by the multiplicity one theorem [AG09, SZ12], there exists a unique entire function $\Xi_{\mu,\nu,j}(s)$ such that the following diagram commutes for all $s \in \mathbb{C}$.



(36)

In the rest of this section we compute the function $\Xi_{\mu,\nu,j}(s)$ and show that it is a nonzero constant whose inverse is equal to $\Omega'_{\mu,\nu,j}$ given by (34). The main ingredient of the computation is [LLSS23].

4.2 Integral over the open orbit

For the convenience of the reader, we describe the main result of [LLSS23]. Write $\widehat{\mathbb{K}^\times}$ for the set of all (unitary or not) characters of $\mathbb{K}^\times$. Let $\varrho = (\varrho_1, \ldots, \varrho_n) \in (\widehat{\mathbb{K}^\times})^n$, viewed as a character of $\bar{\mathrm{B}}_n(\mathbb{K})$ as usual, and let

$$I(\varrho) := \mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})} \varrho$$

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be the corresponding principal series representation of $\mathrm{GL}_n(\mathbb{K})$. Similarly let $\varrho' = (\varrho'_1, \dots, \varrho'_{n-1}) \in (\widehat{\mathbb{K}^\times})^{n-1}$ and let $I(\varrho')$ be the corresponding principal series representation of $\mathrm{GL}_{n-1}(\mathbb{K})$. We have a meromorphic family of unnormalized Rankin–Selberg integrals

$$\mathrm{Z}(\cdot, s, \chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(I(\varrho)\widehat{\otimes} I(\varrho') \otimes \mathfrak{M}_{n-1,\mathbb{K}},\, \chi_{\mathbb{K},-s+1/2}\big)$$

such that

$$\begin{aligned}&\mathrm{Z}(f \otimes f' \otimes m, s, \chi_{\mathbb{K}})\\ &= \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} \lambda'_{\varrho}\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} .f \right) \lambda'_{\varrho'}(g.f') \cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s-1/2} \,\mathrm{d}\bar{m}(g)\end{aligned}$$

for all $f \in I(\varrho)$, $f' \in I(\varrho')$, $m \in \mathfrak{M}_{n-1,\mathbb{K}}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large, where $\lambda'_{\varrho} \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(I(\varrho), \psi_{n,\mathbb{K}})$ and $\lambda'_{\varrho'} \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{K})}(I(\varrho'), \psi_{n-1,\mathbb{K}})$ are defined in the way similar to (18).

Let

$$z := (z_n, z_{n-1}) \in \mathrm{GL}_n(\mathbb{Z}) \times \mathrm{GL}_{n-1}(\mathbb{Z}), \tag{37}$$

where $z_n \in \mathrm{GL}_n(\mathbb{Z})$ is defined inductively in (29). The right action of $\mathrm{GL}_{n-1}(\mathbb{K})$ on the flag variety $(\bar{\mathrm{B}}_n(\mathbb{K}) \times \bar{\mathrm{B}}_{n-1}(\mathbb{K}))\backslash(\mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_{n-1}(\mathbb{K}))$ has a unique open orbit

$$\big((\bar{\mathrm{B}}_n(\mathbb{K}) \times \bar{\mathrm{B}}_{n-1}(\mathbb{K}))z\big) \cdot \mathrm{GL}_{n-1}(\mathbb{K}). \tag{38}$$

Note that

$$I(\varrho)\widehat{\otimes} I(\varrho') = \mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathbb{K})\times\bar{\mathrm{B}}_{n-1}(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})\times\mathrm{GL}_{n-1}(\mathbb{K})} \varrho \otimes \varrho'.$$

Following [LLSS23], we first formally define

$$\Lambda(\cdot, s, \chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(I(\varrho)\widehat{\otimes} I(\varrho') \otimes \mathfrak{M}_{n-1,\mathbb{K}},\, \chi_{\mathbb{K},-s+1/2}\big)$$

as the integral over the above open orbit, that is,

$$\begin{aligned}&\Lambda(\phi \otimes m, s, \chi_{\mathbb{K}})\\ &:= \int_{\mathrm{GL}_{n-1}(\mathbb{K})} \phi\left(z_n \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, z_{n-1} g \right) \cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s-1/2} \,\mathrm{d}m(g)\end{aligned} \tag{39}$$

for $\phi \in I(\varrho)\widehat{\otimes} I(\varrho')$ and $m \in \mathfrak{M}_{n-1,\mathbb{K}}$.

Define

$$\mathrm{sgn}(\varrho, \varrho', \chi_{\mathbb{K}}) := \prod_{i>k,\, i+k\leqslant n} (\varrho_i \cdot \varrho'_k \cdot \chi_{\mathbb{K}})(-1),$$

and a meromorphic function

$$\gamma_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho, \varrho', \chi_{\mathbb{K}}) := \prod_{i+k\leqslant n} \gamma(s, \varrho_i \cdot \varrho'_k \cdot \chi_{\mathbb{K}}, \psi_{\mathbb{K}}^{(n)}),$$

where $\psi_{\mathbb{K}}^{(n)}$ is the additive character $\mathbb{K} \to \mathbb{C}^\times$, $x \mapsto \psi_{\mathbb{K}}((-1)^n x)$,

$$\gamma(s, \omega, \psi_{\mathbb{K}}^{(n)}) = \varepsilon(s, \omega, \psi_{\mathbb{K}}^{(n)}) \cdot \frac{\mathrm{L}(1-s, \omega^{-1})}{\mathrm{L}(s, \omega)}$$

is the local gamma factor of a character $\omega \in \widehat{\mathbb{K}^\times}$, and $\varepsilon(s, \omega, \psi_{\mathbb{K}}^{(n)})$ is the local epsilon factor, defined following [Tat79, Jac79, Kud03]. For convenience also define

$$\varepsilon_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho, \varrho', \chi_{\mathbb{K}}) := \prod_{i+k\leqslant n} \varepsilon(s, \varrho_i \cdot \varrho'_k \cdot \chi_{\mathbb{K}}, \psi_{\mathbb{K}}^{(n)}).$$

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Finally, define a meromorphic function

$$\Gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho,\varrho',\chi_{\mathbb{K}}) := \mathrm{sgn}(\varrho,\varrho',\chi_{\mathbb{K}})\cdot\gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho,\varrho',\chi_{\mathbb{K}}).$$

For a character $\omega\in\widehat{\mathbb{K}^\times}$, denote by $\mathrm{ex}(\omega)$ the real number such that

$$|\omega| = |\cdot|_{\mathbb{K}}^{\mathrm{ex}(\omega)}.$$

Consider the complex manifold

$$\mathcal{M} := \mathbb{C}\times(\widehat{\mathbb{K}^\times})^n\times(\widehat{\mathbb{K}^\times})^{n-1}$$

and its nonempty open subset

$$\Omega := \left\{(s,\varrho,\varrho')\in\mathcal{M}\;\middle|\;\begin{array}{l}\mathrm{ex}(\varrho_i)+\mathrm{ex}(\varrho'_k)+\mathrm{Re}(s)<1 \text{ whenever } i+k\leqslant n,\\ \mathrm{ex}(\varrho_i)+\mathrm{ex}(\varrho'_k)+\mathrm{Re}(s)>0 \text{ whenever } i+k>n\end{array}\right\}.$$

Theorem 4.2. *[LLSS23, Theorem 1.6(b)] Assume that $(s,\varrho,\varrho')\in\Omega$. Then the integral (39) converges absolutely, and*

$$\Lambda(\phi\otimes m,s,\chi_{\mathbb{K}}) = \Gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho,\varrho',\chi_{\mathbb{K}})\cdot\mathrm{Z}(\phi\otimes m,s,\chi_{\mathbb{K}}). \tag{40}$$

We remark that the right-hand side of (40) is holomorphic as a function of the variable $s\in\Omega_{\varrho,\varrho'} := \{s\in\mathbb{C} : (s,\varrho,\varrho')\in\Omega\}$ (see [LLSS23, Remark 1.7]).

Let $(I(\varrho)\widehat{\otimes}I(\varrho'))^\sharp\subset I(\varrho)\widehat{\otimes}I(\varrho')$ be the subspace of $\phi\in I(\varrho)\widehat{\otimes}I(\varrho')$ such that

$$\phi|_{z\cdot\mathrm{GL}_{n-1}(\mathbb{K})}\in\mathcal{S}(z\cdot\mathrm{GL}_{n-1}(\mathbb{K})).$$

Then for every $\varrho\in(\widehat{\mathbb{K}^\times})^n$, $\varrho'\in(\widehat{\mathbb{K}^\times})^{n-1}$ and $\phi\in(I(\varrho)\widehat{\otimes}I(\varrho'))^\sharp$, the integral (39) converges absolutely and is an entire function of $s\in\mathbb{C}$. We deduce the following consequence of Theorem 4.2.

Corollary 4.3. *For every $\varrho\in(\widehat{\mathbb{K}^\times})^n$, $\varrho'\in(\widehat{\mathbb{K}^\times})^{n-1}$ and $\phi\in(I(\varrho)\widehat{\otimes}I(\varrho'))^\sharp$, the equality (40) holds as entire functions of $s\in\mathbb{C}$.*

Proof. Let $\mathcal{C}$ be the connected component of $(\widehat{\mathbb{K}^\times})^n$ containing ϱ, and let $\mathcal{C}'$ be the connected component of $(\widehat{\mathbb{K}^\times})^{n-1}$ containing ϱ'. Write $K_{\mathbb{K}} := K_{n,\mathbb{K}}\times K_{n-1,\mathbb{K}}$. Define

$$C^\infty_{\mathcal{C},\mathcal{C}'}(K_{\mathbb{K}}) := \left\{f\in C^\infty(K_{\mathbb{K}})\;\middle|\;\begin{array}{l}f(b\cdot k)=(\varrho\otimes\varrho')(b)\cdot f(k),\\ \text{for all } b\in K_{\mathbb{K}}\cap(\bar{\mathrm{B}}_n(\mathbb{K})\times\bar{\mathrm{B}}_{n-1}(\mathbb{K})),\ k\in K_{\mathbb{K}}\end{array}\right\},$$

which only depends on $\mathcal{C}$ and $\mathcal{C}'$, not on the particular choices of ϱ and ϱ'.

Consider the natural map

$$K_{\mathbb{K}}\to(\bar{\mathrm{B}}_n(\mathbb{K})\times\bar{\mathrm{B}}_{n-1}(\mathbb{K}))\backslash(\mathrm{GL}_n(\mathbb{K})\times\mathrm{GL}_{n-1}(\mathbb{K})),$$

which is surjective by the Iwasawa decomposition. Let $K_{\mathbb{K}}^\sharp\subset K_{\mathbb{K}}$ be the preimage of the open orbit (38) under the above map. Fix $f\in C^\infty_{\mathcal{C},\mathcal{C}'}(K_{\mathbb{K}})$ such that $f|_{K_{\mathbb{K}}^\sharp}\in\mathcal{S}(K_{\mathbb{K}}^\sharp)$. Then there is a unique

$$\phi_{\varrho,\varrho'} := \phi_{f,\varrho,\varrho'}\in(I(\varrho)\widehat{\otimes}I(\varrho'))^\sharp$$

such that

$$\phi_{\varrho,\varrho'}|_{K_{\mathbb{K}}} = f.$$

Let $\mathcal{M}^\circ := \mathbb{C}\times\mathcal{C}\times\mathcal{C}'$, which is a connected component of $\mathcal{M}$. When (ϱ,ϱ') varies in $\mathcal{C}\times\mathcal{C}'$, the integral $\Lambda(\phi_{\varrho,\varrho'}\otimes m,s,\chi_{\mathbb{K}})$ is clearly holomorphic on $\mathcal{M}^\circ$. By [Jac09, § 8.1], we also have that

$$\Gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho,\varrho',\chi_{\mathbb{K}})\cdot\mathrm{Z}(\phi_{\varrho,\varrho'}\otimes m,s,\chi_{\mathbb{K}})$$

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is meromorphic on $\mathcal{M}^\circ$. Since the equality

$$\Lambda(\phi_{\varrho,\varrho'} \otimes m, s, \chi_{\mathbb{K}}) = \Gamma_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho, \varrho', \chi_{\mathbb{K}}) \cdot \mathrm{Z}(\phi_{\varrho,\varrho'} \otimes m, s, \chi_{\mathbb{K}})$$

holds on $\Omega \cap \mathcal{M}^\circ$, which is nonempty and open, it holds over all $\mathcal{M}^\circ$ by the uniqueness of meromorphic continuation. The corollary then follows by noting that every $\phi \in (I(\varrho)\widehat{\otimes} I(\varrho'))^\sharp$ equals $\phi_{f,\varrho,\varrho'}$ for some $f \in C^\infty_{\mathcal{C},\mathcal{C}'}(K_{\mathbb{K}})$ such that $f|_{K_{\mathbb{K}}^\sharp} \in \mathcal{S}(K_{\mathbb{K}}^\sharp)$. □

4.3 A commutative diagram

We now specify the above discussion to the principal series representations I_μ and I_ν. Define $\varrho^\mu = (\varrho_1^\mu, \ldots, \varrho_n^\mu) \in (\widehat{\mathbb{K}^\times})^n$, where

$$\varrho_i^\mu := \varepsilon_{n,\mathbb{K}} |\cdot|_{\mathbb{K}}^{(n+1)/2-i} \prod_{\iota \in \mathcal{E}_{\mathbb{K}}} \iota^{\mu_i^\iota} \in \widehat{\mathbb{K}^\times}, \quad i = 1, \ldots, n,$$

so that $I_\mu = I(\varrho^\mu)$ in the above notation. Likewise we define $\varrho^\nu = (\varrho_1^\nu, \ldots, \varrho_{n-1}^\nu) \in (\widehat{\mathbb{K}^\times})^{n-1}$ so that $I_\nu = I(\varrho^\nu)$, and put $I_\xi^\sharp := (I_\mu \widehat{\otimes} I_\nu)^\sharp$. Similar to $\varepsilon_{n,\mathbb{K}}$, one defines $\varepsilon_{n-1,\mathbb{K}}$ as the common central character of $F_\nu^\vee \otimes \pi_\nu$ and $\pi_{0_{n-1,\mathbb{K}}}$.

The integral (39) defines a nonzero linear functional

$$\Lambda_\xi(\cdot, s, \chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}\big(I_\xi^\sharp, \chi_{\mathbb{K},-s+1/2} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}}\big).$$

By Corollary 4.3,

$$\begin{aligned}\Lambda_\xi(\cdot, s, \chi_{\mathbb{K}}) &= \Gamma_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho^\mu, \varrho^\nu, \chi_{\mathbb{K}}) \cdot \mathrm{Z}_\xi(\cdot, s, \chi_{\mathbb{K}}) \\ &= \Gamma_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho^\mu, \varrho^\nu, \chi_{\mathbb{K}}) \cdot \mathrm{L}(s, \pi_\mu \times \pi_\nu) \cdot \mathrm{Z}_\xi^\circ(\cdot, s, \chi_{\mathbb{K}})\end{aligned} \tag{41}$$

holds on $I_\xi^\sharp \otimes \mathfrak{M}_{n-1,\mathbb{K}}$. Recall that $\imath_\xi = \imath_\mu \otimes \imath_\nu \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}(I_{\xi_0}, F_\xi^\vee \otimes I_\xi)$. It is clear that

$$\imath_\xi(I_{\xi_0}^\sharp) \subset F_\xi^\vee \otimes I_\xi^\sharp.$$

Proposition 4.4. *The following diagram commutes.*

$$\begin{array}{ccc} F_\xi^\vee \otimes I_\xi^\sharp & \xrightarrow{\phi_{\xi,j} \otimes \Lambda_\xi(\cdot, s+j, \chi_{\mathbb{K}})} & \chi_{\mathbb{K},\frac{1}{2}-s}^{(j)} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}} \\ {\scriptstyle \imath_\xi}\uparrow & & \| \\ I_{\xi_0}^\sharp & \xrightarrow{\Lambda_{\xi_0}(\cdot, s, \chi_{\mathbb{K}}^{(j)})} & \chi_{\mathbb{K},\frac{1}{2}-s}^{(j)} \otimes \mathfrak{M}^*_{n-1,\mathbb{K}} \end{array}$$

Proof. Recall from Proposition 3.1 that $\phi_{\xi,j} \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K}\otimes_{\mathbb{R}}\mathbb{C})}(F_\xi^\vee, \otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \det^j)$ and

$$\phi_{\xi,j}(z^{-1}.v_\xi^\vee) = 1,$$

where $v_\xi^\vee := v_\mu^\vee \otimes v_\nu^\vee$. For $\phi \in I_{\xi_0}$ and $g \in \mathrm{GL}_{n-1}(\mathbb{K}) \subset \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_{n-1}(\mathbb{K})$ (see (27) for the inclusion), we have that

$$\begin{aligned}\phi_{\xi,j}(\imath_\xi(\phi)(zg)) &= \phi_{\xi,j}\big(\phi(zg) \cdot (g^{-1}z^{-1}.v_\xi^\vee)\big) \quad \text{(see (19))} \\ &= \phi(zg) \cdot \phi_{\xi,j}(g^{-1}z^{-1}.v_\xi^\vee) \\ &= \phi(zg) \cdot (\otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \det{}^{-j})(g).\end{aligned}$$

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Assume that $\phi \in I^{\sharp}_{\xi_0}$. By (39), we have that

$$\begin{aligned}&\big(\phi_{\xi,j} \otimes \Lambda_\xi(\cdot, s+j, \chi_{\mathbb{K}})\big)(\imath_\xi(\phi) \otimes m)\\ &= \int_{\mathrm{GL}_{n-1}(\mathbb{K})} \phi_{\xi,j}(\imath_\xi(\phi)(zg)) \cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s+j-1/2} \,\mathrm{d}m(g)\\ &= \int_{\mathrm{GL}_{n-1}(\mathbb{K})} \phi(zg) \cdot (\otimes_{\iota \in \mathcal{E}_{\mathbb{K}}} \det{}^{-j})(g) \cdot \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s+j-1/2} \,\mathrm{d}m(g)\\ &= \int_{\mathrm{GL}_{n-1}(\mathbb{K})} \phi(zg) \cdot \chi_{\mathbb{K}}^{(j)}(\det g) \cdot |\det g|_{\mathbb{K}}^{s-1/2} \,\mathrm{d}m(g)\\ &= \Lambda_{\xi_0}(\phi \otimes m, s, \chi_{\mathbb{K}}^{(j)}),\end{aligned}$$

where $m \in \mathfrak{M}_{n-1,\mathbb{K}}$. This proves the proposition. □

Corollary 4.5. *Let the notation be as above. Then*

$$\Xi_{\mu,\nu,j}(s) \cdot \frac{\Gamma_{\psi_{\mathbb{K}}^{(n)}}(s+j, \varrho^\mu, \varrho^\nu, \chi_{\mathbb{K}})}{\Gamma_{\psi_{\mathbb{K}}^{(n)}}(s, \varrho^{0_{n,\mathbb{K}}}, \varrho^{0_{n-1,\mathbb{K}}}, \chi_{\mathbb{K}}^{(j)})} \cdot \frac{\mathrm{L}(s+j, \pi_\mu \times \pi_\nu)}{\mathrm{L}(s, \pi_{0_{n,\mathbb{K}}} \times \pi_{0_{n-1,\mathbb{K}}})} = 1 \tag{42}$$

as meromorphic functions of the variable $s \in \mathbb{C}$.

Proof. This follows from (36), (41) and Proposition 4.4. □

4.4 Archimedean local factors

To finish the proof, it remains to evaluate the function $\Xi_{\mu,\nu,j}(s)^{-1}$ given by (42) and show that it is equal to the constant $\Omega'_{\mu,\nu,j}$ given by (34) as in Theorem 3.2. To this end, we first recall some standard facts about archimedean local L-factors and epsilon factors that we need, following [Kna94]. Let

$$\Gamma_{\mathbb{K}}(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2), & \text{if } \mathbb{K} \cong \mathbb{R},\\ 2(2\pi)^{-s}\Gamma(s), & \text{if } \mathbb{K} \cong \mathbb{C},\end{cases}$$

where $\Gamma(s)$ is the standard gamma function. Recall the Legendre duplication formula

$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1). \tag{43}$$

Recall the additive character $\psi_{\mathbb{K}}$ that is defined in (10) by using $\psi_{\mathbb{R}}$.

If $\mathbb{K} \cong \mathbb{R}$, then the following hold true.

– For all $t \in \mathbb{C}$ and $\delta \in \{0,1\}$,

$$\mathrm{L}(s, |\cdot|_{\mathbb{K}}^t \,\mathrm{sgn}_{\mathbb{K}^\times}^\delta) = \Gamma_{\mathbb{R}}(s+t+\delta),$$

and

$$\varepsilon(s, |\cdot|_{\mathbb{K}}^t \,\mathrm{sgn}_{\mathbb{K}^\times}^\delta, \psi_{\mathbb{K}}^{(n)}) = ((-1)^n \mathrm{i})^\delta.$$

– For all $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z} \setminus \{0\}$, and t, δ as above,

$$\mathrm{L}(s, D_{a,b} \times |\cdot|_{\mathbb{K}}^t \,\mathrm{sgn}_{\mathbb{K}^\times}^\delta) = \Gamma_{\mathbb{C}}\big(s+t+\max\{a,b\}\big).$$

Here and henceforth, for any $a', b' \in \mathbb{C}$ with $a' - b' \in \mathbb{Z}$,

$$\max\{a', b'\} := \begin{cases} a', & \text{if } a' - b' \geqslant 0;\\ b', & \text{otherwise.}\end{cases}$$

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– For all $a, b, a', b' \in \mathbb{C}$ with $a - b, a' - b' \in \mathbb{Z} \setminus \{0\}$,

$$\mathrm{L}(s, D_{a,b} \times D_{a',b'}) = \Gamma_{\mathbb{C}}\big(s + \max\{a + a', b + b'\}\big) \cdot \Gamma_{\mathbb{C}}\big(s + \max\{a + b', b + a'\}\big).$$

If $\mathbb{K} \cong \mathbb{C}$, then for all $a, b \in \mathbb{C}$ with $a - b \in \mathbb{Z}$,

$$\mathrm{L}(s, \iota^a \bar{\iota}^b) = \Gamma_{\mathbb{C}}\big(s + \max\{a, b\}\big)$$

and

$$\varepsilon(s, \iota^a \bar{\iota}^b, \psi_{\mathbb{K}}^{(n)}) = ((-1)^n \mathrm{i})^{|a-b|}.$$

By the well-known branching rule for $\mathrm{GL}_n(\mathbb{C})$, we have that $j \in \mathbb{Z}$ is a balanced place for ξ if and only if

$$-\mu_n^\iota \geqslant \nu_1^\iota + j \geqslant -\mu_{n-1}^\iota \geqslant \nu_2^\iota + j \geqslant \cdots \geqslant \nu_{n-1}^\iota + j \geqslant -\mu_1^\iota$$

for every $\iota \in \mathcal{E}_{\mathbb{K}}$. Equivalently, by [Rag16, Corollary 2.35], $j \in \mathbb{Z}$ is a balanced place for ξ if and only if

$$m_{\mu,\nu}^- \leqslant j \leqslant m_{\mu,\nu}^+,$$

where

$$m_{\mu,\nu}^- := \max\{-\mu_{n-i}^\iota - \nu_i^\iota : 1 \leqslant i \leqslant n-1, \iota \in \mathcal{E}_{\mathbb{K}}\},$$
$$m_{\mu,\nu}^+ := \min\{-\mu_{n+1-i}^\iota - \nu_i^\iota : 1 \leqslant i \leqslant n-1, \iota \in \mathcal{E}_{\mathbb{K}}\}.$$

Recall the half-integers $\tilde{\mu}_i^\iota$ $(i = 1, \ldots, n)$ given by (14). The following result can be easily checked by using the above result (see the proof of [Rag16, Lemma 2.24]).

Lemma 4.6. *Assume that $\xi = (\mu, \nu)$ is balanced. Then*

$$\tilde{\mu}_i^\iota + \tilde{\nu}_k^\iota - \tilde{\mu}_{n+1-i}^{\bar{\iota}} - \tilde{\nu}_{n-k}^{\bar{\iota}}$$

is positive if $i + k \leqslant n$, and is negative otherwise, for every $\iota \in \mathcal{E}_{\mathbb{K}}$.

Now we establish the following result, which thereby finishes the proof of Theorem 4.1.

Proposition 4.7. *The function $\Xi_{\mu,\nu,j}(s)^{-1}$ given by (42) is equal to the constant $\Omega'_{\mu,\nu,j}$ given by (34).*

Proof. We use the notation of Theorem 3.2. It is clear that

$$\frac{\mathrm{sgn}(\varrho^\mu, \varrho^\nu, \chi_{\mathbb{K}})}{\mathrm{sgn}(\varrho^{0_{n,\mathbb{K}}}, \varrho^{0_{n-1,\mathbb{K}}}, \chi_{\mathbb{K}}^{(j)})} = \prod_{i > k,\, i+k \leqslant n} (-1)^{\sum_{\iota \in \mathcal{E}_{\mathbb{K}}} (\mu_i^\iota + \nu_k^\iota + j)} = (-1)^{j(n^2(n-1)/2)[\mathbb{K}:\mathbb{R}]} \cdot \varepsilon_{\mu,\nu}.$$

To evaluate the contribution from the local gamma and L-factors, we consider the real and complex cases separately.

(i) Assume that $\mathbb{K} \cong \mathbb{R}$. Then $\mathcal{E}_{\mathbb{K}} = \{\iota\}$. Using Lemma 4.6, it is easy to check that

$$\mathrm{L}(s, \pi_\mu \times \pi_\nu) = \prod_{i+k \leqslant n} \Gamma_{\mathbb{C}}(s + \tilde{\mu}_i^\iota + \tilde{\nu}_k^\iota).$$

For a character $\omega \in \widehat{\mathbb{K}^\times}$, write $\delta(\omega) \in \{0, 1\}$ such that $\omega(-1) = (-1)^{\delta(\omega)}$. Then

$$\varepsilon(s, \omega, \psi_{\mathbb{K}}^{(n)}) = ((-1)^n \mathrm{i})^{\delta(\omega)}, \tag{44}$$

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and it is clear that

$$j+\mu_i^\iota+\nu_k^\iota-\delta(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}})+\delta(\varrho_i^{0_{n,\mathbb{K}}}\varrho_k^{0_{n-1,\mathbb{K}}}\chi_{\mathbb{K}}^{(j)})\in 2\mathbb{Z}. \tag{45}$$

We have that

$$\frac{\mathrm{L}(1-s,(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}})^{-1})}{\mathrm{L}(s,\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}})}=\frac{\Gamma_{\mathbb{R}}(1-s-\tilde\mu_i^\iota-\tilde\nu_k^\iota+\delta(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}}))}{\Gamma_{\mathbb{R}}(s+\tilde\mu_i^\iota+\tilde\nu_k^\iota+\delta(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}}))}.$$

It follows from (43) that

$$\begin{aligned}&\left(\prod_{i+k\leqslant n}\frac{\mathrm{L}(1-s,(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}})^{-1})}{\mathrm{L}(s,\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}})}\right)\cdot\mathrm{L}(s,\pi_\mu\times\pi_\nu)\\&\quad=\prod_{i+k\leqslant n}\big(\Gamma_{\mathbb{R}}(s+\tilde\mu_i^\iota+\tilde\nu_k^\iota+1-\delta(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}}))\cdot\Gamma_{\mathbb{R}}(1-s-\tilde\mu_i^\iota-\tilde\nu_k^\iota+\delta(\varrho_i^\mu\varrho_k^\nu\chi_{\mathbb{K}}))\big).\end{aligned} \tag{46}$$

By (44), (45), (46) and the formula

$$\Gamma_{\mathbb{R}}(s+\ell)\cdot\Gamma_{\mathbb{R}}(2-s-\ell)=\mathrm{i}^\ell\cdot\Gamma_{\mathbb{R}}(s)\cdot\Gamma_{\mathbb{R}}(2-s),\quad \ell\in 2\mathbb{Z},$$

we find that

$$\begin{aligned}&\frac{\gamma_{\psi_{\mathbb{K}}^{(n)}}(s+j,\varrho^\mu,\varrho^\nu,\chi_{\mathbb{K}})}{\gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho^{0_{n,\mathbb{K}}},\varrho^{0_{n-1,\mathbb{K}}},\chi_{\mathbb{K}}^{(j)})}\cdot\frac{\mathrm{L}(s+j,\pi_\mu\times\pi_\nu)}{\mathrm{L}(s,\pi_{0_{n,\mathbb{K}}}\times\pi_{0_{n-1,\mathbb{K}}})}\\&\quad=\prod_{i+k\leqslant n}\left(\frac{\varepsilon(s+j,\varrho^\mu\varrho^\nu\chi_{\mathbb{K}},\psi_{\mathbb{K}}^{(n)})}{\varepsilon(s,\varrho^{0_{n,\mathbb{K}}}\varrho^{0_{n-1,\mathbb{K}}}\chi_{\mathbb{K}}^{(j)},\psi_{\mathbb{K}}^{(n)})}\cdot\mathrm{i}^{j+\mu_i^\iota+\nu_k^\iota-\delta(\varrho^\mu\varrho^\nu\chi_{\mathbb{K}})+\delta(\varrho^{0_{n,\mathbb{K}}}\varrho^{0_{n-1,\mathbb{K}}}\chi_{\mathbb{K}}^{(j)})}\right)\\&\quad=\prod_{i+k\leqslant n}((-1)^n\mathrm{i})^{j+\mu_i^\iota+\nu_k^\iota}\\&\quad=(-1)^{jn^2(n-1)/2}\cdot\mathrm{i}^{jn(n-1)/2}\cdot c'_\mu\cdot c_\nu.\end{aligned}$$

(ii) Assume that $\mathbb{K}\cong\mathbb{C}$. Then $\chi_{\mathbb{K}}^{(j)}$ is trivial, which will be omitted from the notation for convenience. Using Lemma 4.6 again, we find that

$$\mathrm{L}(s,\pi_\mu\times\pi_\nu)=\prod_{i+k\leqslant n,\,\iota\in\mathcal{E}_{\mathbb{K}}}\Gamma_{\mathbb{C}}(s+\tilde\mu_i^\iota+\tilde\nu_k^\iota).$$

We have that

$$\frac{\mathrm{L}(1-s,(\varrho_i^\mu\varrho_k^\nu)^{-1})}{\mathrm{L}(s,\varrho_i^\mu\varrho_k^\nu)}=\frac{\Gamma_{\mathbb{C}}(1-s-\min_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\tilde\mu_i^\iota+\tilde\nu_k^\iota\})}{\Gamma_{\mathbb{C}}(s+\max_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\tilde\mu_i^\iota+\tilde\nu_k^\iota\})}.$$

It follows that

$$\begin{aligned}&\left(\prod_{i+k\leqslant n}\frac{\mathrm{L}(1-s,(\varrho_i^\mu\varrho_k^\nu)^{-1})}{\mathrm{L}(s,\varrho_i^\mu\varrho_k^\nu)}\right)\cdot\mathrm{L}(s,\pi_\mu\times\pi_\nu)\\&\quad=\prod_{i+k\leqslant n}\left(\Gamma_{\mathbb{C}}\Big(s+\min_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\tilde\mu_i^\iota+\tilde\nu_k^\iota\}\Big)\cdot\Gamma_{\mathbb{C}}\Big(1-s-\min_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\tilde\mu_i^\iota+\tilde\nu_k^\iota\}\Big)\right).\end{aligned} \tag{47}$$

Using (47) and the formula

$$\Gamma_{\mathbb{C}}(s+\ell)\cdot\Gamma_{\mathbb{C}}(1-s-\ell)=(-1)^\ell\cdot\Gamma_{\mathbb{C}}(s)\cdot\Gamma_{\mathbb{C}}(1-s),\quad \ell\in\mathbb{Z},$$

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we find that

$$\frac{\gamma_{\psi_{\mathbb{K}}^{(n)}}(s+j,\varrho^{\mu},\varrho^{\nu})}{\gamma_{\psi_{\mathbb{K}}^{(n)}}(s,\varrho^{0_{n,\mathbb{K}}},\varrho^{0_{n-1,\mathbb{K}}})}\cdot\frac{\mathrm{L}(s+j,\pi_{\mu}\times\pi_{\nu})}{\mathrm{L}(s,\pi_{0_{n,\mathbb{K}}}\times\pi_{0_{n-1,\mathbb{K}}})}$$

$$=\prod_{i+k\leqslant n}\big(\varepsilon(s+j,\varrho_i^{\mu}\varrho_k^{\nu},\psi_{\mathbb{K}}^{(n)})\cdot(-1)^{j+\min_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\mu_i^{\iota}+\nu_k^{\iota}\}}\big). \tag{48}$$

We have the local epsilon factor

$$\varepsilon(s+j,\varrho_i^{\mu}\varrho_k^{\nu},\psi_{\mathbb{K}}^{(n)})=((-1)^n\mathrm{i})^{\max_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\mu_i^{\iota}+\nu_k^{\iota}\}-\min_{\iota\in\mathcal{E}_{\mathbb{K}}}\{\mu_i^{\iota}+\nu_k^{\iota}\}}.$$

Hence, (48) is equal to

$$\prod_{i+k\leqslant n,\,\iota\in\mathcal{E}_{\mathbb{K}}}((-1)^n\mathrm{i})^{j+\mu_i^{\iota}+\nu_k^{\iota}}=(-1)^{jn^2(n-1)}\cdot\mathrm{i}^{jn(n-1)}\cdot c'_{\mu}\cdot c_{\nu}.$$

This finishes the proof of the proposition. □

5. Non-archimedean period relations

In this section, let $\mathbb{K}$ be a non-archimedean local field of characteristic zero. Fix a nontrivial unitary character $\psi_{\mathbb{K}}:\mathbb{K}\to\mathbb{C}^{\times}$, and define the character $\psi_{n,\mathbb{K}}$ of $\mathrm{N}_n(\mathbb{K})$ as in (11) ($n\geqslant 1$).

5.1 Preliminaries

Denote $\mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}}$ the smooth induction which consists of all functions $f:\mathrm{GL}_n(\mathbb{K})\to\mathbb{C}$ such that:

- f is right invariant under some open compact subgroup of $\mathrm{GL}_n(\mathbb{K})$;
- $f(ug)=\psi_{n,\mathbb{K}}(u)f(g)$ for all $u\in\mathrm{N}_n(\mathbb{K})$ and $g\in\mathrm{GL}_n(\mathbb{K})$.

This is a smooth representation of $\mathrm{GL}_n(\mathbb{K})$ under the right translation.

Let p be the residue characteristic of $\mathbb{K}$, and $\mu_{p^\infty}\subset\mathbb{C}^{\times}$ be the subgroup of pth power roots of unity. Recall the cyclotomic character

$$\mathrm{Aut}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})\to\mathbb{Z}_p^{\times},\quad \sigma\mapsto t_{\sigma,p}$$

defined by requiring that

$$\sigma(\zeta)=\zeta^{t_{\sigma,p}}\quad\text{for all }\zeta\in\mu_{p^\infty}. \tag{49}$$

Write $\sigma\mapsto t_{\sigma,\mathbb{K}}$ for the composition

$$\mathrm{Aut}(\mathbb{C}/\mathbb{Q})\xrightarrow{\text{restriction}}\mathrm{Aut}(\mathbb{Q}(\mu_p^{\infty})/\mathbb{Q})\xrightarrow{\sigma\mapsto t_{\sigma,p}}\mathbb{Z}_p^{\times}\subset\mathbb{K}^{\times}.$$

Following [Har83, pp. 79–80] and [Mah05, p. 594], define

$$\mathbf{t}_{n,\sigma,\mathbb{K}}:=\mathrm{diag}(t_{\sigma,\mathbb{K}}^{-(n-1)},\dots,t_{\sigma,\mathbb{K}}^{-1},1)\in\mathrm{GL}_n(\mathbb{K}), \tag{50}$$

and define an action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}}$ by

$${}^{\sigma}f(g):=\sigma\big(f(\mathbf{t}_{n,\sigma,\mathbb{K}}\cdot g)\big), \tag{51}$$

where $\sigma\in\mathrm{Aut}(\mathbb{C})$, $f\in\mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}}$, and $g\in\mathrm{GL}_n(\mathbb{K})$.

Let $\Pi_{\mathbb{K}}$ be a generic irreducible smooth representations of $\mathrm{GL}_n(\mathbb{K})$, with a fixed Whittaker functional

$$\lambda_{\mathbb{K}} \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{K})}(\Pi_{\mathbb{K}}, \psi_{n,\mathbb{K}}) \setminus \{0\}.$$

Using $\lambda_{\mathbb{K}}$, we realize $\Pi_{\mathbb{K}}$ as a subrepresentation of $\mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}}$ by

$$\Pi_{\mathbb{K}} \to \mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}}, \quad u \mapsto \big(g \mapsto \lambda_{\mathbb{K}}(g.u)\big). \tag{52}$$

Put

$${}^{\sigma}\Pi_{\mathbb{K}} := \sigma(\Pi_{\mathbb{K}}) \subset \mathrm{Ind}_{\mathrm{N}_n(\mathbb{K})}^{\mathrm{GL}_n(\mathbb{K})}\psi_{n,\mathbb{K}},$$

which is also a generic irreducible smooth representations of $\mathrm{GL}_n(\mathbb{K})$ with a fixed Whittaker functional (the evaluation map at the identity matrix).

Let $\chi_{\mathbb{K}} : \mathbb{K}^{\times} \to \mathbb{C}^{\times}$ be a character. Let $\mathfrak{c}(\chi_{\mathbb{K}})$ and $\mathfrak{c}(\psi_{\mathbb{K}})$ be the conductors of $\chi_{\mathbb{K}}$ and $\psi_{\mathbb{K}}$, which are ideal and fractional ideal of $\mathcal{O}_{\mathbb{K}}$, respectively. Here $\mathcal{O}_{\mathbb{K}}$ denotes the ring of integers of $\mathbb{K}$. Fix $y_{\mathbb{K}} \in \mathbb{K}^{\times}$ such that

$$\mathfrak{c}(\psi_{\mathbb{K}}) = y_{\mathbb{K}} \cdot \mathfrak{c}(\chi_{\mathbb{K}}).$$

The local Gauss sum is defined by

$$\mathcal{G}(\chi_{\mathbb{K}}) := \mathcal{G}(\chi_{\mathbb{K}}, \psi_{\mathbb{K}}, y_{\mathbb{K}}) := \int_{\mathcal{O}_{\mathbb{K}}^{\times}} \chi_{\mathbb{K}}(x)^{-1} \cdot \psi_{\mathbb{K}}(y_{\mathbb{K}} x)\, \mathrm{d}x, \tag{53}$$

where $\mathrm{d}x$ is the normalized Haar measure so that $\mathcal{O}_{\mathbb{K}}^{\times}$ has total volume 1. Note that $\mathcal{G}(\chi_{\mathbb{K}}) = 1$ when $\mathfrak{c}(\chi_{\mathbb{K}}) = \mathfrak{c}(\psi_{\mathbb{K}}) = \mathcal{O}_{\mathbb{K}}$. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, it is easily checked that $\mathfrak{c}({}^{\sigma}\chi_{\mathbb{K}}) = \mathfrak{c}(\chi_{\mathbb{K}})$, and

$$\mathcal{G}(\chi_{\mathbb{K}}, \psi_{\mathbb{K}}, y_{\mathbb{K}}) = {}^{\sigma}\chi_{\mathbb{K}}(t_{\sigma,\mathbb{K}}) \cdot \mathcal{G}({}^{\sigma}\chi_{\mathbb{K}}, \psi_{\mathbb{K}}, y_{\mathbb{K}}). \tag{54}$$

5.2 Non-archimedean period relation

Suppose that $n \geqslant 2$, and $\Sigma_{\mathbb{K}}$ is a generic irreducible smooth representation of $\mathrm{GL}_{n-1}(\mathbb{K})$ with a fixed Whittaker functional

$$\lambda'_{\mathbb{K}} \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{K})}(\Sigma_{\mathbb{K}}, \psi_{n-1,\mathbb{K}}) \setminus \{0\}.$$

As before, we use $\lambda'_{\mathbb{K}}$ to realize $\Sigma_{\mathbb{K}}$ as a subrepresentation of $\mathrm{Ind}_{\mathrm{N}_{n-1}(\mathbb{K})}^{\mathrm{GL}_{n-1}(\mathbb{K})}\psi_{n-1,\mathbb{K}}$, and we have a subrepresentation ${}^{\sigma}\Sigma_{\mathbb{K}} \subset \mathrm{Ind}_{\mathrm{N}_{n-1}(\mathbb{K})}^{\mathrm{GL}_{n-1}(\mathbb{K})}\psi_{n-1,\mathbb{K}}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

As in the archimedean case, denote by $\mathfrak{M}_{n-1,\mathbb{K}}$ the one-dimensional space of invariant measures on $\mathrm{GL}_{n-1}(\mathbb{K})$. Fix the Haar measure on $\mathrm{N}_{n-1}(\mathbb{K})$ to be the product of self-dual Haar measures on $\mathbb{K}$ with respect to $\psi_{\mathbb{K}}$, as in (16). Then each $m \in \mathfrak{M}_{n-1,\mathbb{K}}$ induces a quotient measure $\bar{m}$ on $\mathrm{N}_{n-1}(\mathbb{K})\backslash\mathrm{GL}_{n-1}(\mathbb{K})$.

Let $\chi_{\Sigma_{\mathbb{K}}}$ denote the central character of $\Sigma_{\mathbb{K}}$. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, it is clear that ${}^{\sigma}(\chi_{\Sigma_{\mathbb{K}}}) = \chi_{{}^{\sigma}\Sigma_{\mathbb{K}}}$. Similar to (54), we also have that

$$\mathcal{G}(\chi_{\Sigma_{\mathbb{K}}}, \psi_{\mathbb{K}}, y'_{\mathbb{K}}) = {}^{\sigma}\chi_{\mathbb{K}}(t_{\sigma,\mathbb{K}}) \cdot \mathcal{G}(\chi_{{}^{\sigma}\Sigma_{\mathbb{K}}}, \psi_{\mathbb{K}}, y'_{\mathbb{K}}), \tag{55}$$

where $y'_{\mathbb{K}} \in \mathbb{K}^{\times}$ satisfies that $\mathfrak{c}(\psi_{\mathbb{K}}) = y'_{\mathbb{K}} \cdot \mathfrak{c}(\chi_{\Sigma_{\mathbb{K}}})$.

We call an invariant measure m on $\mathrm{GL}_{n-1}(\mathbb{K})$ rational if $m(K) \in \mathbb{Q}$ for every open compact subgroup K of $\mathrm{GL}_{n-1}(\mathbb{K})$. All rational measures on $\mathrm{GL}_{n-1}(\mathbb{K})$ form a rational structure of $\mathfrak{M}_{n-1,\mathbb{K}}$. By using this rational structure, we get a σ-linear isomorphism $\sigma : \mathfrak{M}_{n-1,\mathbb{K}} \to \mathfrak{M}_{n-1,\mathbb{K}}$. By taking the tensor product of the above σ-linear isomorphism with the σ-linear isomorphisms as defined in (51), we get a σ-linear isomorphism

$$\sigma : \Pi_{\mathbb{K}} \otimes \Sigma_{\mathbb{K}} \otimes \chi_{\mathbb{K},s-1/2} \otimes \mathfrak{M}_{n-1,\mathbb{K}} \to {}^{\sigma}\Pi_{\mathbb{K}} \otimes {}^{\sigma}\Sigma_{\mathbb{K}} \otimes {}^{\sigma}(\chi_{\mathbb{K},s-1/2}) \otimes \mathfrak{M}_{n-1,\mathbb{K}}, \tag{56}$$

where $s \in \mathbb{C}$.

Similar to (31), we have the normalized Rankin–Selberg integrals

$$\mathrm{Z}^{\circ}(\cdot, s, \chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}(\Pi_{\mathbb{K}} \otimes \Sigma_{\mathbb{K}} \otimes \chi_{\mathbb{K},s-1/2} \otimes \mathfrak{M}_{n-1,\mathbb{K}}, \mathbb{C})$$

and

$$\mathrm{Z}^{\circ}(\cdot, s, {}^{\sigma}\chi_{\mathbb{K}}) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{K})}({}^{\sigma}\Pi_{\mathbb{K}} \otimes {}^{\sigma}\Sigma_{\mathbb{K}} \otimes ({}^{\sigma}\chi_{\mathbb{K}})_{s-1/2} \otimes \mathfrak{M}_{n-1,\mathbb{K}}, \mathbb{C}),$$

where $({}^{\sigma}\chi_{\mathbb{K}})_{s-1/2} := {}^{\sigma}\chi_{\mathbb{K}} \cdot |\cdot|_{\mathbb{K}}^{s-1/2}$, which equals ${}^{\sigma}(\chi_{\mathbb{K},s-1/2})$ when $s \in \frac{1}{2} + \mathbb{Z}$.

Following the idea of Harder [Har83, § III] (for $n = 2$), Mahnkopf [Mah05, § 3.4] and Raghuram [Rag10, § 3.3], we formulate the non-archimedean period relation as in the following proposition.

PROPOSITION 5.1. *For all $s_0 \in \frac{1}{2} + \mathbb{Z}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, the following diagram commutes.*

$$\begin{array}{ccc} \Pi_{\mathbb{K}} \otimes \Sigma_{\mathbb{K}} \otimes \chi_{\mathbb{K},s_0-\frac{1}{2}} \otimes \mathfrak{M}_{n-1,\mathbb{K}} & \xrightarrow{\mathcal{G}(\chi_{\Sigma_{\mathbb{K}}},\psi_{\mathbb{K}},y'_{\mathbb{K}})\cdot\mathcal{G}(\chi_{\mathbb{K}},\psi_{\mathbb{K}},y_{\mathbb{K}})^{\frac{n(n-1)}{2}}\cdot\mathrm{Z}^{\circ}(\cdot,s_0,\chi_{\mathbb{K}})} & \mathbb{C} \\ \sigma\downarrow & & \downarrow\sigma \\ {}^{\sigma}\Pi_{\mathbb{K}} \otimes {}^{\sigma}\Sigma_{\mathbb{K}} \otimes ({}^{\sigma}\chi_{\mathbb{K}})_{s_0-\frac{1}{2}} \otimes \mathfrak{M}_{n-1,\mathbb{K}} & \xrightarrow{\mathcal{G}(\chi_{{}^{\sigma}\Sigma_{\mathbb{K}}},\psi_{\mathbb{K}},y'_{\mathbb{K}})\cdot\mathcal{G}({}^{\sigma}\chi_{\mathbb{K}},\psi_{\mathbb{K}},y_{\mathbb{K}})^{\frac{n(n-1)}{2}}\cdot\mathrm{Z}^{\circ}(\cdot,s_0,{}^{\sigma}\chi_{\mathbb{K}})} & \mathbb{C} \end{array}$$

Proof. Note that

$$\mathrm{L}(s, \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}} \times \chi_{\mathbb{K}}) = P(q^{1/2-s})^{-1}$$

for a polynomial $P(X) \in \mathbb{C}[X]$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, denote by ${}^{\sigma}P(X) \in \mathbb{C}[X]$ the polynomial obtained by applying σ to the coefficients of the polynomial $P(X)$. Following the proof of [Clo90, Lemma 4.6], and by noting that the local Rankin–Selberg L-function does not depend on $\psi_{\mathbb{K}}$, it is easy to show that

$$\mathrm{L}(s, {}^{\sigma}\Pi_{\mathbb{K}} \times {}^{\sigma}\Sigma_{\mathbb{K}} \times {}^{\sigma}\chi_{\mathbb{K}}) = {}^{\sigma}P(q^{1/2-s})^{-1}.$$

Specifying s to $s_0 \in \frac{1}{2} + \mathbb{Z}$, we obtain that

$$\mathrm{L}(s_0, {}^{\sigma}\Pi_{\mathbb{K}} \times {}^{\sigma}\Sigma_{\mathbb{K}} \times {}^{\sigma}\chi_{\mathbb{K}}) = \sigma(\mathrm{L}(s_0, \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}} \times \chi_{\mathbb{K}})). \tag{57}$$

For $f \in \Pi_{\mathbb{K}}$, $f' \in \Sigma_{\mathbb{K}}$, $m \in \mathfrak{M}_{n-1,\mathbb{K}}$, and $s_0 \in \frac{1}{2} + \mathbb{Z}$ large enough, by (51) and (57) we have that

$$\begin{aligned} &\mathrm{Z}^{\circ}({}^{\sigma}f \otimes {}^{\sigma}f' \otimes {}^{\sigma}m, s_0, {}^{\sigma}\chi_{\mathbb{K}})) \\ &\quad = \frac{1}{\mathrm{L}(s_0, {}^{\sigma}\Pi_{\mathbb{K}} \times {}^{\sigma}\Sigma_{\mathbb{K}} \times {}^{\sigma}\chi_{\mathbb{K}})} \\ &\qquad \cdot \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} {}^{\sigma}f\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot {}^{\sigma}f'(g) \cdot {}^{\sigma}\chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s_0-1/2}\, \mathrm{d}\overline{{}^{\sigma}m}(g) \end{aligned}$$

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$$= \frac{1}{\sigma(\mathrm{L}(s_0, \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}} \times \chi_{\mathbb{K}}))} \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} \sigma\left(f\left(\mathbf{t}_{n,\sigma,\mathbb{K}} \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right)\right)$$

$$\cdot\, \sigma\big(f'(\mathbf{t}_{n-1,\sigma,\mathbb{K}} \cdot g)\big) \cdot {}^{\sigma}\chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s_0 - 1/2}\, \mathrm{d}\overline{{}^{\sigma}m}(g)$$

$$= \sigma\Bigg(\frac{1}{\mathrm{L}(s_0, \Pi_{\mathbb{K}} \times \Sigma_{\mathbb{K}} \times \chi_{\mathbb{K}})} \cdot \int_{\mathrm{N}_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} f\left(\begin{bmatrix} t_{\sigma,\mathbb{K}}^{-1} \mathbf{t}_{n-1,\sigma,\mathbb{K}}\, g & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\cdot\, f'(\mathbf{t}_{n-1,\sigma,\mathbb{K}} \cdot g) \chi_{\mathbb{K}}(\det g) \cdot |\det g|_{\mathbb{K}}^{s_0 - 1/2}\, \mathrm{d}\overline{m}(g) \Bigg)$$

$$= {}^{\sigma}\chi_{\Sigma_{\mathbb{K}}}(t_{\sigma,\mathbb{K}}) \cdot {}^{\sigma}\chi_{\mathbb{K}}(t_{\sigma,\mathbb{K}})^{n(n-1)/2} \cdot \sigma(\mathrm{Z}^{\circ}(f \otimes f' \otimes m, s_0, \chi_{\mathbb{K}})).$$

By [JPSS83, Theorem 2.7] the map $s \mapsto \mathrm{Z}^{\circ}(f \otimes f' \otimes m, s, \chi_{\mathbb{K}})$ is an element of the ring $\mathbb{C}[q^{s-1/2}, q^{1/2-s}]$. Therefore, the above equality holds for all $s_0 \in \frac{1}{2} + \mathbb{Z}$. Hence, by (54) and (55), the diagram in the proposition is commutative. □

6. Whittaker periods

Let k be a number field with adele ring $\mathbb{A}$ as in the introduction. In this section we define the Whittaker periods for irreducible subrepresentations Π of $\mathcal{A}^{\infty}(\mathrm{GL}_n(\mathrm{k})\backslash \mathrm{GL}_n(\mathbb{A}))$ which will be assumed to be *tamely isobaric* (see (63)) and regular algebraic.

6.1 Canonical generators of the cohomology spaces

Put $K_{n,\infty} := \prod_{v|\infty} K_{n,\mathrm{k}_v}$ $(n \geqslant 1)$, where K_{n,k_v} is the standard maximal compact subgroup of $\mathrm{GL}_n(\mathrm{k}_v)$ as in (23) for an archimedean place v of k. Define a one-dimensional real vector space

$$\omega_{n,\infty}(\mathbb{R}) := \wedge^{d_{n,\infty}}(\mathfrak{gl}_n(\mathrm{k}_{\infty})/\mathfrak{k}_{n,\infty}),$$

where

$$d_{n,\infty} := \sum_{v|\infty} d_{n,\mathrm{k}_v} = \dim_{\mathbb{R}}(\mathfrak{gl}_n(\mathrm{k}_{\infty})/\mathfrak{k}_{n,\infty}).$$

Put

$$\omega_{n,\infty} := \omega_{n,\infty}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Similar to (25) in the archimedean case, denote by $\mathfrak{O}_{n,\infty}$ the complex orientation space of $\omega_{n,\infty}$, and put

$$\widetilde{\mathfrak{O}}_{n,\infty} := \mathfrak{O}_{n-1,\infty} \otimes \cdots \otimes \mathfrak{O}_{1,\infty} \otimes \mathfrak{O}_{0,\infty}. \tag{58}$$

By convention, we set $\widetilde{\mathfrak{O}}_{0,\infty} := \mathfrak{O}_{0,\infty} := \mathbb{C}$. For any $m \geqslant 0$, we identify $\mathfrak{O}_{m,\infty} \otimes \mathfrak{O}_{m,\infty}$ with $\mathbb{C}$ in the obvious way. Then we have that

$$\widetilde{\mathfrak{O}}_{n,\infty} \otimes \widetilde{\mathfrak{O}}_{n-1,\infty} = \mathfrak{O}_{n-1,\infty}. \tag{59}$$

Let $\mu = \{\mu^{\iota}\}_{\iota \in \mathcal{E}_{\mathrm{k}}} \in (\mathbb{Z}^n)^{\mathcal{E}_{\mathrm{k}}}$ be a highest weight that is pure as in the introduction. For every archimedean place v of k, view $\mathcal{E}_{\mathrm{k}_v}$ as a subset of $\mathcal{E}_{\mathrm{k}}$ in the obvious way, and set

$$\mu_v := \{\mu^{\iota}\}_{\iota \in \mathcal{E}_{\mathrm{k}_v}} \in (\mathbb{Z}^n)^{\mathcal{E}_{\mathrm{k}_v}}. \tag{60}$$

Put

$$\Omega(\mu) := \left\{ \widehat{\otimes}_{v|\infty} \pi_{\mu_v} : \pi_{\mu_v} \in \Omega(\mu_v) \right\}$$

and

$$\mathcal{H}_{\mu} := \bigoplus_{\pi_{\mu} \in \Omega(\mu)} \mathcal{H}(\pi_{\mu}),$$

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where

$$\mathcal{H}(\pi_\mu) := \mathrm{H}_{\mathrm{ct}}^{b_{n,\infty}}(\mathbb{R}_+^\times\backslash \mathrm{GL}_n(\mathrm{k}_\infty)^0; F_\mu^\vee \otimes \pi_\mu) \otimes \widetilde{\mathfrak{O}}_{n,\infty}.$$

Here $b_{n,\infty} = \sum_{v|\infty} b_{n,\mathrm{k}_v}$ is as in the introduction, and $\mathbb{R}_+^\times$ is identified with a central subgroup of $\mathrm{GL}_n(\mathrm{k}_\infty)$ via the diagonal embedding.

Recall from the introduction that F_μ is an irreducible algebraic representation of $\mathrm{GL}_n(\mathrm{k} \otimes_{\mathbb{Q}} \mathbb{C})$ of highest weight μ. It has a decomposition

$$F_\mu = \otimes_{v|\infty} F_{\mu_v}.$$

For every archimedean place v of k, we have fixed a generator $v_{\mu_v} \in (F_{\mu_v})^{\mathrm{N}_n(\mathrm{k}_v \otimes_{\mathbb{R}} \mathbb{C})}$. This yields a generator

$$v_\mu := \otimes v_{\mu_v} \in (F_\mu)^{\mathrm{N}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}.$$

We remark that the representation F_μ is unique up to isomorphism, and the pair (F_μ, v_μ) is more rigid in the sense that it is unique up to a unique isomorphism. Also recall from (12) the Whittaker functional λ_{μ_v} on $\pi_{\mu_v} \in \Omega(\mu_v)$. By tensor product, this induces the Whittaker functional λ_μ on every $\pi_\mu \in \Omega(\mu)$.

Let $\varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)}$. Denote the ε-isotypic component of $\mathcal{H}_\mu$ by $\mathcal{H}_\mu[\varepsilon]$ (similar notation will be used without further explanation). Then $\mathcal{H}_\mu[\varepsilon]$ is one-dimensional. In what follows, we will define a canonical generator $\kappa_{\mu,\varepsilon}$ of $\mathcal{H}_\mu[\varepsilon]$, which is determined by the pairs (F_μ, v_μ) and (π_μ, λ_μ). Here we suppose that π_μ is the unique representation in $\Omega(\mu)$ such that $\mathcal{H}(\pi_\mu)[\varepsilon] \neq \{0\}$.

We first consider the case that $\mu = 0_{n,\infty}$, the zero weight. For $n = 1$, we naturally identify $\mathcal{H}_{0_{1,\infty}}[\varepsilon]$ with $\mathbb{C}$, and put $\kappa_{0_{1,\infty},\varepsilon} := 1$ under this identification.

For $n \geqslant 2$, fix

$$\pi_{0_{n,\infty}} = \widehat{\otimes}_{v|\infty} \pi_{0_{n,\mathrm{k}_v}} \in \Omega(0_{n,\infty}) \quad \text{and} \quad \pi_{0_{n-1,\infty}} = \widehat{\otimes}_{v|\infty} \pi_{0_{n-1,\mathrm{k}_v}} \in \Omega(0_{n-1,\infty}),$$

with fixed Whittaker functionals

$$\lambda_{0_{n,\infty}} \in \mathrm{Hom}_{\mathrm{N}_n(\mathrm{k}_\infty)}(\pi_{0_{n,\infty}}, \psi_{n,\infty}) \setminus \{0\}$$

and

$$\lambda_{0_{n-1,\infty}} \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathrm{k}_\infty)}(\pi_{0_{n-1,\infty}}, \psi_{n-1,\infty}) \setminus \{0\}.$$

Denote by $\mathfrak{M}_{n-1,\infty}$ the one-dimensional space of invariant measures on $\mathrm{GL}_{n-1}(\mathrm{k}_\infty)$. Similar to the local case at each archimedean place, we have an identification

$$\mathfrak{M}_{n-1,\infty} = \omega_{n-1,\infty}^* \otimes \mathfrak{O}_{n-1,\infty}$$

by push-forward of measures. Similar to (31), we have the normalized Rankin–Selberg integral

$$\mathrm{Z}^\circ(\cdot, s) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k}_\infty)}(\pi_{0_{n,\infty}} \widehat{\otimes} \pi_{0_{n-1,\infty}} \otimes |\det|_{\mathrm{k}_\infty}^{s-1/2} \otimes \mathfrak{M}_{n-1,\infty}, \mathbb{C}).$$

In view of (59), we define a map $\mathcal{P}_{\infty,0}$ to be the composition of

$$\begin{aligned}
\mathcal{P}_{\infty,0} : \mathcal{H}(\pi_{0_{n,\infty}}) \otimes \mathcal{H}(\pi_{0_{n-1,\infty}}) \\
&\to \mathrm{H}_{\mathrm{ct}}^{d_{n-1,\infty}}(\mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \pi_{0_{n,\infty}} \widehat{\otimes} \pi_{0_{n-1,\infty}}) \otimes \mathfrak{O}_{n-1,\infty} \\
&\to \mathrm{H}_{\mathrm{ct}}^{d_{n-1,\infty}}(\mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \mathfrak{M}_{n-1,\infty}^*) \otimes \mathfrak{O}_{n-1,\infty} = \mathbb{C},
\end{aligned}$$

where the first arrow is the restriction of cohomology composed with the cup product, and the last arrow is the map induced by the linear functional

$$\mathrm{Z}^\circ(\cdot, \tfrac{1}{2}) : \pi_{0_{n,\infty}} \widehat{\otimes} \pi_{0_{n-1,\infty}} \to \mathfrak{M}_{n-1,\infty}^*.$$

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By the non-vanishing hypothesis that is proved in [Sun17], these modular symbols for all $\pi_{0_{n,\infty}} \in \Omega(0_{n,\infty})$ and $\pi_{0_{n-1,\infty}} \in \Omega(0_{n-1,\infty})$ give the following non-degenerate pairings for all $\varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)}$, still denoted by

$$\mathcal{P}_{\infty,0} : \mathcal{H}_{0_{n,\infty}}[\varepsilon] \times \mathcal{H}_{0_{n-1,\infty}}[\varepsilon] \to \mathbb{C}.$$

We inductively define $\kappa_{0_{n,\infty},\varepsilon}$ by requiring that

$$\mathcal{P}_{\infty,0}(\kappa_{0_{n,\infty},\varepsilon}, \kappa_{0_{n-1,\infty},\varepsilon}) = 1.$$

In general, we define

$$\kappa_{\mu,\varepsilon} := \jmath_\mu(\kappa_{0_{n,\infty},\varepsilon}),$$

where

$$\jmath_\mu : \mathcal{H}_{0_{n,\infty}}[\varepsilon] \to \mathcal{H}_\mu[\varepsilon]$$

is the isomorphism induced by the local ones in Proposition 2.1.

6.2 Some actions of Aut($\mathbb{C}$)

Recall the additive character $\psi_\mathbb{R}$ from (9). Denote by $\mathbb{A}_\mathbb{Q}$ the adele ring of $\mathbb{Q}$. Fix a nontrivial additive character of $\mathrm{k}\backslash\mathbb{A}$ as the composition of

$$\psi : \mathrm{k}\backslash\mathbb{A} \xrightarrow{\mathrm{Tr}_{\mathrm{k}/\mathbb{Q}}} \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q} \to \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}/\widehat{\mathbb{Z}} = \mathbb{R}/\mathbb{Z} \xrightarrow{\psi_\mathbb{R}} \mathbb{C}^\times, \tag{61}$$

where $\mathrm{Tr}_{\mathrm{k}/\mathbb{Q}}$ is the trace map, and $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. Write $\psi = \otimes_v \psi_v$, where ψ_v is a character of k_v for each place v of k. By using ψ, we define the character ψ_n of $\mathrm{N}_n(\mathbb{A})$ as in (11) ($n \geqslant 1$). Then we have a decomposition $\psi_n = \psi_{n,f} \otimes \psi_{n,\infty}$, where $\psi_{n,f}$ and $\psi_{n,\infty}$ are characters of $\mathrm{N}_n(\mathbb{A}_f)$ and $\mathrm{N}_n(\mathrm{k}_\infty)$, respectively. Here $\mathbb{A}_f$ denotes the finite adele ring of k so that $\mathbb{A} = \mathbb{A}_f \times \mathrm{k}_\infty$.

For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, put

$$\mathbf{t}_{n,\sigma} := (\mathbf{t}_{n,\sigma,\mathrm{k}_v})_{v\nmid\infty} \in \mathrm{GL}_n(\mathbb{A}_f) \qquad \text{(see (50))},$$

and define an action of Aut($\mathbb{C}$) on $\mathrm{Ind}_{\mathrm{N}_n(\mathbb{A}_f)}^{\mathrm{GL}_n(\mathbb{A}_f)} \psi_{n,f}$ (the smooth induction) by

$${}^\sigma f(g) := \sigma\big(f(\mathbf{t}_{n,\sigma} \cdot g)\big), \tag{62}$$

where $f \in \mathrm{Ind}_{\mathrm{N}_n(\mathbb{A}_f)}^{\mathrm{GL}_n(\mathbb{A}_f)} \psi_{n,f}$ and $g \in \mathrm{GL}_n(\mathbb{A}_f)$.

Let Π_f be a generic irreducible smooth representation of $\mathrm{GL}_n(\mathbb{A}_f)$, with a fixed Whittaker functional

$$\lambda_f \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A}_f)}(\Pi_f, \psi_{n,f}) \setminus \{0\}.$$

As before, we use λ_f to realize Π_f as a subrepresentation of $\mathrm{Ind}_{\mathrm{N}_n(\mathbb{A}_f)}^{\mathrm{GL}_n(\mathbb{A}_f)} \psi_{n,f}$, namely the Whittaker model of Π_f. The rationality field of Π_f, denoted by $\mathbb{Q}(\Pi_f)$, is the fixed field of the group of field automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma\Pi_f := \sigma(\Pi_f) = \Pi_f$.

Let Π be an irreducible subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$. If Π is cuspidal, then the *exponent* of Π is defined to be the real number ex(Π) such that $\Pi \otimes |\det|_\mathbb{A}^{-\mathrm{ex}(\Pi)}$ is unitarizable, where $|\cdot|_\mathbb{A}$ is the normalized absolute value on $\mathbb{A}$.

We say that Π is *tamely isobaric* if

$$\Pi \cong \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\Pi_1 \widehat{\otimes}_\mathrm{i} \cdots \widehat{\otimes}_\mathrm{i} \Pi_r) \quad \text{(cf. (15))}, \tag{63}$$

for a standard parabolic subgroup P of GL_n with Levi subgroup $M_P = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, and irreducible cuspidal subrepresentations Π_i of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_{n_i}(\mathbb{A}))$, $i = 1, \ldots, r$, that have the same exponent. Here $\widehat{\otimes}_{\mathrm{i}}$ denotes the completed inductive tensor product (see [Trè67, Definition 43.5]). We can view the right-hand side of (63) as a space of smooth automorphic forms by using the Eisenstein series (see [Lan79, Proposition 2] and [GL21, § 1.4.3] for more details).

Suppose that Π_f and Π_∞ are the finite and infinite part of Π, respectively, so that $\Pi = \Pi_f \otimes \Pi_\infty$. Now we assume that Π is tamely isobaric and regular algebraic (in the sense of [Clo90] such that (1) holds). By the proof of [Gro18, Lemma 1.2] (see also [GL21, § 1.4.3]), for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma\Pi_f := \sigma(\Pi_f)$ given by (62) is the finite part of a unique irreducible subrepresentation ${}^\sigma\Pi$ of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$. Moreover, ${}^\sigma\Pi$ is also tamely isobaric and regular algebraic.

Remark 6.1. More precisely, the above assertion holds when Π is cuspidal and regular algebraic by [Clo90, Theorem 3.13]. In general, if Π is tamely isobaric as in (63) and is regular algebraic, then

$$\Xi_1\widehat{\otimes}_{\mathrm{i}}\cdots\widehat{\otimes}_{\mathrm{i}}\Xi_r := (\Pi_1\widehat{\otimes}_{\mathrm{i}}\cdots\widehat{\otimes}_{\mathrm{i}}\Pi_r)\otimes\rho_P$$

is a regular algebraic irreducible cuspidal subrepresentation of $\mathcal{A}^\infty(M_P(\mathrm{k})\backslash M_P(\mathbb{A}))$, where $\rho_P := \delta_P^{1/2}$ is the square root of the modular character δ_P of $P(\mathbb{A})$. Then we have that

$${}^\sigma\Pi \cong \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}\left({}^\sigma\Xi_1\widehat{\otimes}_{\mathrm{i}}\cdots\widehat{\otimes}_{\mathrm{i}}{}^\sigma\Xi_r\right)\otimes\rho_P^{-1}.$$

Recall that the rationality field of Π is defined to be $\mathbb{Q}(\Pi) := \mathbb{Q}(\Pi_f)$. Let $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$ act on Π_f by (62). It is known that $\mathbb{Q}(\Pi)$ is a number field and $(\Pi_f)^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$ is a $\mathbb{Q}(\Pi)$-rational structure of Π_f (see [RS08b, Lemma 3.2]).

As in the introduction, suppose that F_μ is an irreducible algebraic representation of $\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})$ whose highest weight $\mu = \{\mu^\iota\}_{\iota\in\mathcal{E}_{\mathrm{k}}} \in (\mathbb{Z}^n)^{\mathcal{E}_{\mathrm{k}}}$ is pure of weight $w_\mu \in \mathbb{Z}$ so that

$$\mu_1^\iota + \mu_n^{\bar\iota} = \cdots = \mu_n^\iota + \mu_1^{\bar\iota} = w_\mu \quad \text{for all } \iota \in \mathcal{E}_{\mathrm{k}}.$$

Similar to the local case (8), we realize F_μ as the algebraic induction

$$F_\mu = {}^{\mathrm{alg}}\mathrm{Ind}_{\bar{\mathrm{B}}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}^{\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}\chi_\mu,$$

and realize $v_\mu \in F_\mu$ as the $\mathrm{N}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})$-invariant function that has value 1 at the identity matrix. Then the generator $v_\mu^\vee \in (F_\mu^\vee)^{\bar{\mathrm{N}}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}$ is identified with the evaluation map at the identity matrix. Similarly, $F_\mu^\vee$ is realized as the algebraic induction

$$F_\mu^\vee = {}^{\mathrm{alg}}\mathrm{Ind}_{\mathrm{B}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}^{\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})}\chi_{-\mu}.$$

For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, write

$${}^\sigma\mu = \{\mu^{\sigma^{-1}\circ\iota}\}_{\iota\in\mathcal{E}_{\mathrm{k}}}.$$

As a consequence of the purity lemma [Clo90, Lemma 4.9], μ necessarily satisfies the condition that (see [Gro18, Lemma 1.3])

$$\mu^{\sigma\circ\bar\iota} = \mu^{\overline{\sigma\circ\iota}} \quad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}) \text{ and } \iota \in \mathcal{E}_{\mathrm{k}}.$$

Therefore, ${}^\sigma\mu$ is also pure of weight w_μ. The rationality field $\mathbb{Q}(F_\mu)$ is defined to be the fixed field of the group of field automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that ${}^\sigma\mu = \mu$.

Let $\mathrm{Aut}(\mathbb{C})$ act on the space of algebraic functions on $\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})$ by

$$({}^\sigma f)(x) := \sigma(f(\sigma^{-1}x)) \quad (x \in \mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})), \tag{64}$$

where $\mathrm{Aut}(\mathbb{C})$ acts on $\mathrm{GL}_n(\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C})$ through its action on the second factor of $\mathrm{k}\otimes_{\mathbb{Q}}\mathbb{C}$. Then

$$\sigma(F_\mu) = F_{{}^\sigma\mu} \quad \text{and} \quad \sigma(F_\mu^\vee) = F_{{}^\sigma\mu}^\vee.$$

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Define

$$\mathcal{X}_n := (\mathbb{R}_+^\times \cdot \mathrm{GL}_n(\mathrm{k}))\backslash \mathrm{GL}_n(\mathbb{A})/K_{n,\infty}^0. \tag{65}$$

For every open compact subgroup K_f of $\mathrm{GL}_n(\mathbb{A}_f)$, the finite-dimensional representation $F_\mu^\vee$ defines a sheaf on $\mathcal{X}_n/K_f$, which is still denoted by $F_\mu^\vee$. Let $\mathrm{H}^{b_{n,\infty}}(\mathcal{X}_n/K_f, F_\mu^\vee)$ be the sheaf cohomology group, and define

$$\begin{aligned}\mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) &:= \mathrm{H}^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \otimes \widetilde{\mathfrak{O}}_{n,\infty} \\ &:= \varinjlim_{K_f} \mathrm{H}^{b_{n,\infty}}(\mathcal{X}_n/K_f, F_\mu^\vee) \otimes \widetilde{\mathfrak{O}}_{n,\infty},\end{aligned} \tag{66}$$

where K_f runs over the directed system of open compact subgroups of $\mathrm{GL}_n(\mathbb{A}_f)$.

Note that $\widetilde{\mathfrak{O}}_{n,\infty}$ has a natural $\mathbb{Q}$-structure. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the map (64) induces a σ-linear map

$$\sigma : \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \to \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F_{\sigma\mu}^\vee). \tag{67}$$

Put

$$\mathrm{GL}_n(\mathbb{A})^\natural := \mathrm{GL}_n(\mathbb{A}_f) \times \pi_0(\mathrm{k}_\infty^\times).$$

Then both the domain and codomain of the map (67) are naturally smooth representations of $\mathrm{GL}_n(\mathbb{A})^\natural$, and the map (67) is $\mathrm{GL}_n(\mathbb{A})^\natural$-equivariant.

6.3 Definition of the Whittaker periods

As above, Π is an irreducible subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash \mathrm{GL}_n(\mathbb{A}))$ which is assumed to be tamely isobaric and regular algebraic. Fix the Haar measure on $\mathrm{N}_n(\mathbb{A})$ to be the product of self-dual Haar measures on $\mathbb{A}$ with respect to ψ, as in (16). Then we have a nonzero continuous linear functional

$$\lambda \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A})}(\Pi, \psi_n), \quad \varphi \mapsto \int_{\mathrm{N}_n(\mathrm{k})\backslash \mathrm{N}_n(\mathbb{A})} \varphi(u) \cdot \overline{\psi_n}(u)\, \mathrm{d}u. \tag{68}$$

By the uniqueness of Whittaker models, we have a factorization

$$\lambda = \lambda_f \otimes \lambda_\infty, \tag{69}$$

where

$$\lambda_f \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A}_f)}(\Pi_f, \psi_{n,f})$$

as before, and

$$\lambda_\infty \in \mathrm{Hom}_{\mathrm{N}_n(\mathrm{k}_\infty)}(\Pi_\infty, \psi_{n,\infty}).$$

More generally, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, let ${}^\sigma\lambda \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A})}({}^\sigma\Pi, \psi_n)$ be the Whittaker functional defined by the integrals as in (68). Similar to (69), we also have factorizations

$${}^\sigma\Pi = {}^\sigma\Pi_f \otimes {}^\sigma\Pi_\infty \quad \text{and} \quad {}^\sigma\lambda = {}^\sigma\lambda_f \otimes {}^\sigma\lambda_\infty.$$

Recall that ${}^\sigma\Pi_f := \sigma(\Pi_f)$ is realized as a space of Whittaker functions so that ${}^\sigma\lambda_f$ is realized as the evaluation map at the identity matrix.

Suppose that F_μ is the coefficient system of Π as in the introduction. Then $F_{\sigma\mu}$ is the coefficient system of ${}^\sigma\Pi$ (cf. [Clo90, Theorem 3.13] and [Gro18, Corollary 1.4]) so that

$$\mathcal{H}({}^\sigma\Pi_\infty) := \mathrm{H}_{\mathrm{ct}}^{b_{n,\infty}}(\mathbb{R}_+^\times\backslash \mathrm{GL}_n(\mathrm{k}_\infty)^0; F_{\sigma\mu}^\vee \otimes {}^\sigma\Pi_\infty) \otimes \widetilde{\mathfrak{O}}_{n,\infty} \neq \{0\}.$$

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Consequently, $\mathbb{Q}(F_\mu) \subset \mathbb{Q}(\Pi)$. Put

$$\mathcal{H}({}^\sigma\Pi) := \mathrm{H}^{b_{n,\infty}}_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash \mathrm{GL}_n(\mathrm{k}_\infty)^0; F^\vee_{{}^\sigma\mu} \otimes {}^\sigma\Pi) \otimes \widetilde{\mathfrak{O}}_{n,\infty}.$$

Then we have the canonical isomorphism

$$\iota_{\mathrm{can}} : {}^\sigma\Pi_f \otimes \mathcal{H}({}^\sigma\Pi_\infty) \to \mathcal{H}({}^\sigma\Pi).$$

Following [Clo90, Lemma 3.15] and [Gro18, Proposition 1.6], we have a $\mathrm{GL}_n(\mathbb{A})^\natural$-equivariant embedding

$$\iota_\Pi : \Pi_f \otimes \mathcal{H}(\Pi_\infty) = \mathcal{H}(\Pi) \hookrightarrow \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F^\vee_\mu). \tag{70}$$

Let $\varepsilon \in \widehat{\pi_0(\mathrm{k}^\times_\infty)}$ be the character $\varepsilon_{\Pi_\infty} \cdot \mathrm{sgn}_\infty^{(n-1)(n-2)/2}$ when n is odd, and be arbitrary when n is even. Then we have a $\mathrm{GL}_n(\mathbb{A})^\natural$-equivariant linear embedding

$$\iota_{\Pi,\varepsilon} : \Pi_f \otimes \mathcal{H}(\Pi_\infty)[\varepsilon] = \mathcal{H}(\Pi)[\varepsilon] \hookrightarrow \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F^\vee_\mu).$$

Proposition 6.2. *Let the notation and assumptions be as above. Then*

$$\dim_{\mathrm{GL}_n(\mathbb{A})^\natural}(\mathcal{H}(\Pi)[\varepsilon], \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F^\vee_\mu)) = 1,$$

and for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ the map (67) induces the following commutative diagram.

$$\begin{array}{ccc} \mathcal{H}(\Pi)[\varepsilon] & \xrightarrow{\iota_{\Pi,\varepsilon}} & \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F^\vee_\mu) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathcal{H}({}^\sigma\Pi)[\varepsilon] & \xrightarrow{\iota_{{}^\sigma\Pi,\varepsilon}} & \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F^\vee_{{}^\sigma\mu}) \end{array}$$

Moreover, under the action given by the left vertical arrow of the above diagram, $(\mathcal{H}(\Pi)[\varepsilon])^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$ is a $\mathbb{Q}(\Pi)$-rational structure of $\mathcal{H}(\Pi)[\varepsilon]$.

Proof. The commutative diagram follows from [GL21, Propositions 1.19 and 1.21], and the fact that the map (67) commutes with the actions of $\pi_0(\mathrm{k}^\times_\infty) \cong \pi_0(K_{n,\infty})$. The last assertion is implied by Drinfeld–Manin principle (see [Clo90, Proposition 3.16]) and [Clo90, Lemma 3.2.1]. □

Remark 6.3. It follows from Proposition 6.2 that for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the central character of $F^\vee_{{}^\sigma\mu} \otimes ({}^\sigma\Pi)_\infty$ equals that of $F^\vee_\mu \otimes \Pi_\infty$. Consequently, $({}^\sigma\Pi)_\infty$ is uniquely determined by σ and Π_∞. Specifying to the case that $n = 1$, we know that the infinite part of ${}^\sigma\chi$ equals that of χ, for every finite-order Hecke character $\chi : \mathrm{k}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$.

We equip $\mathcal{H}(\Pi)[\varepsilon]$ with the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$ given by Proposition 6.2. Write

$$\Pi^\natural := \Pi_f \otimes \varepsilon \cong \mathcal{H}(\Pi)[\varepsilon],$$

where $\varepsilon \in \widehat{\pi_0(\mathrm{k}^\times_\infty)}$ is identified with $\mathbb{C}$ as a vector space. We equip $\Pi^\natural$ with the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$ given by its action on Π_f as in (62) and its natural action on $\mathbb{C}$.

Lemma 6.4. *There exists a generator*

$$\omega_{\Pi^\natural} \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}(\Pi^\natural, \mathcal{H}(\Pi)[\varepsilon])$$

that is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$-equivariant. Moreover, such a generator is unique up to multiplication by scalar in $\mathbb{Q}(\Pi)^\times$.

Proof. Recall that $(\Pi^\natural)^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$ is a $\mathbb{Q}(\Pi)$-rational structure of $\Pi^\natural$ (see [RS08b, Lemma 3.2]), and $(\mathcal{H}(\Pi)[\varepsilon])^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$ is a $\mathbb{Q}(\Pi)$-rational structure of $\mathcal{H}(\Pi)[\varepsilon]$ (Proposition 6.2). Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$. By the multiplicity one property of new vectors, the $\overline{\mathbb{Q}}$-rational structure of Π_f is unique up to homotheties (see the proof of [Clo90, Theorem 3.13], and [Wal85, Chapter I]). It follows that $\Pi^\natural$ and $\mathcal{H}(\Pi)[\varepsilon]$ are isomorphic over $\overline{\mathbb{Q}}$. Since $(\Pi^\natural)^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}$ is irreducible (as a smooth representation of $\mathrm{GL}_n(\mathbb{A})^\natural$ over $\overline{\mathbb{Q}}$), $\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\Pi))$ acts continuously on the one-dimensional $\overline{\mathbb{Q}}$-vector space (with the discrete topology)

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}\big((\Pi^\natural)^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}, (\mathcal{H}(\Pi)[\varepsilon])^{\mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})}\big).$$

This implies the existence of $\omega_{\Pi^\natural}$ by [Spr98, Proposition 11.1.6]. The uniqueness is obvious. □

Fix $\omega_{\Pi^\natural}$ as in Lemma 6.4. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, put

$${}^\sigma\Pi^\natural := {}^\sigma\Pi_f \otimes \varepsilon.$$

The σ-linear isomorphisms $\sigma : \Pi^\natural \to {}^\sigma\Pi^\natural$ and

$$\sigma : \mathcal{H}(\Pi)[\varepsilon] \to \mathcal{H}({}^\sigma\Pi)[\varepsilon] \quad \text{(see Proposition 6.2)}$$

induce a σ-linear isomorphism

$$\sigma : \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}(\Pi^\natural, \mathcal{H}(\Pi)[\varepsilon]) \to \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}({}^\sigma\Pi^\natural, \mathcal{H}({}^\sigma\Pi)[\varepsilon]).$$

Using this isomorphism, we define

$$\omega_{{}^\sigma\Pi^\natural} := \sigma(\omega_{\Pi^\natural}) \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}({}^\sigma\Pi^\natural, \mathcal{H}({}^\sigma\Pi)[\varepsilon]).$$

Unraveling definitions, we have the following commutative diagram.

$$\begin{array}{ccc} \Pi^\natural & \xrightarrow{\omega_{\Pi^\natural}} & \mathcal{H}(\Pi)[\varepsilon] \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ {}^\sigma\Pi^\natural & \xrightarrow{\omega_{{}^\sigma\Pi^\natural}} & \mathcal{H}({}^\sigma\Pi)[\varepsilon] \end{array} \tag{71}$$

Recall form §6.1 that the pairs (F_μ, v_μ) and $(\Pi_\infty, \lambda_\infty)$ determine a generator $\kappa_{\mu,\varepsilon}$ of $\mathcal{H}(\Pi_\infty)[\varepsilon] = \mathcal{H}_\mu[\varepsilon]$. More generally, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the pairs $(F_{{}^\sigma\mu}, v_{{}^\sigma\mu})$ and $({}^\sigma\Pi_\infty, {}^\sigma\lambda_\infty)$ determine a generator $\kappa_{{}^\sigma\mu,\varepsilon}$ of $\mathcal{H}({}^\sigma\Pi_\infty)[\varepsilon] = \mathcal{H}_{{}^\sigma\mu}[\varepsilon]$.

DEFINITION 6.5. For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the Whittaker period $\Omega_\varepsilon({}^\sigma\Pi) \in \mathbb{C}^\times$ is the unique scalar such that the following diagram commutes.

$$\begin{array}{ccc} {}^\sigma\Pi_f \otimes \varepsilon & \xrightarrow{\mathrm{id}\otimes\kappa_{{}^\sigma\mu,\varepsilon}} & {}^\sigma\Pi_f \otimes \mathcal{H}({}^\sigma\Pi_\infty)[\varepsilon] \\ \Omega_\varepsilon({}^\sigma\Pi)\big\downarrow & & \big\downarrow\iota_{\mathrm{can}} \\ {}^\sigma\Pi_f \otimes \varepsilon & \xrightarrow{\omega_{{}^\sigma\Pi^\natural}} & \mathcal{H}({}^\sigma\Pi)[\varepsilon] \end{array} \tag{72}$$

Up to scalar multiplication by $\mathbb{Q}(\Pi)^\times$, the Whittaker periods defined above are independent of the choice of the generator $\omega_{\Pi^\natural}$. More precisely, we have the following lemma.

LEMMA 6.6. *Let $c \in \mathbb{Q}(\Pi)^\times$ so that $\omega'_{\Pi^\natural} := c \cdot \omega_{\Pi^\natural} \in \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{A})^\natural}(\Pi^\natural, \mathcal{H}(\Pi)[\varepsilon])$ is another generator that is $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))$-equivariant, which defines a corresponding family of Whittaker periods $\{\Omega'_\varepsilon({}^\sigma\Pi)\}_{\sigma\in\mathrm{Aut}(\mathbb{C})}$. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{\Omega'_\varepsilon(\Pi)}{\Omega_\varepsilon(\Pi)}\right) = \frac{\Omega'_\varepsilon({}^\sigma\Pi)}{\Omega_\varepsilon({}^\sigma\Pi)} = \sigma(c^{-1}).$$

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Proof. This is an easy consequence of the commutative diagrams (71) and (72). □

For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define a σ-linear map

$$\sigma : \mathcal{H}(\Pi_\infty)[\varepsilon] \to \mathcal{H}({}^\sigma\Pi_\infty)[\varepsilon] \tag{73}$$

such that

$$\sigma(\kappa_{\mu,\varepsilon}) = \kappa_{{}^\sigma\mu,\varepsilon}.$$

Proposition 6.7. *For all* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *the diagram*

$$\begin{array}{ccc} \Pi_f \otimes \mathcal{H}(\Pi_\infty)[\varepsilon] & \xrightarrow{\Omega_\varepsilon(\Pi)^{-1}\cdot\iota_{\mathrm{can}}} & \mathcal{H}(\Pi)[\varepsilon] \\ \sigma\downarrow & & \downarrow\sigma \\ {}^\sigma\Pi_f \otimes \mathcal{H}({}^\sigma\Pi_\infty)[\varepsilon] & \xrightarrow{\Omega_\varepsilon({}^\sigma\Pi)^{-1}\cdot\iota_{\mathrm{can}}} & \mathcal{H}({}^\sigma\Pi)[\varepsilon] \end{array} \tag{74}$$

commutes, where the left vertical arrow is the σ*-linear map induced by the map* $\sigma : \Pi_f \to {}^\sigma\Pi_f$ *and the map* (73).

Proof. This follows easily from (71) and (72). □

7. Modular symbols and proof of Theorem 1.2

7.1 Rankin–Selberg integrals

In this subsection, let Π be an irreducible cuspidal subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_n(\mathrm{k})\backslash\mathrm{GL}_n(\mathbb{A}))$ $(n \geqslant 2)$, and let Σ be an irreducible tamely isobaric subrepresentation of $\mathcal{A}^\infty(\mathrm{GL}_{n-1}(\mathrm{k})\backslash \mathrm{GL}_{n-1}(\mathbb{A}))$.

As in (68), we have Whittaker functionals

$$\lambda \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A})}(\Pi, \psi_n) \quad \text{and} \quad \lambda' \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{A})}(\Sigma, \psi_{n-1})$$

defined by integrals, and as in (69), we have decompositions

$$\lambda = \lambda_f \otimes \lambda_\infty \quad \text{and} \quad \lambda' = \lambda'_f \otimes \lambda'_\infty,$$

with

$$\lambda_f \in \mathrm{Hom}_{\mathrm{N}_n(\mathbb{A}_f)}(\Pi_f, \psi_{n,f}), \quad \lambda_\infty \in \mathrm{Hom}_{\mathrm{N}_n(\mathrm{k}_\infty)}(\Pi_\infty, \psi_{n,\infty}), \quad \Pi = \Pi_f \otimes \Pi_\infty,$$

and

$$\lambda'_f \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathbb{A}_f)}(\Sigma_f, \psi_{n-1,f}), \quad \lambda'_\infty \in \mathrm{Hom}_{\mathrm{N}_{n-1}(\mathrm{k}_\infty)}(\Sigma_\infty, \psi_{n-1,\infty}), \quad \Sigma = \Sigma_f \otimes \Sigma_\infty.$$

Let $\chi : \mathrm{k}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a Hecke character. Similar to (30), for each $t \in \mathbb{C}$ define a character

$$\chi_t := \chi \cdot |\cdot|_{\mathbb{A}}^t : \mathbb{A}^\times \to \mathbb{C}^\times. \tag{75}$$

As usual, write

$$\chi = \otimes_v \chi_v = \chi_f \otimes \chi_\infty \quad \text{and} \quad \chi_t = \chi_{f,t} \otimes \chi_{\infty,t}.$$

Denote by $\mathfrak{M}_{n-1}$ and $\mathfrak{M}_{n-1,f}$ the one-dimensional spaces of invariant measures on $\mathrm{GL}_{n-1}(\mathbb{A})$ and $\mathrm{GL}_{n-1}(\mathbb{A}_f)$, respectively, so that

$$\mathfrak{M}_{n-1} = \mathfrak{M}_{n-1,f} \otimes \mathfrak{M}_{n-1,\infty}.$$

Similar to (31), we have the finite part of the normalized Rankin–Selberg integral

$$\mathrm{Z}^\circ(\cdot, s, \chi_f) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{A}_f)}(\Pi_f \otimes \Sigma_f \otimes \chi_{f,s-1/2} \otimes \mathfrak{M}_{n-1,f}, \mathbb{C}),$$

and the normalized Rankin–Selberg integral at infinity

$$\mathrm{Z}^\circ(\cdot, s, \chi_\infty) \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k}_\infty)}(\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty, s-1/2} \otimes \mathfrak{M}_{n-1,\infty}, \mathbb{C}).$$

Define the global Rankin–Selberg period integral

$$\mathrm{Z}(\cdot, s, \chi) : \Pi \widehat{\otimes} \Sigma \otimes \chi_{s-1/2} \otimes \mathfrak{M}_{n-1} \to \mathbb{C}$$

by

$$\mathrm{Z}(\varphi \otimes \varphi' \otimes 1 \otimes m) := \int_{\mathrm{GL}_{n-1}(\mathrm{k})\backslash \mathrm{GL}_{n-1}(\mathbb{A})} \varphi\left(\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right) \cdot \varphi'(g) \cdot \chi(g) \cdot |\det g|_{\mathbb{A}}^{s-1/2} \, \mathrm{d}\bar{m}(g),$$

where $\varphi \in \Pi$, $\varphi' \in \Sigma$, $m \in \mathfrak{M}_{n-1}$ and $\bar{m}$ is the quotient measure of m. Here

$$\Pi \widehat{\otimes} \Sigma := (\Pi_\infty \widehat{\otimes} \Sigma_\infty) \otimes (\Pi_f \otimes \Sigma_f).$$

The following proposition reformulates the Euler factorization of Rankin–Selberg period integrals established in [JS81b, p. 796, (7)].

PROPOSITION 7.1. *For Π, Σ, χ as above, and all $s \in \mathbb{C}$, the following diagram commutes.*

$$\begin{array}{ccc} (\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty, s-\frac{1}{2}} \otimes \mathfrak{M}_{n-1,\infty}) \otimes (\Pi_f \otimes \Sigma_f \otimes \chi_{f, s-\frac{1}{2}} \otimes \mathfrak{M}_{n-1,f}) & \xrightarrow{\mathrm{Z}^\circ(\cdot, s, \chi_\infty) \otimes \mathrm{Z}^\circ(\cdot, s, \chi_f)} & \mathbb{C} \\ {=}\big\downarrow & & \big\downarrow \mathrm{L}(s, \Pi \times \Sigma \times \chi) \\ \Pi \widehat{\otimes} \Sigma \otimes \chi_{s-\frac{1}{2}} \otimes \mathfrak{M}_{n-1} & \xrightarrow{\mathrm{Z}(\cdot, s, \chi)} & \mathbb{C} \end{array}$$

7.2 Modular symbols and modular symbols at infinity

From now on, further assume that Π and Σ are regular algebraic with balanced coefficient systems F_μ and F_ν, respectively, and assume that χ has finite order.

Similar to (66) we have the following space given by sheaf cohomology with compact support:

$$\begin{aligned} \mathcal{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) &:= \mathrm{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \otimes \widetilde{\mathfrak{O}}_{n,\infty} \\ &:= \varinjlim_{K_f} \mathrm{H}_c^{b_{n,\infty}}(\mathcal{X}_n/K_f, F_\mu^\vee) \otimes \widetilde{\mathfrak{O}}_{n,\infty}, \end{aligned} \tag{76}$$

where K_f runs over the directed system of open compact subgroups of $\mathrm{GL}_n(\mathbb{A}_f)$. This is also a smooth representation of $\mathrm{GL}_n(\mathbb{A})^\natural$ and we have a natural $\mathrm{GL}_n(\mathbb{A})^\natural$-equivariant linear map

$$\iota_\mu : \mathcal{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \to \mathcal{H}^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee).$$

Since Π is cuspidal, there is a natural embedding

$$\iota'_\Pi : \mathcal{H}(\Pi) \hookrightarrow \mathcal{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee)$$

such that $\iota_\mu \circ \iota'_\Pi = \iota_\Pi$ (see [Clo90, Lemma 3.15]).

Put

$$\widetilde{\mathcal{X}}_{n-1} := \mathrm{GL}_{n-1}(\mathrm{k})\backslash \mathrm{GL}_{n-1}(\mathbb{A})/K^0_{n-1,\infty}.$$

The embedding $\imath : \mathrm{GL}_{n-1}(\mathbb{A}) \hookrightarrow \mathrm{GL}_n(\mathbb{A})$ given by (26) induces a proper map, still denoted by

$$\imath : \widetilde{\mathcal{X}}_{n-1} \to \mathcal{X}_n,$$

which induces a map

$$\imath^* : \mathcal{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \to \mathcal{H}_c^{b_{n,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\mu^\vee).$$

The natural map $\wp : \widetilde{\mathcal{X}}_{n-1} \to \mathcal{X}_{n-1}$ induces a map

$$\wp^* : \mathcal{H}^{b_{n-1,\infty}}(\mathcal{X}_{n-1}, F_\nu^\vee) \to \mathcal{H}^{b_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\nu^\vee).$$

Since $\xi := (\mu, \nu)$ is assumed to be balanced, by [Rag16, Theorem 2.21] (see also [KS13, Theorem 2.3] and [GH16, Lemma 4.7]) we have that

$$\{j \in \mathbb{Z} : j \text{ is balanced for } \xi\} = \left\{j \in \mathbb{Z} : \tfrac{1}{2} + j \text{ is a critical place of } \Pi \times \Sigma\right\}.$$

Recall that a half-integer $\frac{1}{2} + j$ is a critical place of $\Pi \times \Sigma$ if it is a pole of neither $\mathrm{L}(s, \Pi_\infty \times \Sigma_\infty)$ nor $\mathrm{L}(1 - s, \Pi_\infty^\vee \times \Sigma_\infty^\vee)$.

Let j be a balanced place for ξ. Define an algebraic character

$$\delta_j := \otimes_{\iota \in \mathcal{E}_{\mathrm{k}}} \det{}^j$$

of $\mathrm{GL}_{n-1}(\mathrm{k} \otimes_{\mathbb{Q}} \mathbb{C})$. Put

$$\mathrm{H}(\chi_j) := \mathrm{H}^0_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash \mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \delta_j^\vee \otimes \chi_j) \quad \text{(see (75) for the definition of } \chi_j).$$

Then we have a natural injective map

$$\iota_j : \mathrm{H}(\chi_j) \hookrightarrow \mathrm{H}^0(\mathcal{X}_{n-1}, \delta_j^\vee).$$

With the notation as before, we have the generators

$$v_\mu^\vee := \otimes_{\iota \in \mathcal{E}_{\mathrm{k}}} v_{\mu^\iota}^\vee \in (F_\mu^\vee)^{\bar{\mathrm{N}}_n(\mathrm{k} \otimes_{\mathbb{Q}} \mathbb{C})} \quad \text{and} \quad v_\nu^\vee := \otimes_{\iota \in \mathcal{E}_{\mathrm{k}}} v_{\nu^\iota}^\vee \in (F_\nu^\vee)^{\bar{\mathrm{N}}_n(\mathrm{k} \otimes_{\mathbb{Q}} \mathbb{C})}.$$

Put $F_\xi := F_\mu \otimes F_\nu$ and $v_\xi^\vee := v_\mu^\vee \otimes v_\nu^\vee$. Recall from (37) the element

$$z = (z_n, z_{n-1}) \in \mathrm{GL}_n(\mathbb{Z}) \times \mathrm{GL}_{n-1}(\mathbb{Z}) \subset \mathrm{GL}_n(\mathrm{k}) \times \mathrm{GL}_{n-1}(\mathrm{k}).$$

By Proposition 3.1, we have a unique element

$$\phi_{\xi,j} \in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k} \otimes_{\mathbb{R}} \mathbb{C})}(F_\xi^\vee \otimes \delta_j^\vee, \mathbb{C})$$

such that $\phi_{\xi,j}\big(z.v_\xi^\vee \otimes 1\big) = 1$. Then $\phi_{\xi,j}$ induces a linear map

$$\phi_{\xi,j} : \mathrm{H}_c^{d_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\xi^\vee) \otimes \mathrm{H}^0(\widetilde{\mathcal{X}}_{n-1}, \delta_j^\vee) \to \mathrm{H}_c^{d_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, \mathbb{C}).$$

Put

$$\mathfrak{M}_{n-1}^\natural := \mathfrak{M}_{n-1,f} \otimes \mathfrak{O}_{n-1,\infty}.$$

Note that $\widetilde{\mathcal{X}}_{n-1}/K_f$ is an orientable manifold when K_f is a sufficiently small open compact subgroup of $\mathrm{GL}_{n-1}(\mathbb{A}_f)$, and pairing with the fundamental class yields a linear map

$$\int_{\widetilde{\mathcal{X}}_{n-1}} : \mathrm{H}_c^{d_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, \mathbb{C}) \otimes \mathfrak{M}_{n-1}^\natural \to \mathbb{C}.$$

See [Mah05, § 5.1] for more explanations.

In view of (59), define the modular symbol $\mathcal{P}_j$ to be the composition of

$$\begin{aligned}
&\mathcal{P}_j : \mathcal{H}(\Pi) \otimes \mathcal{H}(\Sigma) \otimes \mathrm{H}(\chi_j) \otimes \mathfrak{M}_{n-1,f} \\
&\xrightarrow{\iota'_\Pi \otimes \iota_\Sigma \otimes \iota_j \otimes \mathrm{id}} \mathcal{H}_c^{b_{n,\infty}}(\mathcal{X}_n, F_\mu^\vee) \otimes \mathcal{H}^{b_{n-1,\infty}}(\mathcal{X}_{n-1}, F_\nu^\vee) \otimes \mathrm{H}^0(\mathcal{X}_{n-1}, \delta_j^\vee) \otimes \mathfrak{M}_{n-1,f} \\
&\xrightarrow{\imath^* \otimes \wp^* \otimes \wp^* \otimes \mathrm{id}} \mathcal{H}_c^{b_{n,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\mu^\vee) \otimes \mathcal{H}^{b_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\nu^\vee) \otimes \mathrm{H}^0(\widetilde{\mathcal{X}}_{n-1}, \delta_j^\vee) \otimes \mathfrak{M}_{n-1,f}
\end{aligned}$$

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$$\xrightarrow{\cup\otimes\mathrm{id}} \mathrm{H}_c^{d_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, F_\xi^\vee \otimes \delta_j^\vee) \otimes \mathfrak{O}_{n-1,\infty} \otimes \mathfrak{M}_{n-1,f}$$
$$\xrightarrow{\phi_{\xi,j}\otimes\mathrm{id}} \mathrm{H}_c^{d_{n-1,\infty}}(\widetilde{\mathcal{X}}_{n-1}, \mathbb{C}) \otimes \mathfrak{M}_{n-1,f}^{\natural}$$
$$\xrightarrow{\int_{\widetilde{\mathcal{X}}_{n-1}}} \mathbb{C}. \tag{77}$$

Recall that we have the normalized Rankin–Selberg integral at infinity

$$\begin{aligned}\mathrm{Z}^\circ(\cdot, s, \chi_\infty) &\in \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k}_\infty)}(\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty,s-1/2} \otimes \mathfrak{M}_{n-1,\infty}, \mathbb{C}) \\ &= \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathrm{k}_\infty)}(\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty,s-1/2}, \mathfrak{M}_{n-1,\infty}^*),\end{aligned}$$

where, as before, $*$ stands for the dual space. Put

$$\begin{aligned}\mathrm{H}(\chi_{\infty,j}) :&= \mathrm{H}^0_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash \mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \delta_j^\vee \otimes \chi_{\infty,j}) \\ &= \mathrm{H}^0_{\mathrm{ct}}(\mathbb{R}_+^\times \backslash \mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \chi_\infty \cdot \mathrm{sgn}_\infty^j),\end{aligned}$$

where sgn_∞ is given as in the introduction.

Analogous to the archimedean modular symbol defined in § 3.2, we define the *modular symbol at infinity*, which is denoted by $\mathcal{P}_{\infty,j}$, to be the composition of

$$\begin{aligned}\mathcal{P}_{\infty,j} &: \mathcal{H}(\Pi_\infty) \otimes \mathcal{H}(\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) \\ &\to \mathrm{H}_{\mathrm{ct}}^{d_{n-1,\infty}}(\mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; (\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty,j}) \otimes (F_\xi^\vee \otimes \delta_j^\vee)) \otimes \mathfrak{O}_{n-1,\infty} \\ &\to \mathrm{H}_{\mathrm{ct}}^{d_{n-1,\infty}}(\mathrm{GL}_{n-1}(\mathrm{k}_\infty)^0; \mathfrak{M}_{n-1,\infty}^*) \otimes \mathfrak{O}_{n-1,\infty} = \mathbb{C},\end{aligned}$$

where the first arrow is the restriction of cohomology composed with the cup product, and the last arrow is the map induced by the linear functional

$$\mathrm{Z}^\circ(\cdot, \tfrac{1}{2}+j, \chi_\infty) \otimes \phi_{\xi,j} : (\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty,j}) \otimes (F_\xi^\vee \otimes \delta_j^\vee) \to \mathfrak{M}_{n-1,\infty}^*.$$

PROPOSITION 7.2. *(Cf. [Mah05, (5.3)] and [Jan19, § 4.6]) Let the notation and assumptions be as above. Then the diagram*

$$\begin{array}{ccc}\Pi_f \otimes \Sigma_f \otimes \chi_{f,j} \otimes \mathfrak{M}_{n-1,f} \otimes \mathcal{H}(\Pi_\infty) \otimes \mathcal{H}(\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) & \xrightarrow{\mathrm{Z}^\circ(\cdot,\frac{1}{2}+j,\chi_f)\otimes\mathcal{P}_{\infty,j}} & \mathbb{C} \\ \downarrow{\scriptstyle \iota_{\mathrm{can}}} & & \downarrow{\scriptstyle \mathrm{L}(\frac{1}{2}+j,\Pi\times\Sigma\times\chi)} \\ \mathcal{H}(\Pi) \otimes \mathcal{H}(\Sigma) \otimes \mathrm{H}(\chi_j) \otimes \mathfrak{M}_{n-1,f} & \xrightarrow{\mathcal{P}_j} & \mathbb{C}\end{array}$$

commutes, where the left vertical arrow ι_{can} is the natural isomorphism.

Proof. Define $\mathfrak{q}_{n,\infty} := \big(\mathfrak{gl}_n(\mathrm{k}_\infty)/(\mathbb{R} \oplus \mathfrak{k}_{n,\infty})\big) \otimes_{\mathbb{R}} \mathbb{C}$. We have a map

$$(\wedge^{b_{n,\infty}} \mathfrak{q}_{n,\infty})^* \otimes (\wedge^{b_{n-1,\infty}} \mathfrak{q}_{n-1,\infty})^* \to \omega_{n-1,\infty}^* = \wedge^{d_{n-1,\infty}}\big((\mathfrak{gl}_{n-1}(\mathrm{k}_\infty)/\mathfrak{k}_{n-1,\infty}) \otimes_{\mathbb{R}} \mathbb{C}\big)^*$$

induced by restriction. By the identification of continuous cohomology and relative Lie algebra cohomology [HM62, Theorem 6.1], as well as the explicit determination of the relative Lie algebra cohomology [Wal88, Proposition 9.4.3], we have that

$$\mathcal{H}(\Pi_\infty) = \big((\wedge^{b_{n,\infty}} \mathfrak{q}_{n,\infty})^* \otimes \Pi_\infty \otimes F_\mu^\vee\big)^{K_{n,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n,\infty}$$

and

$$\mathcal{H}(\Sigma_\infty) = \big((\wedge^{b_{n-1,\infty}} \mathfrak{q}_{n-1,\infty})^* \otimes \Sigma_\infty \otimes F_\nu^\vee\big)^{K_{n-1,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n-1,\infty}.$$

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By definition of $\mathcal{P}_{\infty,j}$, the top horizontal arrow of the diagram is identified with the composition of

$$\begin{aligned}
&\Pi_f \otimes \Sigma_f \otimes \chi_{f,j} \otimes \mathfrak{M}_{n-1,f} \otimes \big((\wedge^{b_{n,\infty}} \mathfrak{q}_{n,\infty})^* \otimes \Pi_\infty \otimes F_\mu^\vee\big)^{K_{n,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n,\infty} \\
&\quad \otimes \big((\wedge^{b_{n-1,\infty}} \mathfrak{q}_{n-1,\infty})^* \otimes \Sigma_\infty \otimes F_\nu^\vee\big)^{K_{n-1,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n-1,\infty} \otimes \delta_j^\vee \otimes \chi_{\infty,j} \\
&\xrightarrow{\text{restriction}} \Pi_f \otimes \Sigma_f \otimes \chi_{f,j} \otimes \mathfrak{M}_{n-1,f} \otimes \omega_{n-1,\infty}^* \otimes \Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes F_\xi^\vee \otimes \mathfrak{O}_{n-1,\infty} \otimes \delta_j^\vee \otimes \chi_{\infty,j} \\
&= (\Pi_f \otimes \Sigma_f \otimes \chi_{f,j} \otimes \mathfrak{M}_{n-1,f}) \otimes (\Pi_\infty \widehat{\otimes} \Sigma_\infty \otimes \chi_{\infty,j} \otimes \mathfrak{M}_{n-1,\infty}) \otimes (F_\xi^\vee \otimes \delta_j^\vee) \\
&\to \mathbb{C},
\end{aligned}$$

where the last map is given by

$$\mathrm{Z}^\circ(\cdot, \tfrac{1}{2}+j, \chi_f) \otimes \mathrm{Z}^\circ(\cdot, \tfrac{1}{2}+j, \chi_\infty) \otimes \phi_{\xi,j}.$$

Using fast decreasing differential forms as in [Bor81, § 5.6], the bottom arrow of the diagram is identified with the composition of

$$\begin{aligned}
&\big((\wedge^{b_{n,\infty}} \mathfrak{q}_{n,\infty})^* \otimes \Pi \otimes F_\mu^\vee\big)^{K_{n,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n,\infty} \\
&\quad \otimes \big((\wedge^{b_{n-1,\infty}} \mathfrak{q}_{n-1,\infty})^* \otimes \Sigma \otimes F_\nu^\vee\big)^{K_{n-1,\infty}^0} \otimes \widetilde{\mathfrak{O}}_{n-1,\infty} \otimes \delta_j^\vee \otimes \chi_j \otimes \mathfrak{M}_{n-1,f} \\
&\xrightarrow{\text{restriction}} \omega_{n-1,\infty}^* \otimes \Pi \widehat{\otimes} \Sigma \otimes F_\xi^\vee \otimes \mathfrak{O}_{n-1,\infty} \otimes \delta_j^\vee \otimes \chi_j \otimes \mathfrak{M}_{n-1,f} \\
&= (\Pi \widehat{\otimes} \Sigma \otimes \chi_j \otimes \mathfrak{M}_{n-1}) \otimes (F_\xi^\vee \otimes \delta_j^\vee) \\
&\xrightarrow{\mathrm{Z}(\cdot, \frac{1}{2}+j, \chi) \otimes \phi_{\xi,j}} \mathbb{C}.
\end{aligned}$$

The proposition then follows from Proposition 7.1. □

7.3 Two commutative diagrams

For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we note that the infinite part of $({}^\sigma\chi)_j$ coincides with $\chi_{\infty,j}$. Denote the corresponding modular symbol at infinity by

$${}^\sigma\mathcal{P}_{\infty,j} : \mathcal{H}({}^\sigma\Pi_\infty) \otimes \mathcal{H}({}^\sigma\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) \to \mathbb{C},$$

and introduce the normalized modular symbol at infinity

$${}^\sigma\mathcal{P}^\circ_{\infty,j} := \Omega'_{\mu,\nu,j} \cdot {}^\sigma\mathcal{P}_{\infty,j}, \quad \text{where } \Omega'_{\mu,\nu,j} := \prod_{v|\infty} \Omega'_{\mu_v,\nu_v,j}. \tag{78}$$

In particular, we have the normalized modular symbol at infinity

$$\mathcal{P}^\circ_{\infty,j} := \Omega'_{\mu,\nu,j} \cdot \mathcal{P}_{\infty,j}.$$

As in (73), we have a σ-linear isomorphism

$$\sigma : \mathcal{H}(\Pi_\infty) \to \mathcal{H}({}^\sigma\Pi_\infty)$$

such that

$$\sigma(\kappa_{\mu,\varepsilon}) = \kappa_{{}^\sigma\mu,\varepsilon} \quad \text{for all } \varepsilon \in \widehat{\pi_0(\mathrm{k}_\infty^\times)} \text{ that occur in } \mathcal{H}(\Pi_\infty).$$

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We have a similar σ-linear isomorphism

$$\sigma : \mathcal{H}(\Sigma_\infty) \to \mathcal{H}({}^\sigma\Sigma_\infty),$$

as well as a σ-linear isomorphism

$$\sigma : \mathrm{H}(\chi_{\infty,j}) \to \mathrm{H}(\chi_{\infty,j})$$

such that $\sigma(1) = 1$. By tensor product, we get a σ-linear isomorphism

$$\sigma : \mathcal{H}(\Pi_\infty) \otimes \mathcal{H}(\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) \to \mathcal{H}({}^\sigma\Pi_\infty) \otimes \mathcal{H}({}^\sigma\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}).$$

Proposition 7.3. *For all $\sigma \in \mathrm{Aut}(\mathbb{C})$, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{H}(\Pi_\infty) \otimes \mathcal{H}(\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) & \xrightarrow{\mathcal{P}^\circ_{\infty,j}} & \mathbb{C} \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ \mathcal{H}({}^\sigma\Pi_\infty) \otimes \mathcal{H}({}^\sigma\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) & \xrightarrow{{}^\sigma\mathcal{P}^\circ_{\infty,j}} & \mathbb{C} \end{array} \tag{79}$$

Proof. Let $\Pi_{0_{n,\infty}} := \widehat{\otimes}_{v|\infty} \Pi_{0_{n,\mathrm{k}_v}}$ and $\Sigma_{0_{n-1,\infty}} := \widehat{\otimes}_{v|\infty} \Sigma_{0_{n-1,\mathrm{k}_v}}$ be the cohomological representations (as in § 2) of $\mathrm{GL}_n(\mathrm{k}_\infty)$ and $\mathrm{GL}_{n-1}(\mathrm{k}_\infty)$ that have trivial coefficient systems and respectively have the same central characters as that of $F_\mu^\vee \otimes \Pi_\infty$ and $F_\nu^\vee \otimes \Sigma_\infty$.

Applying Theorem 3.2 for all $v|\infty$, we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathcal{H}(\Pi_\infty) \otimes \mathcal{H}(\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) & \xrightarrow{\Omega'_{\mu,\nu,j} \cdot \mathcal{P}_{\infty,j}} & \mathbb{C} \\ \jmath_\mu \otimes \jmath_\nu \otimes \mathrm{id} \big\uparrow & & \big\| \\ \mathcal{H}(\Pi_{0_{n,\infty}}) \otimes \mathcal{H}(\Sigma_{0_{n-1,\infty}}) \otimes \mathrm{H}(\chi_{\infty,j}) & \xrightarrow{\mathcal{P}_{\infty,0}} & \mathbb{C} \end{array} \tag{80}$$

This easily implies the proposition. □

Pick an element $y = (y_v)_{v\nmid\infty} \in \mathbb{A}_f^\times$ such that

$$\mathfrak{c}(\psi_v) = y_v \cdot \mathfrak{c}(\chi_v) \quad \text{for all } v \nmid \infty.$$

Define the Gauss sum

$$\mathcal{G}(\chi) := \mathcal{G}(\chi, \psi, y) := \prod_{v\nmid\infty} \mathcal{G}(\chi_v, \psi_v, y_v), \tag{81}$$

where $\mathcal{G}(\chi_v, \psi_v, y_v)$ is the local Gauss sum given by (53). Similarly, pick an element $y' = (y'_v)_{v\nmid\infty} \in \mathbb{A}_f^\times$ such that

$$\mathfrak{c}(\psi_v) = y'_v \cdot \mathfrak{c}(\chi_{\Sigma_v}) \quad \text{for all } v \nmid \infty,$$

and define the Gauss sum

$$\mathcal{G}(\chi_\Sigma) := \mathcal{G}(\chi_\Sigma, \psi, y') := \prod_{v\nmid\infty} \mathcal{G}(\chi_{\Sigma_v}, \psi_v, y'_v). \tag{82}$$

Here we write $\Sigma_f = \otimes'_{v\nmid\infty} \Sigma_v$ as usual, and χ_Σ and χ_{Σ_v} denote the central characters of Σ and Σ_v, respectively. More generally, we have the Gauss sums

$$\mathcal{G}({}^\sigma\chi) := \mathcal{G}({}^\sigma\chi, \psi, y) \quad \text{and} \quad \mathcal{G}(\chi_{{}^\sigma\Sigma}) := \mathcal{G}(\chi_{{}^\sigma\Sigma}, \psi, y'),$$

where $\chi_{{}^\sigma\Sigma}$ denotes the central character of ${}^\sigma\Sigma$.

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Similar to (56), for all $s \in \mathbb{C}$ we have a σ-linear isomorphism

$$\sigma : \Pi_f \otimes \Sigma_f \otimes \chi_{f,s-1/2} \otimes \mathfrak{M}_{n-1,f} \to {}^\sigma\Pi_f \otimes {}^\sigma\Sigma_f \otimes {}^\sigma(\chi_{f,s-1/2}) \otimes \mathfrak{M}_{n-1,f}.$$

Note that ${}^\sigma(\chi_{f,s-1/2}) = ({}^\sigma\chi)_{f,s-1/2}$ when $s \in \frac{1}{2} + \mathbb{Z}$.

Proposition 7.4. *For all $s_0 \in \frac{1}{2} + \mathbb{Z}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, the following diagram commutes.*

$$\begin{array}{ccc} \Pi_f \otimes \Sigma_f \otimes \chi_{f,s_0-\frac{1}{2}} \otimes \mathfrak{M}_{n-1,f} & \xrightarrow{\mathcal{G}(\chi_\Sigma)\cdot\mathcal{G}(\chi)^{\frac{n(n-1)}{2}}\cdot \mathrm{Z}^\circ(\cdot,s_0,\chi_f)} & \mathbb{C} \\ \sigma\downarrow & & \downarrow\sigma \\ {}^\sigma\Pi_f \otimes {}^\sigma\Sigma_f \otimes {}^\sigma(\chi_{f,s_0-\frac{1}{2}}) \otimes \mathfrak{M}_{n-1,f} & \xrightarrow{\mathcal{G}(\chi_{{}^\sigma\Sigma})\cdot\mathcal{G}({}^\sigma\chi)^{\frac{n(n-1)}{2}}\cdot \mathrm{Z}^\circ(\cdot,s_0,{}^\sigma\chi_f)} & \mathbb{C} \end{array}$$

Proof. Write $\Pi_f = \otimes'_{v\nmid\infty}\Pi_v$ as usual. By the uniqueness of Whittaker functionals, we write $\lambda_f = \otimes_{v\nmid\infty}\lambda_v$, $\lambda'_f = \otimes_{v\nmid\infty}\lambda'_v$ and assume that

$$\lambda_v(e_v) = \lambda'_v(e'_v) = 1$$

for all but finitely many $v \nmid \infty$ such that Π_v and Σ_v are unramified, where $e_v \in \Pi_v$ and $e'_v \in \Sigma_v$ are the spherical vectors used in the definition of the restricted tensor products Π_f and Σ_f. For places v as above, if moreover χ_v is unramified and ψ_v has conductor $\mathcal{O}_{\mathrm{k}_v}$, then it is known that (see [JS81a, Proposition 2.4])

$$\mathrm{Z}^\circ(e_v \otimes e'_v \otimes m^\circ_{n-1,\mathrm{k}_v}, s, \chi_v) = 1,$$

where $m^\circ_{n-1,\mathrm{k}_v} \in \mathfrak{M}_{n-1,\mathrm{k}_v}$ is the Haar measure on $\mathrm{GL}_{n-1}(\mathrm{k}_v)$ such that a maximal open compact subgroup has total volume 1. The proposition then follows from Proposition 5.1. □

In analogy to (78), for the finite part we introduce

$${}^\sigma\mathcal{P}^\circ_{f,j} := \mathcal{G}(\chi_{{}^\sigma\Sigma}) \cdot \mathcal{G}({}^\sigma\chi)^{n(n-1)/2} \cdot \mathrm{Z}^\circ(\cdot, \tfrac{1}{2} + j, {}^\sigma\chi_f).$$

Specifically, we have

$$\mathcal{P}^\circ_{f,j} := \mathcal{G}(\chi_\Sigma) \cdot \mathcal{G}(\chi)^{n(n-1)/2} \cdot \mathrm{Z}^\circ(\cdot, \tfrac{1}{2} + j, \chi_f).$$

Then Proposition 7.4 can be rephrased as the following commutative diagram.

$$\begin{array}{ccc} \Pi_f \otimes \Sigma_f \otimes \chi_{f,j} \otimes \mathfrak{M}_{n-1,f} & \xrightarrow{\mathcal{P}^\circ_{f,j}} & \mathbb{C} \\ \sigma\downarrow & & \downarrow\sigma \\ {}^\sigma\Pi_f \otimes {}^\sigma\Sigma_f \otimes {}^\sigma(\chi_{f,j}) \otimes \mathfrak{M}_{n-1,f} & \xrightarrow{{}^\sigma\mathcal{P}^\circ_{f,j}} & \mathbb{C} \end{array} \tag{83}$$

7.4 Proof of Theorem 1.2

As in (77), we have the modular symbol map

$${}^\sigma\mathcal{P}_j : \mathcal{H}({}^\sigma\Pi) \otimes \mathcal{H}({}^\sigma\Sigma) \otimes \mathrm{H}({}^\sigma\chi_j) \otimes \mathfrak{M}_{n-1,f} \to \mathbb{C}.$$

Put

$${}^\sigma\,\mathrm{L}^*_j := \frac{\mathrm{L}(\frac{1}{2} + j, {}^\sigma\Pi \times {}^\sigma\Sigma \times {}^\sigma\chi)}{\Omega'_{\mu,\nu,j} \cdot \mathcal{G}(\chi_{{}^\sigma\Sigma}) \cdot \mathcal{G}({}^\sigma\chi)^{n(n-1)/2}}.$$

Then by Proposition 7.2 the following diagram commutes.

$$\begin{CD}
{}^\sigma\Pi_f \otimes {}^\sigma\Sigma_f \otimes {}^\sigma(\chi_{f,j}) \otimes \mathfrak{M}_{n-1,f} \otimes \mathcal{H}({}^\sigma\Pi_\infty) \otimes \mathcal{H}({}^\sigma\Sigma_\infty) \otimes \mathrm{H}(\chi_{\infty,j}) @>{{}^\sigma\mathcal{P}^\circ_{f,j} \otimes {}^\sigma\mathcal{P}^\circ_{\infty,j}}>> \mathbb{C} \\
@V{\iota_{\mathrm{can}}}VV @VV{{}^\sigma \mathrm{L}^*_j}V \\
\mathcal{H}({}^\sigma\Pi) \otimes \mathcal{H}({}^\sigma\Sigma) \otimes \mathrm{H}({}^\sigma\chi_j) \otimes \mathfrak{M}_{n-1,f} @>{{}^\sigma\mathcal{P}_j}>> \mathbb{C}
\end{CD}$$

We are now ready to prove Theorem 1.2. It is clear that (3) is a consequence of (4), and we will prove the latter. To save space, denote the subspaces of $\pi_0(\mathrm{k}_\infty^\times)$-fixed vectors in the two spaces in the left vertical arrow of the last diagram by $\mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{loc}}$ and $\mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{glob}}$, respectively, so that the last diagram reads as follows.

$$\begin{CD}
\mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{loc}} @>{{}^\sigma\mathcal{P}^\circ_{f,j} \otimes {}^\sigma\mathcal{P}^\circ_{\infty,j}}>> \mathbb{C} \\
@V{\iota_{\mathrm{can}}}VV @VV{{}^\sigma \mathrm{L}^*_j}V \\
\mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{glob}} @>{{}^\sigma\mathcal{P}_j}>> \mathbb{C}
\end{CD} \tag{84}$$

By (74), we have a commutative diagram

$$\begin{CD}
\mathcal{H}(\Pi, \Sigma, \chi, j)_{\mathrm{loc}} @>{\sigma}>> \mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{loc}} \\
@V{\Omega_{(j)}^{-1} \cdot \iota_{\mathrm{can}}}VV @VV{{}^\sigma\Omega_{(j)}^{-1} \cdot \iota_{\mathrm{can}}}V \\
\mathcal{H}(\Pi, \Sigma, \chi, j)_{\mathrm{glob}} @>{\sigma}>> \mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{glob}}
\end{CD} \tag{85}$$

where the top horizontal arrow is the tensor product of the left vertical arrows in (79) and (83), and for short we write

$$\Omega_{(j)} := \Omega_{\varepsilon_n}(\Pi) \cdot \Omega_{\varepsilon_{n-1}}(\Sigma) \quad \text{and} \quad {}^\sigma\Omega_{(j)} := \Omega_{\varepsilon_n}({}^\sigma\Pi) \cdot \Omega_{\varepsilon_{n-1}}({}^\sigma\Sigma).$$

It is well-known that the global modular symbol is $\mathrm{Aut}(\mathbb{C})$-equivariant (see [Rag10, Proposition 3.14]), that is, the following diagram commutes.

$$\begin{CD}
\mathcal{H}(\Pi, \Sigma, \chi, j)_{\mathrm{glob}} @>{\mathcal{P}_j}>> \mathbb{C} \\
@V{\sigma}VV @VV{\sigma}V \\
\mathcal{H}({}^\sigma\Pi, {}^\sigma\Sigma, {}^\sigma\chi, j)_{\mathrm{glob}} @>{{}^\sigma\mathcal{P}_j}>> \mathbb{C}
\end{CD} \tag{86}$$

Since $\Omega_{\mu,\nu,j}$ and $\Omega'_{\mu,\nu,j}$ only differ by a sign, (4) is equivalent to the equation

$$\sigma\left(\frac{\mathrm{L}^*_j}{\Omega_{(j)}}\right) = \frac{{}^\sigma\mathrm{L}^*_j}{{}^\sigma\Omega_{(j)}},$$

which amounts to the commutativity of the following diagram.

$$\begin{CD}
\mathbb{C} @>{\sigma}>> \mathbb{C} \\
@V{\frac{\mathrm{L}^*_j}{\Omega_{(j)}}}VV @VV{\frac{{}^\sigma\mathrm{L}^*_j}{{}^\sigma\Omega_{(j)}}}V \\
\mathbb{C} @>{\sigma}>> \mathbb{C}
\end{CD} \tag{87}$$

Here

$$\mathrm{L}^*_j := \frac{\mathrm{L}(\frac{1}{2} + j, \Pi \times \Sigma \times \chi)}{\Omega'_{\mu,\nu,j} \cdot \mathcal{G}(\chi_\Sigma) \cdot \mathcal{G}(\chi)^{n(n-1)/2}}.$$

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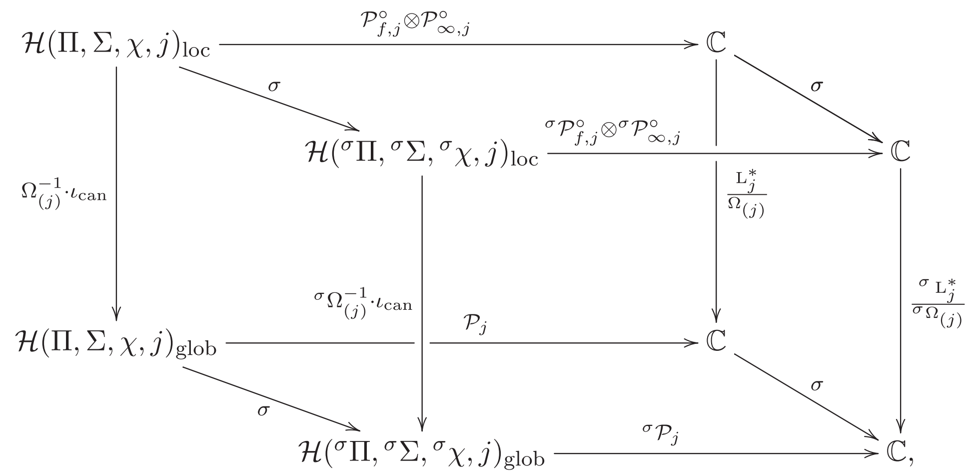
The commutative diagrams (84), (85) and (86), together with (79) and (83), give us the following diagram



where all squares are commutative except (87). This forces (87) to be commutative as well. This proves (4), hence finishes the proof of Theorem 1.2.

Conflicts of interest
None.

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