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REWRITABLE PRODUCTS IN FC-BY-FINITE GROUPS

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1. Introduction. Let n be an integer greater than 1. The group G has the property $\mathbf{Q}_n$, or is *n-rewritable*, if for each n-element subset $\{x_1, x_2, \dots, x_n\}$ of G, there exist permutations $\sigma \neq \tau$ in S_n such that

$$x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)} = x_{\tau(1)}x_{\tau(2)} \dots x_{\tau(n)}.$$

If one of σ, τ can always be chosen to be the identity, then G has $\mathbf{P}_n$, or is *totally n-rewritable*. We also use $\mathbf{P}_n$ and $\mathbf{Q}_n$ to denote the classes of groups having these properties. Making use of the obvious inclusions, we define

$$\mathbf{P} = \bigcup_{n=2,3,\dots} \mathbf{P}_n \quad \text{and} \quad \mathbf{Q} = \bigcup_{n=2,3,\dots} \mathbf{Q}_n,$$

which are the classes of *totally rewritable* and *rewritable* groups respectively.

The classes $\mathbf{P}_n$ for semigroups were introduced by Restivo and Reutenauer in [**12**], and for groups by Curzio, Longobardi and Maj [**6**]. A classification for $\mathbf{P}$-groups was given by Curzio, Longobardi, Maj and Robinson [**7**] and for $\mathbf{Q}$-groups by Blyth [**2**]; in fact, the classes $\mathbf{P}$ and $\mathbf{Q}$ are precisely the class of finite-by-abelian-by-finite groups (recall that a group G is finite-by-abelian-by-finite if it has subgroups H and K, where H is a normal subgroup of G of finite index, K is a finite normal subgroup of H, and the quotient H/K is abelian). Classifications for $\mathbf{P}_n$-groups and $\mathbf{Q}_n$-groups for small n are given in [**1**], [**3**], [**5**], [**8**], [**9**], and [**10**]. A summary of the results for groups is given in [**4**].

The purpose of this article is to discuss the following properties: we say that the group G has the property $\mathbf{P}_\infty$, or is *eventually totally rewritable*, if for each infinite sequence $x_1, x_2, \dots$ of elements of G, there is an integer n and a nonidentity permutation $\sigma \in S_n$ such that

$$x_1x_2 \dots x_n = x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)}.$$

Similarly, the group G has the property $\mathbf{Q}_\infty$, or is *eventually rewritable*, if for each infinite sequence $x_1, x_2, \dots$ of elements of G, there is an integer n and distinct permutations $\sigma, \tau \in S_n$ such that

$$x_{\sigma(1)}x_{\sigma(2)} \dots x_{\sigma(n)} = x_{\tau(1)}x_{\tau(2)} \dots x_{\tau(n)}.$$

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We use some elementary theory of FC-groups (see [**15**]). A group G is an FC-*group* if every element of G has a finite number of conjugates in the group, or equivalently, if the centralizer $C_G(x)$ of every element $x \in G$ has finite index in G. The FC-*center* of any group is the characteristic subgroup consisting of its FC-*elements*, that is, of the elements which have a finite number of conjugates. A group is FC-*by-finite* if it has a normal FC-subgroup of finite index.

The main results are

PROPOSITION 1. $\mathbf{P} = \mathbf{P}_\infty$.

PROPOSITION 2. $\mathbf{Q}_\infty$ *is the class of* FC-*by-finite groups.*

At first glance, these two results are unexpected considering that the classes $\mathbf{P}$ and $\mathbf{Q}$ are the same. Observe that a $\mathbf{Q}_\infty$-group is locally a $\mathbf{Q}$-group and $\mathbf{Q} \subset \mathbf{Q}_\infty \subset \mathbf{LQ}$ with both inclusions being strict. An infinite direct product of finite nonabelian groups is not a $\mathbf{Q}$-group [**2**], but it is a $\mathbf{Q}_\infty$-group, since it is an FC-group. Since a locally FC-by-finite group does not have to be FC-by-finite, the class $\mathbf{Q}_\infty$ is not $\mathbf{L}$-closed. Proposition 1 is surprising because by definition the length m of the product $x_1 \dots x_m$ that can be rewritten depends on the given sequence $x_1, x_2, \dots$. Yet $G \in \mathbf{P}_n$ for some n and hence for any group G in $\mathbf{P}_\infty$, this number m is bounded above. A direct proof showing $\mathbf{P}_\infty = \mathbf{P}$ is not likely without knowledge of the structure of such groups.

2. $\mathbf{P}_\infty$-groups. The proof that every $\mathbf{P}_\infty$-group is finite-by-abelian-by-finite mimics parts of the corresponding proofs for $\mathbf{P}$-groups and $\mathbf{Q}$-groups.

LEMMA 2.1. *Suppose that G is a $\mathbf{P}_\infty$-group. Then the* FC-*center F of G has finite index in G.*

Proof. Choose a sequence $x_1, x_2, \dots$ of elements of G in the following way:
(i) $x_1 \in G \backslash F$
(ii) for $j \geqq 1$, $x_{j+1} \in G \backslash \{F \cup x_{i_1}^{-1} \dots x_{i_r}^{-1} F \mid (i_1, \dots, i_r)$ an arrangement chosen from $\{1, \dots j\}, 1 \leqq r \leqq j\}$, and $x_1 \dots x_{j+1}$ does not rewrite.

This sequence must stop, say at $x_1, \dots, x_m$, since $G \in \mathbf{P}_\infty$ (that is, $x_1, \dots x_m$ is a sequence of this type of maximal length).

The remainder of the proof follows that of (2.1) of [**7**]. Let

$$S = \{1_G\} \cup \{x_{i_1}^{-1} \dots x_{i_r}^{-1} \mid (i_1, \dots, i_r) \text{ an arrangement chosen from } \{1, \dots m\}, 1 \leqq r \leqq m\}.$$

If $x_{m+1} \in G \backslash SF$, the sequence $x_1, \dots, x_m, x_{m+1}$ rewrites, that is, there is a $\sigma \neq 1$ in S_{m+1} such that

$$x_1 x_2 \dots x_{m+1} = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m+1)}.$$

Clearly $\sigma(m+1) \neq m+1$ here. For each $\sigma \in S_{m+1}$ such that $\sigma(m+1) \neq m+1$ define A_σ to be the (possibly empty) set of all $x_{m+1} \in G \backslash SF$ such that

$$x_1 x_2 \dots x_{m+1} = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m+1)}.$$

Then

$$G = \bigcup_{s\in S} sF \bigcup \bigcup_{\sigma} A_\sigma.$$

Let a_σ be a fixed element of A_σ (if the latter is nonempty) and let b be any element of A_σ. If $\sigma(i) = m+1$, then we have the equations

$$x_1x_2\dots x_m a_\sigma = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(i-1)}a_\sigma x_{\sigma(i+1)}\dots x_{\sigma(m+1)},$$
$$x_1x_2\dots x_m b = x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(i-1)}bx_{\sigma(i+1)}\dots x_{\sigma(m+1)}.$$

These equations yield

$$a_\sigma d_\sigma a_\sigma^{-1} = bd_\sigma b^{-1}$$

where

$$d_\sigma = x_{\sigma(i+1)}\dots x_{\sigma(m+1)}.$$

Hence $b \in a_\sigma C_G(d_\sigma)$ and it follows that

$$G \bigcup_{s\in S} sF \bigcup \bigcup_{\sigma} a_\sigma C_G(d_\sigma).$$

Suppose that $d_\sigma = x_{i_1}\dots x_{i_k} \in F$. Let

$$r = \max_{1\leqq j\leqq k} i_j \quad \text{and} \quad r = i_s.$$

Since $F \triangleleft G$ we may solve for x_r, obtaining

$$x_r \in x_{i_{s-1}}^{-1}\dots x_{i_1}^{-1}x_{i_{s+1}}^{-1}\dots x_{i_k}^{-1}F.$$

This contradicts condition (ii). The $C_G(d_\sigma)$ thus have infinite index in G and can be omitted from the above union by a well-known theorem of B.H. Neumann ([**11**], or [**13**], Lemma 4.17). Therefore

$$G = \bigcup_{s\in S} sF$$

and hence

$$|G:F| \leqq |S| \leqq 1 + m + m(m-1) + \dots + m!$$

This completes the proof.

It now remains to show

PROPOSITION 2.2. *If $G \in \mathbf{P}_\infty$ is an* FC*-group, then G is finite-by-abelian.*

We shall essentially mimic the corresponding proof for $\mathbf{Q}$-groups (see [**2**]), with one major departure.

LEMMA 2.3. *Let $G = G_1G_2\ldots$ be an infinite product of nonabelian subgroups such that the derived subgroup $G' = [G,G]$ of G is the direct product*

$$G' = \underset{i=1,2,\ldots}{Dr}\ G'_i$$

of the derived subgroups G'_i of the G_i and $[G_i,G_j] = 1$ whenever $i \neq j$. Then $G \notin \mathbf{P}_\infty$.

Proof. Choose elements g_i, h_i from G_i so that

$$[g_i,h_i] = g_i^{-1}h_i^{-1}g_ih_i = c_i \neq 1$$

and select the elements $x_1, x_2, \ldots$ in G to be

$$x_1 = g_1,$$
$$x_2 = h_1g_2,$$
$$x_3 = h_2g_3, \ldots.$$

Consider decomposing $[x_q,x_l]$ into a product of commutators of the h_i and g_j using the commutator identities

$$[x,yz] = [x,z][x,y]^z \quad \text{and} \quad [xy,z] = [x,z]^y[y,z].$$

Since $[G_i,G_j] = 1$ whenever $i \neq j$, we conclude for $q < l$ that

$$[x_q,x_l] = \begin{cases} c_q \neq 1 & \text{if } q+1 = l \\ 1 & \text{if } q+1 \neq l. \end{cases}$$

If $q > l$, then

$$[x_q,x_l] = \begin{cases} c_q^{-1} \neq 1 & \text{if } l+1 = q \\ 1 & \text{if } l+1 \neq q. \end{cases}$$

For each n, and each $\sigma \neq 1$ in S_n, consider $x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$. Let j be the smallest integer such that x_j appears to the right of x_{j+1} in the product; that is, j is the smallest integer such that $\sigma^{-1}(j) > \sigma^{-1}(j+1)$. Then

$$x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)} \equiv x_1x_2\ldots x_nc_j^{-1} \bmod H_j,$$

using the identity $xy = yx[x,y]$, where

$$H_j = \operatorname*{Dr}_{\substack{i=1\\ i\neq j}}^{n} G_i',$$

and hence, since $c_j^{-1} \notin H_j$, we deduce that

$$x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)} \not\equiv x_1, x_2 \dots x_n \bmod H_j.$$

Thus the product $x_1x_2\dots x_n$ does not rewrite. Since n is arbitrary, $G \notin \mathbf{P}_\infty$.

LEMMA 2.5. *If G is an* FC *torsion $\mathbf{P}_\infty$-group which is nilpotent of class at most 2, then the derived subgroup G' has finite exponent.*

Proof. G is torsion and nilpotent, so

$$G = \operatorname*{Dr}_{i} G_{p_i}$$

for various primes p_i ([**14**], 5.2.7) and

$$G' = \operatorname*{Dr}_{i} G'_{p_i}.$$

(i) Suppose that $G'_{p_i} \neq 1$ for infinitely many odd primes p_i, say $p_1, p_2, \dots$. Choose $x_i, y_i \in G_{p_i}$ such that $c_i = [x_i, y_i] \neq 1$. Let $z_1 = x_1$ and for $i > 1$, let $z_i = y_{i-1}x_i$. Since $[G_i, G_j] = 1$ whenever $i \neq j$,

$$[z_i, z_j] = \begin{cases} c_i & \text{if } j = i+1 \\ c_i^{-1} & \text{if } j = i-1 \\ 1 & \text{if } |i-j| \neq 1. \end{cases}$$

Then

$$z_1 \dots z_n = x_1y_1x_2y_2\dots y_{n-1}x_n = x_1\dots x_ny_1\dots y_{n-1}.$$

For $\sigma \neq 1$ in S_n, let j be the least integer such that $\sigma(j) = m > j$. Let

$$K = \langle G'_{p_i}, i = 1, 2, \dots; i \neq m-1 \rangle.$$

Hence, using the commutator identity $xy = yx[x,y]$, we have

$$\begin{aligned} z_{\sigma(1)}\dots z_{\sigma(n)} &\equiv z_1\dots z_{m-2}z_mz_{m-1}z_{m+1}\dots z_n \bmod K \\ &\equiv z_1\dots z_nc_{m-1}^{-1} \bmod K \not\equiv z_1\dots z_n \bmod K. \end{aligned}$$

Therefore $z_1\dots z_n$ does not rewrite, and we conclude that $G'_{p_i} = 1$ for all but finitely many p_i. If the exponent of G'_{p_i} is finite for each p_i then the exponent of

G' is finite. If the result is false then there is a prime p such that the exponent of G'_p is not finite, so we reduce to considering this case.

(ii) Suppose that $G \in \mathbf{P}_\infty$ is an FC torsion p-group which is nilpotent of class at most 2. Suppose to the contrary that the exponent of G' is infinite. Then the exponent of $G/Z(G)$ is not finite, since $x, y \in G$, with $x^n \in Z(G)$ implies that (as G is nilpotent of class at most 2) $[x, y]^n = [x^n, y] = 1$. Set $G_1 = G$. Pick $x_1, y_1 \in G_1$ such that $1 \neq [x_1, y_1] = c_1$ is of order p. For each integer $i > 1$ we pick x_i, y_i from

$$G_i = C_G\langle x_1, y_1, \dots, x_{i-1}, y_{i-1}\rangle$$

such that $1 \neq [x_i, y_i] = c_i$ is of order p^i. This is possible since G is FC, so G_i has finite index in G, and hence the exponent of

$$G_iZ(G)/Z(G) = G_i/Z(G)$$

is not finite. Now let $z_1 = x_1$ and for $i > 1$ choose $z_i = y_{i-1}x_i$. Then

$$z_1 \dots z_n = x_1y_1x_2y_2 \dots y_{n-1}x_n = x_1x_2 \dots x_ny_1 \dots y_{n-1}.$$

Suppose that $\sigma \neq 1$ and consider $z_{\sigma(1)} \dots z_{\sigma(n)}$. Observe that

$$[z_i, z_j] = \begin{cases} 1 & \text{if } i = j \text{ or } |i - j| > 1 \\ c_i & \text{if } j - i = 1 \\ c_{i-1}^{-1} & \text{if } i - j = 1. \end{cases}$$

Let j be the largest integer such that z_j comes to the left of z_{j-1} in the product $z_{\sigma(1)} \dots z_{\sigma(n)}$. This is the same as saying that j is the largest integer such that $\sigma^{-1}(j) < \sigma^{-1}(j-1)$. As in (i) above, we conclude that

$$z_{\sigma(1)} \dots z_{\sigma(n)} \equiv z_1 \dots z_n c_{j-1}^{-1} \bmod \Omega_{j-2},$$

where

$$\Omega_{j-2} = \langle a \in G', a^{p^{j-2}} = 1\rangle.$$

Thus

$$z_1 \dots z_n \neq z_{\sigma(1)} \dots z_{\sigma(n)}.$$

Since n is arbitrary, $G \notin \mathbf{P}_\infty$.

Proof of (2.2). Assume that $G \in \mathbf{P}_\infty$ is an FC-group. We proceed through several special cases.

(i) *Case. G is residually finite and torsion.* Suppose to the contrary that G is not finite-by-abelian. Thus G is also not abelian-by-finite. We construct a sequence $G_1, G_2, \dots$ such that $\langle G_1, G_2, \dots \rangle = G_1 \times G_2 \times \dots$. Let g_1 and g_2 be noncommuting elements of G, and define G_1 to be

$$\langle g_1, g_2 \rangle^G = \langle g_1^x, g_2^x | g_i \in G_i, x \in G \rangle.$$

By Dicman's Lemma ([**14**], 14.5.7, or [**15**], Lemma 1.3), the subgroup G_1 is a nonabelian finite normal subgroup. Suppose that the finite nonabelian subgroups $G_1, G_2, \dots, G_n$ have been defined, with $\langle G_1, \dots, G_n \rangle = G_1 \times \dots \times G_n$. Let

$$C = C_G(\langle G_1, \dots, G_n \rangle^G).$$

Since, by Dicman's Lemma,

$$\langle G_1, \dots, G_n \rangle^G = \langle g_i^x | g_i \in G_i, x \in G \rangle$$

is finite, then $|G : C|$ is finite because G is an FC-group. By residual finiteness, there is a normal subgroup H of G of finite index with

$$\langle G_1, \dots G_n \rangle \cap H = 1.$$

Then $H \cap C$ has finite index in G and so is not abelian. We can therefore find a finite nonabelian subgroup G_{n+1} of $H \cap C$ which is normal in G. It follows that

$$\langle G_1, \dots G_{n+1} \rangle = G_1 \times \dots \times G_{n+1}.$$

This construction is continued ad infinitum to produce an infinite product $K = G_1 \times G_2 \times \dots$ which is a subgroup of G. By (2.3) K is not eventually rewritable, which is a contradiction. Therefore G is finite-by-abelian.

(ii) *Case. G is a p-group which has elementary abelian derived subgroup.* Suppose again that G is not finite-by-abelian and therefore also not abelian-by-finite. It follows that G' is an infinite elementary abelian p-group. We construct a sequence of finite nonabelian subgroups $G_1, G_2, \dots$ such that $[G_i, G_j] = 1$ whenever $i \neq j$, and $\langle G_1, G_2, \dots \rangle' = G_1' \times G_2' \times \dots$. Choose a nontrivial element $h_1 = [a_{11}, b_{11}] \dots [a_{1s}, b_{1s}]$ in G', where each $[a_{1i}, b_{1i}]$ is nontrivial. Let $c_1 = [a_{11}, b_{11}]$ and define the nonabelian p-group G_1 to be $\langle a_{11}, b_{11} \rangle$. The subgroup G_1 is finite, since it is a finitely generated nilpotent torsion group. Suppose now that the finite nonabelian subgroups $G_1, \dots, G_n$ have been defined, with $[G_i, G_j] = 1$ whenever $i \neq j$ and $\langle G_1, \dots G_n \rangle' = G_1' \times \dots \times G_n'$. Let

$$H = \bigcap_{i=1}^{n} C_G(G_i).$$

H has finite index in G and therefore H is not abelian. Suppose that H is finite-by-abelian. Since G is nilpontent of class 2, H is normal in G, and thus G is finite-by-abelian-by finite. But since G is an FC-group, this implies that G is actually finite-by-abelian, a contradiction. Therefore H' is an infinite elementary abelian p-group. We can choose an element $h \in H'$ which is not in $G_1' \times \ldots \times G_n'$. Thus $h = [a_1, b_1] \ldots [a_t, b_t]$ is a product of nontrivial commutators with $a_i, b_i \in H$ and at least one commutator $[a_i, b_i]$ not in $G_1' \times \ldots \times G_n'$. Choose one such commutator $[a_j, b_j]$; since G' is elementary abelian,

$$\langle [a_j, b_j] \rangle \cap (G_1' \times \ldots \times G_n') = 1.$$

Let $c_{n+1} = [a_j, b_j]$, and define the finite nonabelian subgroup G_{n+1} of H to be $\langle a_j, b_j \rangle$. It follows that $[G_{n+1}, G_i] = 1$ for $i = 1, \ldots, n$, and

$$\langle G_1, \ldots G_{n+1} \rangle' = \langle c_1 \rangle \times \ldots \times \langle c_{n+1} \rangle = G_1' \times \ldots \times G_{n+1}'.$$

We continue this construction to produce an infinite product $K = G_1 G_2 \ldots$ of nonabelian subgroups such that $[G_i, G_j] = 1$ whenever $i \neq j$. In addition,

$$K' = \underset{i=1}{\overset{\infty}{Dr}}\, G_i'.$$

By (2.3), K is not rewritable, which is a contradiction. Therefore G is finite-by-abelian.

(iii) *Case. G is nilpotent of class 2 and torsion.* G is the direct product of its maximal p-subgroups G_p. Since by (2.5)

$$G' = \underset{p}{Dr}\, G_i'$$

has finite exponent e, say, it now follows that if p does not divide e, then $G_p' = 1$. Suppose now that G' is not infinite. Then some G_p' is an infinite abelian p-group of finite exponent. By the structure theorem of Kurilov for abelian p-groups **[14]**, G_p' is an infinite direct product of cyclic p-groups. Let

$$\bar{G}_p = G_p / (G_p')^p.$$

On the one hand,

$$(\bar{G}_p)' = G_p' / ((G_p')^p \cap G_p') = G_p' / (G_p')^p$$

is an elementary abelian p-group of infinite order. On the other hand, since $\bar{G}_p$ is a p-group with $(\bar{G}_p)'$ elementary abelian, it follows from (ii) that $\bar{G}_p$ is finite-by-abelian. This contradicts the fact that $(\bar{G}_p)'$ is infinite. Therefore each G_p' is finite, and we conclude that G' is also finite; that is, G is finite-by-abelian.

(iv) *Case. G is a general* FC $\mathbf{P}_\infty$*-group.* Suppose first that G is torsion. The group $\bar{G} = G/Z(G)$ is residually finite **[15]**, and thus by (i) it is finite-by-abelian. Therefore there is a finite normal subgroup $\bar{H}$ of $\bar{G}$ such that $\bar{G}/\bar{H}$ is abelian. Using residual finiteness, let

$$\bar{N} = \bigcap_{\bar{g}\in\bar{H}} \bar{N}_g,$$

where each $\bar{N}_g$ is a normal subgroup of $\bar{G}$ of finite index not containing $\bar{g}$. $\bar{N}$ is normal and has finite index in $\bar{G}$, and meets $\bar{H}$ trivially. From $\bar{N}\cap\bar{H} = 1$ we obtain

$$\bar{N} \cong \bar{N}\bar{H}/\bar{H} \leqq \bar{G}/\bar{H},$$

and therefore $\bar{N}$ is abelian. There is a normal subgroup N of G such that $\bar{N} = N/Z(G)$. Since $Z(N) \geqq Z(G)$, it follows that $N/Z(N)$ is abelian, and thus that N is nilpotent of class at most 2. By (iii), N is finite-by-abelian and therefore G is finite-by-abelian-by-finite, and hence finite-by-abelian [**2**].

Finally, consider a general FC $\mathbf{P}_\infty$-group G. The group $G/Z(G)$ is torsion [**15**], and thus using Zorn's Lemma we can find a maximal torsion-free subgroup H of $Z(G)$. Then H is normal in G and $Z(G)/H$ is a torsion group. Therefore G/H is a torsion group, and so, by the above, it is finite-by-abelian; that is, $(G/H)'$ is finite. Since G' is torsion [**15**],

$$(G/H)' = G'/(G'\cap H) = G'.$$

Therefore G is finite-by-abelian, and the proof is complete.

3. $\mathbf{Q}_\infty$-groups. Lemma 3.1 reflects the essential difference between $\mathbf{Q}$-groups and $\mathbf{Q}_\infty$-groups.

LEMMA 3.1. *Every* FC-*group is eventually rewritable.*

Proof. Let G be an FC-group and $x = x_1, x_2, x_3, \ldots$ be an infinite sequence of elements in G. Let

$$c_r = [x^{-1}, (x_2 \ldots x_r)^{-1}].$$

Since x has finitely many conjugates, $c_i = c_j$ for some $1 < i < j$. Hence

$$x_1 \ldots x_j = c_i x_2 \ldots x_i x_1 x_{i+1} \ldots x_j = c_j x_2 \ldots x_j x_1;$$

that is

$$x_2 \ldots x_i x_1 x_{i+1} \ldots x_j = x_2 \ldots x_j x_1.$$

Thus $G \in \mathbf{Q}_\infty$.

LEMMA 3.2. *Let H be a subgroup of finite index in the group G. Suppose that H is eventually rewritable. Then G is eventually rewritable.*

Proof. Let $x_1, x_2, \ldots$ be a sequence of elements of G. Consider the cosets $H, x_1H, x_1x_2H, \ldots$. Since $|G : H|$ is finite, there is some coset in the list above which appears infinitely often, that is, there is a sequence $0 \leqq i_1 < i_2 < \ldots$ such that $x_1 \ldots x_{i_1}H = x_1 \ldots x_{i_2}H = \ldots$ (if $i_1 = 0$, the first coset is H). The elements $u_1 = x_{i_1+1} \ldots x_{i_2}, u_2 = x_{i_2+1} \ldots x_{i_3}, \ldots$ therefore belong to H. Since $H \in \mathbf{Q}_\infty$, there is an n and purmutations $\sigma \neq \tau \in S_n$ such that

$$u_{\sigma(1)} \ldots u_{\sigma(n)} = u_{\tau(1)} \ldots u_{\tau(n)}.$$

Thus

$$x_1 \ldots x_{i_1} u_{\sigma(1)} \ldots u_{\sigma(n)} = x_1 \ldots x_{i_1} u_{\tau(1)} \ldots u_{\tau(n)}$$

shows that the subset $\{x_1, x_2, \ldots, x_{i_{n+1}}\}$ rewrites, and hence $G \in \mathbf{Q}_\infty$.

Thus every $\mathbf{Q}_\infty$-by-finite group is a $\mathbf{Q}_\infty$-group; in particular, every FC-by-finite group is eventually rewritable.

LEMMA 3.3. *Suppose that G is an eventually rewritable group. Then the* FC-*center F of G has finite index in G.*

Proof. Choose a sequence $x_1, x_2, \ldots$ of elements of G in the following way:
(i) let $x_1 \in G \backslash F$
(ii) for $j \geqq 2$, let x_j be an element of

$$G \backslash \{F \cup x_{i_1}^{-1} \ldots x_{i_r}^{-1} F \mid (i_1, \ldots, i_r) \text{ is an arrangement chosen from } \{1, \ldots, j-1\},\ 1 \leqq r \leqq j-1\}$$

such that $x_{\sigma(1)} \ldots x_{\sigma(j)} = x_{\tau(1)} \ldots x_{\tau(j)}$ only if $\sigma = \tau \in S_j$.

This sequence must stop, say at $x_1, x_2, \ldots, x_m$, since $G \in \mathbf{Q}_\infty$. Thus we assume that $x_1, x_2, \ldots, x_m$ is a maximal sequence of this type. The remainder of the proof follows (3.1) of [**2**].

For each pair of permutations $\sigma \neq \tau$ of S_{m+1}, let $r = \sigma^{-1}(m+1)$ and $s = \tau^{-1}(m+1)$, and define

$$d(\sigma, \tau) = \begin{cases} (x_{\sigma(r+1)} \ldots x_{\sigma(m+1)})(x_{\tau(s+1)} \ldots x_{\tau(m+1)})^{-1} & \text{if } r \neq m+1,\ s \neq m+1 \\ (x_{\tau(s+1)} \ldots x_{\tau(m+1)})^{-1} & \text{if } r = m+1,\ s \neq m+1 \\ x_{\sigma(r+1)} \ldots x_{\sigma(m+1)} & \text{if } r \neq m+1,\ s = m+1 \\ 1 & \text{if } r = m+1,\ s = m+1 \end{cases}$$

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and

$$e(\sigma,\tau) = \begin{cases} (x_{\sigma(1)}\dots x_{\sigma(r-1)})^{-1}(x_{\tau(1)}\dots x_{\tau(s-1)}) & \text{if } r \neq 1,\ s \neq 1 \\ x_{\tau(1)}\dots x_{\tau(s-1)} & \text{if } r = 1,\ s \neq 1 \\ (x_{\sigma(1)}\dots x_{\sigma(r-1)})^{-1} & \text{if } r \neq 1,\ s = 1 \\ 1 & \text{if } r = 1,\ s = 1. \end{cases}$$

For the sake of brevity we shall express $d(\sigma,\tau)$ and $e(\sigma,\tau)$ in the first form (that is, for $r \neq m+1$ and $s \neq m+1$, and for $r \neq 1$ and $s \neq 1$, respectively). For $a \in G$, note that

$$x_{\sigma(1)}\dots x_{\sigma(r-1)}ax_{\sigma(r+1)}\dots x_{\sigma(m+1)} = x_{\tau(1)}\dots x_{\tau(s-1)}ax_{\tau(s+1)}\dots x_{\tau(m+1)}$$

is equivalent to $ad(\sigma,\tau)a^{-1} = e(\sigma,\tau)$.

Let $\mathfrak{S}$ be the set of all sequences of distinct pairs (σ,τ) of permutations $\sigma \neq \tau$ in S_{m+1}, together with the sequence ( ) of length zero; each sequence has length at most

$$f_m = (m+1)![(m+1)! - 1].$$

If $s \in \mathfrak{S}$, then let $l(s)$ be the length of the sequence s, let $s(i)$ be the ith term of s, and let s^- be the subsequence $(s(1),\dots s(l(s)-1))$ of s when $l(s) > 1$, and ( ) when $l(s) = 1$. Corresponding to each sequence $s = ((\sigma_1,\tau_1),\dots,(\sigma_k,\tau_k))$ in $\mathfrak{S}$ we define a sequence $t(s)$ such that either $t(s) = (\ )$, or $t(s) = (a_1(s),\dots,a_k(s))$, where

$$a_i(s)d(\sigma_i,\tau_i)a_i(s)^{-1} = e(\sigma_i,\tau_i) \quad \text{for } 1 \leqq i \leqq k,$$

$$a_i(s) \in \bigcap_{v=1}^{i-1} C_G(d(\sigma_v,\tau_v)) \quad \text{for } 2 \leqq i \leqq k,$$

and $a_i(s) = a_i(s^-)$ for $1 \leqq i \leqq k-1$, when $k \geqq 2$. The elements of

$$\mathfrak{T} = \{t(s) | s \in \mathfrak{S}\}$$

are constructed in order of increasing length of s. For $s = (\ )$, define $t(s)$ to be ( ). Let $s = ((\sigma_1,\tau_1))$ for $\sigma_1 \neq \tau_1$ in S_{m+1}. If there is an element $a \in G$ such that

$$ad(\sigma_1,\tau_1)a^{-1} = e(\sigma_1,\tau_1),$$

then choose one such element and call it $a_1(s)$. In this case, define $t(s)$ to be $(a_1(s))$. If there is no such element $a \in G$, then define $t(s)$ to be ( ). Suppose that $t(s)$ has been defined for all sequences $s \in \mathfrak{S}$ of length at most $k-1$, where $k \geqq 2$. Let $s = ((\sigma_1,\tau_1),\dots,(\sigma_k,\tau_k))$ be a sequence in $\mathfrak{S}$ of length k, where $k < f_m$. Since the sequence $s^- \in \mathfrak{S}$ has length $k-1$, the sequence $t(s^-)$

has been defined. If $t(s^-) = (\ )$, then define $t(s)$ to be $(\ )$ also. Suppose that $t(s^-) \neq (\ )$. If there exists an element $a \in G$ such that

$$ad(\sigma_k, \tau_k)a^{-1} = e(\sigma_k, \tau_k) \quad \text{and}$$

$$a \in \bigcap_{v=1}^{k-1} C_G(d(\sigma_v, \tau_v)),$$

then choose one such element, call it $a_k(s)$, and define $t(s)$ to be $(a_1(s^-), \dots, a_{k-1}(s^-), a_k(s))$. If no such $a \in G$ exists, then define $t(s)$ to be $(\ )$. The set $\mathfrak{T}$ has now been defined inductively.

For each $s \in \mathfrak{S}$ for which $t(s) \neq (\ )$, let

$$u(s) = \prod_{i=1}^{l(t(s))} a_i(s).$$

Let $u(s) = 1$ when $t(s) = (\ )$. Take S to be the set

$$\{(x_{j_1} \dots x_{j_k})^{-1} | (j_1, \dots, j_k) \text{ is an arrangement chosen from } \{1, \dots, m\},\ 1 \leqq k \leqq m\} \cup \{1\}.$$

Let

$$T = \bigcup_{s \in \mathfrak{S}} u(s)S.$$

Suppose that $G \neq TF$. We shall show that this leads to a contraction. Let

$$V = TF \bigcup \bigcup_{(j_1, \dots, j_k)} TC_G(x_{j_1} \dots x_{j_k})$$

where $(j_1, \dots, j_k)$ ranges over all arrangements chosen from $\{1, \dots, m\}$, $1 \leqq k \leqq m$.

Suppose that $G = V$. As in (2.1), each $x_{j_1} \dots x_{j_k} \notin F$, and therefore each $C_G(x_{j_1} \dots x_{j_k})$ has infinite index in G. Since $G = V$ is a finite union, we may, as in (2.1), discard those cosets of infinite index. In other words, $G = TF$, which is a contradiction. Consequently, $G \neq V$.

We now construct a sequence $(g_0, g_1, g_2, \dots)$ of elements of G which has associated with it a sequence $(s_0, s_1, s_2, \dots)$ of elements of $\mathfrak{S}$, such that

$$g_0 = u(s_k)g_k \quad \text{for } k = 0, 1, 2, \dots,$$

where

$$g_{k-1}d(\sigma_k, \tau_k)g_{k-1}^{-1} = e(\sigma_k, \tau_k) \quad \text{for } k = 1, 2, \dots,$$

$$g_k \in \bigcap_{\nu=1}^{k} C_G(d(\sigma_\nu, \tau_\nu)) \quad \text{for } k = 1, 2, \ldots,$$

$$g_k \notin SF \quad \text{for } k = 0, 1, 2, \ldots,$$

$$s_0 = (\ ) \text{ and } s_k = ((\sigma_1, \tau_1), \ldots, (\sigma_k, \tau_k)) \quad \text{for } k = 1, 2, \ldots.$$

and

$$s_k^- = s_{k-1} \quad \text{for } k = 1, 2, \ldots.$$

Choose an element $g_0 \in G \backslash V$, and let $s_0 = (\ )$. The element g_0 and sequence s_0 satisfy the conditions above for $k = 0$. Suppose that the first q terms of the sequences $(g_0, g_1, g_2, \ldots)$ and $(s_0, s_1, s_2, \ldots)$ have been defined and they satisfy the conditions above for $k = 0, \ldots, q-1$. Let $x_{m+1} = g_{q-1}$, and consider the $(m+1)$-tuple $(x_1, \ldots, x_m, x_{m+1})$. By the maximality of $(x_1, \ldots, x_m)$, either $\{x_1, \ldots, x_m, x_{m+1}\}$ is rewritable, or

$$x_{m+1} \in \{F \cup x_{i_1}^{-1} \ldots x_{i_r}^{-1} F | (i_1, \ldots, i_r) \text{ is an arrangement chosen from } \{1, \ldots, i-1\},\ 1 \leqq r \leqq i-1\}.$$

But the latter implies that $x_{m+1} = g_{q-1} \in SF$, which is a contradiction. It follows that there are permutations $\sigma_q \neq \tau_q$ in S_{m+1} such that

$$g_{q-1} d(\sigma_q, \tau_q) g_{q-1}^{-1} = e(\sigma_q, \tau_q).$$

Unless $(\sigma_q, \tau_q) = (\sigma_j, \tau_j)$ for some $1 \leqq j \leqq q-1$, we continue by defining g_q and s_q. Since

$$g_{q-1} \in \bigcap_{\nu=1}^{q-1} C_G(d(\sigma_\nu, \tau_\nu)) \quad \text{whenever } q > 1,$$

we have already defined $t(s_q)$ of nonzero length in $\mathfrak{T}$ corresponding to

$$s_q = ((\sigma_1, \tau_1), \ldots, (\sigma_q, \tau_q))$$

in $\mathfrak{S}$. In fact, we have $t(s_q) = (a_1(s_q), \ldots, a_q(s_q))$, where

$$a_i(s_q) = a_i(s_{q-1}) \text{ for } 1 \leqq i \leqq q-1,$$

$$a_q(s_q) d(\sigma_q, \tau_q) a_q(s_q)^{-1} = e(\sigma_q, \tau_q)$$

and

$$a_q(s_q) \in \bigcap_{\nu=1}^{q-1} C_G(d(\sigma_\nu, \tau_\nu)) \quad \text{whenever } q > 1.$$

It follows that $g_{q-1} = a_q(s_q)g_q$, where $g_q \in C_G(d(\sigma_q, \tau_q))$. Moreover, since both g_{q-1} and $a_q(s_q)$ are elements of

$$\bigcap_{v=1}^{q-1} C_G(d(\sigma_v, \tau_v)) \quad \text{whenever } q > 1,$$

we have that

$$g_q \in \bigcap_{v=1}^{q} C_G(d(\sigma_v, \tau_v)).$$

Therefore,

$$\begin{aligned} g_0 &= u(s_{q-1})g_{q-1} \\ &= a_1(s_{q-1}) \dots a_{q-1}(s_{q-1})a_q(s_q)g_q \\ &= a_1(s_q) \dots a_q(s_q)g_q \\ &= u(s_q)g_q. \end{aligned}$$

Finally, should $g_q \in SF$, then $g_0 \in u(s_q)SF$, which contradicts $g_0 \notin V$. The sequences $(g_0, g_1, g_2, \dots)$ and $(s_0, s_1, s_2, \dots)$ have now been defined inductively.

Construction of the sequences $(g_0, g_1, g_2, \dots)$ and $(s_0, s_1, s_2, \dots)$ halts when $(\sigma_j, \tau_j) = (\sigma_N, \tau_N)$ for some $j < N$; this occurs for some $N \leqq f_m + 1$. To simplify the notation, we write σ and τ in place of σ_N and τ_N, and let $r = \sigma^{-1}(m+1)$ and $s = \tau^{-1}(m+1)$. By definition,

$$d(\sigma_j, \tau_j) = d(\sigma, \tau) \quad \text{and} \quad e(\sigma_j, \tau_j) = e(\sigma, \tau).$$

Furthermore,

$$g_{N-1} \in \bigcap_{v=1}^{N-1} C_G(d(\sigma_v, \tau_v)).$$

In particular,

$$g_{N-1}d(\sigma, \tau)g_{N-1}^{-1} = d(\sigma, \tau).$$

Since $g_{N-1}d(\sigma, \tau)g_{N-1}^{-1} = e(\sigma, \tau)$ by construction, we conclude that $d(\sigma, \tau) = e(\sigma, \tau)$, that is,

$$x_{\sigma(r+1)} \dots x_{\sigma(m+1)}(x_{\tau(s+1)} \dots x_{\tau(m+1)})^{-1} = x_{\sigma(1)} \dots x_{\sigma(r-1)})^{-1}x_{\tau(1)} \dots x_{\tau(s-1)}.$$

We may assume that $r \leqq s$. Rearranging the equation above gives

$$x_{\sigma(1)} \dots x_{\sigma(r-1)}x_{\sigma(r+1)} \dots x_{\sigma(m+1)} = x_{\tau(1)} \dots x_{\tau(s-1)}x_{\tau(s+1)} \dots x_{\tau(m+1)}.$$

If $m \geqq 2$, this expression is a rewriting of $\{x_1, \dots, x_m\}$ and so it follows that

$$(\sigma(1), \dots, \sigma(r-1), \sigma(r+1), \dots, \sigma(m+1))$$
$$= (\tau(1), \dots, \tau(s-1), \tau(s+1), \dots, \tau(m+1)).$$

Therefore $r \neq s$. If $m = 1$, then we must have $r = 1$ and $s = 2$, corresponding to $\sigma = (1, 2)$ and $\tau = 1$. Using $r < s$, we have $\sigma(l) = \tau(l)$ for $1 \leqq l \leqq r - 1$ and for $s + 1 \leqq l \leqq m + 1$. We can now simplify $d(\sigma, \tau)$:

$$d(\sigma, \tau) = x_{\sigma(r+1)} \dots x_{\sigma(s)} x_{\sigma(s+1)} \dots x_{\sigma(m+1)} (x_{\sigma(s+1)} \dots x_{\sigma(m+1)})^{-1}$$
$$= x_{\sigma(r+1)} \dots x_{\sigma(s)}.$$

Since g_{N-1} centralizes $d(\sigma, \tau)$ and $g_0 = u(s_{N-1})g_{N-1}$, it now follows that

$$g_0 \in u(s_{N-1}) C_G(x_{\sigma(r+1)} \dots x_{\sigma(s)}).$$

But this contradicts $g_0 \notin V$, and therefore we conclude that $G = TF$. Since T is a finite set, the proof is complete.

Proposition 2 has not been proved.

COROLLARY 3.4. *Every finitely-generated* $\mathbf{Q}_\infty$*-group* G *is a* $\mathbf{Q}$*-group.*

Proof. Let F be the FC-center of G. Since it has finite index in a finitely generated group, F is finitely-generated and hence is abelian-by-finite [**15**]. Therefore G is finite-by-abelian-by-finite, that is, a $\mathbf{Q}$-group.

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