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AUTOMORPHISMS OF FUCHSIAN GROUPS AND THEIR LIFTS TO FREE GROUPS

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0. Introduction. This paper has been motivated by earlier work of the first two authors (see [3]), where distinct Nielsen classes of generating systems for a Fuchsian group

$$G = \langle s_1, \dots, s_m, a_1, b_1, \dots, a_g, b_g \mid s_i^{\pi(i)} (i = 1, \dots, m), \\ s_1 s_2 \dots s_m [a_1, b_1] \dots [a_g, b_g] \rangle$$

have been established and, in the case of odd and pairwise relative prime exponents $\pi(i)$, classified. As a consequence they could distinguish non-isotopic Heegaard decompositions of Seifert fibred 3-manifolds. In proving that these decompositions are actually non-homeomorphic (see [3], Section 2), they investigated the question whether the different Nielsen classes of generating systems for G remain distinct, if one passes over to the weaker notion of "Nielsen equivalence up to automorphisms" (see [12], p. 3.5, 4.11 a-c): this means that the automorphisms of G are added to the Nielsen equivalence relations on the generators.

This question is equivalent to asking whether every automorphism h of G lifts to an automorphism $\mathscr{H}$ of the free group F_{2g+m-1} with respect to any surjection $F_{2g+m-1} \twoheadrightarrow G$ that describes one of the given Nielsen classes. Our theorem below gives an affirmative answer to the latter question, even in a more general situation than considered [3]: we require only that the exponents $\pi(i)$ in the above presentation of G are pairwise distinct (also G may come from a non-orientable surface as listed in (*) below). The last restriction is necessary in order to get the conclusion of our theorem in fully generality:

Example (communication from M. Boileau/H. Zieschang). Given

$$G = \langle s_1, s_2, s_3 \mid s_1^p, s_2^p, s_3^q, s_1 s_2 s_3 \rangle$$

and $\beta \in \mathbf{Z}$ relatively prime to p with $\beta \neq 1, -1 \bmod p$. Consider the generating systems s_1^β, s_2 and s_1, s_2^β of G. The automorphism $h: G \to G$, defined by $h(s_1) = s_2$, $h(s_2) = s_1$ (and hence $h(s_3) = s_2 s_3 s_2^{-1}$) interchanges these two generating systems. However, an easy computation, using Theo-

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rem 5.4 in [**4**], p. 256, shows that the commutators $[s_1^\beta, s_2]$ and $[s_1, s_2^\beta]$ are not conjugated in G and hence the two systems are not Nielsen equivalent (see [**4**], p. 44).

The main result presented in this paper has been obtained in parallel work by the first two authors and by the third author. It extends earlier work of the latter (see [**8**]), who had proved that an automorphism $\hbar$ of G is induced by an automorphism $\mathscr{H}$ of the free group F_{2g+m-1} with respect to some epimorphism $F_{2g+m-1} \twoheadrightarrow G$. The analysis of his proof (in [**8**]) shows that this epimorphism defines always the Nielsen equivalence class of a standard generating system $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m, a_1, b_1, \ldots, a_g, b_g$ (i.e., with trivial exponents $\beta(i)$). However, an extension of this proof to the other generating systems with nontrivial $\beta(i)$ is possible. This is provided here in form of an Appendix (due to the third author) as an alternative proof of the theorem.

The work of the first two authors is given in Section 1. Their proof of the theorem is short and describes a new and rather elementary approach which yields a slightly stronger result (see the Corollary). On the other hand, the "classical" method of proof (used in [**8**] and here in the Appendix), is based on the difficult papers [**12**] and [**6**] (see also [**7**]), and seems somewhat deeper. In particular it yields Satz 2.2. of [**8**], which states that, for $g \geqq 1$ and $m \geqq 2$, any generating system of G with $2g + m - 1$ elements is Nielsen equivalent to one of the generating systems

$$x_1 = s_1^{\beta(1)}, \ldots, x_{k-1} = s_{k-1}^{\beta(k-1)}, x_{k+1} = s_{k+1}^{\beta(k+1)}, \ldots, x_m = s_m^{\beta(m)},$$
$$a_1, b_1, \ldots, a_g, b_g$$

considered here. (See also the discussion given in the Appendix after the alternative proof of the theorem.)

1. Results. The proofs of this section are based on the following classical result of H. Zieschang (see [**11**], p. 165, Satz 9):

(*) *Given a Fuchsian group*

$$G = \langle s_1, \ldots, s_m, a_1, \ldots, a_p | s_i^{\pi(i)} \ (i = 1, \ldots, m), s_1 s_2 \ldots s_m \Pi(a_j) \rangle$$

with all $\pi(i) \geqq 2$, *which satisfies either*

(a) (*orientable case*)

$$\Pi(a_j) = [a_1, a_2][a_3, a_4] \ldots [a_{p-1}, a_p] \quad (p \textit{ even}),$$

and conditions:

(i) $p \geqq 2$ *or*

(ii) $m \geqq 4$ *or*

(iii) $p = 0$, $m = 3$ *and*

$1/\pi(1) + 1/\pi(2) + 1/\pi(3) \leqq 1,$

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or

(b) (*non-orientable case*)

$$\Pi(a_j) = a_1^2 a_2^2 \dots a_p^2$$

and conditions:

(i) $p \geqq 2$ *or*

(ii) $p = 1$ *and* $m \geqq 2$.

Then every automorphism $\hbar: G \to G$ *is induced by an automorphism* $\mathscr{H}$ *of the free group*

$$F = F(S_1, \dots, S_m, A_1, \dots, A_p)$$

with respect to the epimorphism $F \twoheadrightarrow G$ *given by* $S_i \to s_i$, $A_j \to a_j$. *The automorphism* $\mathscr{H}$ *satisfies furthermore*

1. $\mathscr{H}(S_i) = C_i S_{\sigma(i)}^{t(i)} C_i^{-1}$ (*for* $i = 1, \dots, m$) *and*

2. $\mathscr{H}(S_1 S_2 \dots S_m \Pi(A_j)) = C(S_1 S_2 \dots S_m \Pi(A_j))^t C^{-1}$

for some $t, t_1, \dots, t_m = \pm 1$, $C, C_i \in F$, *and a permutation* $\sigma: i \to \sigma(i)$.

For the proof of our theorem we need the following slight generalization of this fact:

LEMMA. *Let G be as in* (*), *with the additional assumption* $\pi(i) \neq \pi(j)$ *for* $i \neq j$, *and consider some generating system*

$$x_1 = s_1^{\beta(1)}, \dots, x_m = s_m^{\beta(m)}, a_1, \dots, a_p$$

with $\beta(i)$ *relatively prime to* $\pi(i)$ *for all* $i = 1, \dots, m$. *Then every automorphism* $\hbar: G \to G$ *is induced by some automorphism* $\mathscr{H}^*$ *of the free group*

$$F^* = F(X_1, \dots, X_m, A_1, \dots, A_p)$$

with respect to the epimorphism $F^* \twoheadrightarrow G$ *defined by* $X_i \to x_i$, $A_j \to a_j$. *Here* $\mathscr{H}^*$ *satisfies*:

1. $\mathscr{H}^*(X_i) = C_i X_i^{t(i)} C_i^{-1}$ ($i = 1, \dots, m$) *and*

2. $\mathscr{H}^*(X_1^{\mu(1)} X_2^{\mu(2)} \dots X_m^{\mu(m)} \Pi(A_j))$
$$= C(X_1^{\mu(1)} X_2^{\mu(2)} \dots X_m^{\mu(m)} \Pi(A_j))^t C^{-1}$$

for some $t, t(1), \dots, t(m) = \pm 1$, $C, C_i \in F^*$ *and integers* $\mu(i)$ *with*

$$\mu(i)\beta(i) \equiv 1 \bmod \pi(i)$$

and $0 \leqq \mu(i) < \pi(i)$ *for all* $i = 1, \dots, m$.

Note. From the proof below it is immediate that the condition $0 \leqq \mu(i) < \pi(i)$ may be replaced by $1_i \leqq \mu(i) < 1_i + \pi(i)$ for any integer 1_i,

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although that might lead to different automorphisms $\mathscr{H}^*$.

Proof. Let $\hbar:G \rightarrow G$. We extend the automorphism $\mathscr{H}$ of $F = F(S_1, \ldots, S_m, A_1, \ldots, A_p)$, provided by (*), to an automorphism $\mathscr{H}^*$ of the amalgamated product

$$F^* = F * F(X_1, \ldots, X_m)/\langle S_i = X_i^{\mu(i)} \quad (i = 1, \ldots, m) \rangle$$

as follows: We define

$$\mathscr{H}^*(X_i) = C_i X_i^{t(i)} C_i^{-1},$$

using the data $t(i)$ and C_i given in (*). The permutation σ is trivial, since all x_i have distinct order. One verifies easily:

a) F^* is a free group with basis $X_1, \ldots, X_m, A_1, \ldots, A_p$

b) $\mathscr{H}^*(X_1^{\mu(1)} \ldots X_m^{\mu(m)}\Pi(A_j)) = \mathscr{H}(S_1 \ldots S_m \Pi(A_j))$

$$= C(S_1 \ldots S_m \Pi(A_j))^t C^{-1} = C(X_1^{\mu(1)} \ldots X_m^{\mu(m)} \Pi(A_j))^t C^{-1}$$

c) $\mathscr{H}^*$ induces $\hbar$ with respect to the epimorphism $F^* \twoheadrightarrow G$ defined by $X_i \rightarrow x_i$, $A_j \rightarrow a_j$ (which gives $S_i = X_i^{\mu(i)} \rightarrow x_i^{\mu(i)} = s_i^{\beta(i)\mu(i)} = s_i$): There exist integers $q(i)$ such that

$$\begin{aligned}\mathscr{H}^*(X_i) &= C_i X_i^{t(i)} C_i^{-1} \\ &= C_i X_i^{t(i)(\beta(i)\mu(i) + q(i)\pi(i))} C_i^{-1} \\ &= (C_i (X_i^{\mu(i)})^{t(i)} C_i^{-1})^{\beta(i)} (C_i X_i^{\pi(i)} C_i^{-1})^{t(i)q(i)} \\ &\rightarrow \hbar(s_i)^{\beta(i)} = \hbar(x_i).\end{aligned}$$

For later use we would like to point out that the triviality of the permutation σ has been used only for part c); it is easy to see that the hypothesis of pairwise distinct $\pi(i)$ as stated in the Lemma may be weakened to

$$\beta(\sigma(i)) = \beta(i) \quad \text{for } i = 1, \ldots, m.$$

We would like to point out that the statement of the Lemma without conditions 1 and 2 (which we need for the following theorem) is a trivial consequence of fact (*), since the generating systems $x_1, \ldots, x_m$, $a_1, \ldots, a_p$ and $s_1, \ldots, s_m$, $a_1, \ldots, a_p$ of G are Nielsen equivalent. This follows inductively from the fact that any system $x_1, \ldots, x_i, s_{i+1}, \ldots, s_m$, $a_1, \ldots, a_p$ can be transformed by elementary Nielsen operations into $x_1, \ldots, x_{i-1}$, $s_i, \ldots, s_m$, $a_1, \ldots, a_p$ (since the elements $x_1, \ldots, x_{i-1}$, $s_{i+1}, \ldots, s_m$, $a_1, \ldots, a_p$ clearly constitute a generating system for G).

We can now prove:

THEOREM. *Let G be given as in (*), with $m \geqq 2$ and pairwise distinct $\pi(i)$. Consider any generating system*

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$$x_1 = s_1^{\beta(i)}, \dots, x_{k-1} = s_{k-1}^{\beta(k-1)}, x_{k+1} = s_{k+1}^{\beta(k+1)}, \dots, x_m = s_m^{\beta(m)},$$
$$a_1, \dots, a_p \text{ of } G,$$

with relatively prime pairs $\beta(i)$, $\pi(i)$ *for* $i = 1, \dots, k-1, k+1, \dots, m$. *Then any automorphism* $h{:}G \rightarrow G$ *is induced by some automorphism* $\mathscr{H}^\#$ *of the free group*

$$F^\# = F(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_m, A_1, \dots, A_p)$$

with respect to the epimorphism $F^\# \twoheadrightarrow G$ *given by* $X_i \rightarrow x_i$, $A_j \rightarrow a_j$.

(Note that we can omit the assumption that the $\beta(i)$, $\pi(i)$ are pairwise relatively prime. If this were not the case then some quotient group $G/\langle s_i^{\beta(i)}\rangle$ would have smaller rank than determined in [**6**].)

Proof. Apply the Lemma to the enlarged generating system $x_1, \dots, x_{k-1}$, $x_k = s_k$, $x_{k+1}, \dots, x_m$, $a_1, \dots, a_p$, and consider the diagram:

$$\begin{tikzcd}
F^* \arrow[rrr, "\mathscr{H}^*"] \arrow[dr] \arrow[dd] & & & F^* \arrow[dr] & \\
& F^*/\langle\Pi\rangle \arrow[rrr, dashed, "\mathscr{H}^\#"] \arrow[dl, "\varphi"] & & & F^*/\langle\Pi\rangle \arrow[dl, "\varphi"] \\
G \arrow[rrr, "\mathscr{H}"] & & & G &
\end{tikzcd}$$

Here F^* and $\mathscr{H}^*$ are given in the Lemma, and Π denotes the element

$$X_1^{\mu(1)} X_2^{\mu(2)} \dots X_m^{\mu(m)} \Pi(A_j)$$

in F^*, exhibited there. By statement 2 of the Lemma the dashed map $\mathscr{H}^\#$ in the diagram is well defined and an automorphism, since $\mathscr{H}^*$ is. From the condition $x_k = s_k$ and the normalization condition $0 \leqq \mu(i) < \pi(i)$ in the Lemma it follows that $\mu(k)$ equals to 1. Hence $X_1, \dots, X_{k-1}$, Π, $X_{k+1}, \dots, X_m$, $A_1, \dots, A_p$ constitute a basis for F^*, and $F^*/\langle\Pi\rangle$ is naturally isomorphic to the free group

$$F^\# = F(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_m, A_1, \dots, A_p)$$

of rank $m + p - 1$. Since the element Π of F^* projects onto $1 \in G$, the (vertical) epimorphism $F^* \twoheadrightarrow G$ (given by $X_i \rightarrow x_i$, $A_j \rightarrow a_j$) induces an epimorphism $\varphi{:}F^\# \twoheadrightarrow G$ which is described by $\varphi(X_i) = x_i$, $\varphi(A_i) = a_i$ (indicated in the above diagram). The automorphism $\mathscr{H}^\#$ induces the given map $h{:}G \rightarrow G$ since by construction all parts of the diagram commute.

COROLLARY. *The automorphism* $\mathscr{H}^\#{:}F^\# \rightarrow F^\#$ *given by the above theorem satisfies furthermore*:

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1. $\mathscr{H}^{\#}(X_i) = C_i X_i^{t(i)} C_i^{-1}$

(for $i = 1, \ldots, k-1, k+1, \ldots, m$*) and*

2. $\mathscr{H}^{\#}(X_{k+1}^{\mu(k+1)} \ldots X_m^{\mu(m)} \Pi(A_j) X_1^{\mu(1)} \ldots X_{k-1}^{\mu(k-1)})$

$$= C_k (X_{k+1}^{\mu(k+1)} \ldots X_m^{\mu(m)} \Pi(A_j) X_1^{\mu(1)} \ldots X_{k-1}^{\mu(k-1)})^{t(k)} C_k^{-1}$$

for some $t(1), \ldots, t(m) = \pm 1$, $C_i \in F^{\#}$ *and integers* $\mu(i)$ *as stated in the Lemma.*

Proof. The map $F^* \to F^*/\langle \Pi \rangle = F^{\#}$ given in the last proof is described by $X_i \to X_i$ (for $i \neq k$), $A_j \to A_j$ and

$$X_k = X_k^{\mu(k)} \to (X_{k+1}^{\mu(k+1)} \ldots X_m^{\mu(m)} \Pi(A_j) X_1^{\mu(1)} \ldots X_{k-1}^{\mu(k-1)})^{-1}$$

(since $\Pi \to 1$). Hence condition 1 of the Lemma gives immediately the statements of the Corollary.

The above presented line of proof seems to capture (despite its elementaricity) the strongest possible result in the desired direction. In fact, it gives immediately the analogous result for a general cocompact Fuchsian group without reflections (i.e., the $\pi(i)$ do not necessarily need to be distinct), if one assumes that $\mathscr{h}(s_i)$ is conjugated to s_i for all i.

Without this assumption the proof of our theorem still gives the lifting result of **[8]** quoted in the introduction, since it always applies to generating systems $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_m, a_p, \ldots, a_p$, i.e., all $\beta(i)$ are taken equal to 1. More generally, the proof of the lemma shows, that without any restrictions on the exponents $\pi(i)$ of G, for the existence of a lift $\mathscr{H}^{\#}$ of a given automorphism $\mathscr{h}$ as in the theorem one only needs the condition

$$\beta(\sigma(i)) = \beta(i) \quad \text{for all } i = 1, \ldots, m$$

to be satisfied. (Here one defines $\beta(k) = 1$ and uses the permutation σ determined by $\mathscr{h}$ as stated in (*) above.) On the other hand, it follows from the results of [**2**] that $\mathscr{h}:G \to G$ maps any generating system

$$s_1^{\beta(1)}, \ldots, s_{k-1}^{\beta(k-1)}, s_{k+1}^{\beta(k+1)}, \ldots, s_m^{\beta(m)}, a_1, \ldots, a_p,$$

which violates this last condition, into a Nielsen inequivalent generating system (see the Example in Section 0). Thus $\mathscr{h}$ cannot be lifted to any automorphism $\mathscr{H}^{\#}$.

Final remark. In the spirit of the existing terminology (see [**4**], p. 95) the authors propose the attribute "rank n fully almost quasi free" for groups that allow a lift $\mathscr{H}:F_n \to F_n$ for any automorphism $\mathscr{h}:G \to G$ and any epimorphism $F_n \twoheadrightarrow G$. Our theorem, together with Satz 2.2. in [**8**], shows that every Fuchsian group G with pairwise distinct $\pi(i)$ and $p \geqq 2$, $m \geqq 2$ is contained in this newly defined class.

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Appendix. We now give a second (independent) proof of our theorem as was promised in the introduction:

Alternative proof of the theorem. Given G as in (*) with $m \geqq 2$ and pairwise distinct $\pi(i)$. Note first that G has rank $p + m - 1$ (see [**6**]). We distinguish two cases according to the possible values for p and m.

1. Let $p = 0$ and $m = 3$.

For $1/\pi(1) + 1/\pi(2) + 1/\pi(3) < 1$ the theorem follows easily from the classification results in [**9**]. Alternatively one may consider the Teichmueller space of G, described here by the trace point

$$(2\cos(\pi/\pi(1)),\ 2\cos(\pi/\pi(2)),\ -2\cos(\pi/\pi(3))),$$

see for instance [**9**]. Pairwise distinct $\pi(i)$ yield pairwise distinct $2\cos(\pi/\pi(i))$. So any automorphism $\hbar$ of G induces the identical permutation on the triple

$$(2\cos(\pi/\pi(1)),\ 2\cos(\pi/\pi(2)),\ 2\cos(\pi/\pi(3))).$$

Hence any generating pair $s_i^{\beta(i)}$, $s_j^{\beta(j)}$ of G is fixed by $\hbar$ up to inversion and inner automorphisms.

For $1/\pi(1) + 1/\pi(2) + 1/\pi(3) = 1$ we clearly have

$$\beta(i) \equiv \pm 1 \ (\mathrm{mod}\ \pi(i)) \quad \text{and} \quad \beta(j) \equiv \pm 1 \ (\mathrm{mod}\ \pi(j))$$

for all generating pairs $s_i^{\beta(i)}$, $s_j^{\beta(j)}$ of G considered in the Theorem.

2. From now on we assume $m \geqq 4$ whenever $p = 0$.

Let

$$x_1 = g_1 s_1^{\beta(1)} g_1^{-1}, \dots, x_{k-1} = g_{k-1} s_{k-1}^{\beta(k-1)} g_{k-1}^{-1},$$
$$x_{k+1} = g_{k+1} s_{k+1}^{\beta(k+1)} g_{k+1}^{-1}, \dots, x_m = g_m s_m^{\beta(m)} g_m^{-1},$$
$$y_1, \dots, y_p$$

be a generating system of G with arbitrary elements $g_1, \dots, g_{k-1}$, $g_{k+1}, \dots, g_m$ in G. By assumption each $\beta(i)$ is relatively prime to $\pi(i)$. Hence the arguments of [**8**] apply directly (see also [**7**] for a slightly different proof and [**1**] for a refinement of these arguments) to show that our generating system is Nielsen equivalent to the generating system

$$z_1 = s_1^{\beta(1)}, \dots, z_{k-1} = s_{k-1}^{\beta(k-1)}, z_{k+1} = s_{k+1}^{\beta(k+1)}, \dots, z_m = s_m^{\beta(m)},$$
$$a_1, \dots, a_p.$$

This equivalence can be obtained by a sequence of transformations, each of which changes at most one of the x_i, replacing this x_i by a conjugate.

Every automorphism $\hbar$ maps z_i to an element

$$h_i s_i^{\gamma(i)} h_i^{-1}$$

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for some h_i in G ($i = 1, \ldots, k - 1, k + 1, \ldots, m$), since the $\pi(i)$ are pairwise distinct (by assumption). This enables us to use the corresponding arguments of [**8**], Section 3, and we obtain

$$\gamma(i) \equiv \pm\beta(i) \pmod{\pi(i)}$$

for $i = 1, \ldots, k - 1, k + 1, \ldots, m$. For this purpose we only need to notice that for each $\beta(i)$ there exists a $\delta(i)$ with

$$\beta(i)\delta(i) \equiv 1 \pmod{\pi(i)}.$$

Furthermore the transformation

$$\hbar(z_i) \rightarrow \hbar(z_i)^{-1}$$

is a Nielsen transformation, so that we obtain:

The system $\hbar(z_1), \ldots, \hbar(z_{k-1}), \hbar(z_{k+1}), \ldots, \hbar(z_m), \hbar(a_1), \ldots, \hbar(a_p)$ is Nielsen equivalent to the system $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_m, a_1, \ldots, a_p$, which proves the Theorem.

In the context of our theorem it is natural and interesting to ask, for groups G as in (*),

(1) whether there exist Nielsen classes of minimal generating systems other than the "standard" ones (represented by $x_1 = s_1^{\beta(1)}, \ldots, x_{k-1} = s_{k-1}^{\beta(k-1)}, x_{k+1} = s_{k+1}^{\beta(k+1)}, \ldots, x_m = s_m^{\beta(m)}, a_1, \ldots, a_p$), and

(2) whether for an "exotic" (= non-standard) minimal system the assertion of the Theorem holds likewise.

For $p, m \geqq 2$ Satz 2.2. of [**8**] gives a complete (negative) answer to the first (and hence to the second) question (see the above final remark). On the other hand there are exotic generating systems known for $p = 0$ and $m = 3$ (see [**9**]) and others conjectured for $m \geqq 4$, $p = 0, 1$. In order to narrow down the margin for possible non-standard phenomena we would at least like to state the following Remark; its proof is a somewhat complicated extension of the proof of the above mentioned Satz 2.2. of [**8**] (see [**10**]).

Remark. Let G be as in (*) *with $0 \leqq p \leqq 1$ and $\pi(i) \geqq 4$ for all $i = 1, \ldots, m$. Then any generating system $y_1, \ldots, y_{p+m-1}$ of G is Nielsen equivalent to some* (*standard*) *system*

$$x_1 = s_1^{\beta(1)}, \ldots, x_{k-1} = s_{k-1}^{\beta(k-1)}, x_{k+1} = s_{k+1}^{\beta(k+1)}, \ldots, x_m = s_m^{\beta(m)}, \\ a_1, \ldots, a_p,$$

with relatively prime pairs $\beta(i)$, $\pi(i)$.

(For $p = 0$ and $m \geqq 6$ a proof can be found also in [**5**], and for $p = 0$ and $m = 3$ it follows from [**9**]. For $p = 1$ the above Remark can be strengthened further to cases where one requires only $\pi(i) \geqq 3$ for at least two of the i.)

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Notice also that a group G as in (*) with $0 \leqq m \leqq 1$ has exactly one Nielsen class of minimal generating systems (see **[7]** and **[12]**).

About question (2) above we would like to point out that for the case $p = 0$, $m = 3$ and pairwise distinct $\pi(i)$ the group G is always rank 2 fully almost quasi free (defined in the above final remark), although for low $\pi(i)$ there exist frequently exotic Nielsen classes of generating pairs: These are listed in **[9]**, Theorem 1 and the succeeding Remark 2. The above claim follows from the classification given there and the known facts about Out G for rank $G = 2$.

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