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Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems 2. The smooth case

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Abstract. We prove that an arbitrary one dimensional smooth dynamical system with non-degenerate critical points has no wandering intervals.

2.1. *Introduction*

The present paper is the continuation of [1]. In [1] the absence of wandering intervals for maps with negative Schwarzian derivative is proved. Here this result is extended to the smooth case. As a rule, we will use the terminology and notations from [1] without special explanations.

Let M be a one dimensional compact manifold with boundary. Consider a class $\mathscr{A}_d$ of C^2-smooth maps $f: M \to M$ having d critical points $c_k \in \operatorname{int} M$ ('*d-modal*') and satisfying the following conditions.

(U1) In punctured neighbourhoods of the critical points the following estimates hold

$$A_1|x-c_k|^{\beta_k} \le |f'(x)| \le A_2|x-c_k|^{\beta_k}$$

where A_1, A_2, $\beta_k > 0$.

(U2) The critical points c_k are extrema.

(U3) (local). In punctured neighbourhoods of its critical points f has negative Schwarzian derivative.

In particular, a C^∞ map f with non-flat critical points satisfies (U1) and (U3) (local).

Set $\mathscr{A} = \bigcup_{d=0}^{\infty} \mathscr{A}_d$.

MAIN THEOREM (*the smooth case*). *A map $f \in \mathscr{A}$ has no wandering intervals.*

For the history of this result see [1].

A smooth map $f \in \mathscr{A}$ may have uncountably many periodic points of a given period. For that reason the spectral decomposition in the smooth case is more complicated than in the case of negative Schwarzian derivative.

COROLLARY 2.1. *Let $f \in \mathscr{A}$. Then*

$$\operatorname{Per}(f) = \bigcup A_i \cup R_j \cup S,$$

where A_i, R_j have the same sense as in ([1], *Corollary* 1.1) *and S is the union of some non-limit cycles. For every non-preperiodic* $x \in M$ *either* $\omega(x) \subset A_i$ *or* $f^n x \in R_j$ *for some* $n \in \mathbb{N}$.

COROLLARY 2.2. *For generic non-preperiodic* $x \in M$ *the limit set* $\omega(x)$ *is either a limit cycle, a transitive invariant submanifold or a solenoid.*

Remark. Clearly, the number of transitive invariant submanifolds and solenoids is finite. It is unknown whether it is true that f always has only a finite number of limit cycles.

Further the reference to, say, Lemma 1.3 (or § 1.5) means 'Lemma 1.3 from [1]' (correspondingly '§ 1.5 from[1]'). The numeration of Sections and Lemmas of the present paper is as follows: 2.1, 2.2 and so on.

2.2. *Distortion lemmas for smooth maps*

The main analytical tool in what follows will be the results due to de Melo and van Strien [2, 3]. Now we state them.

Let T, J be two closed intervals, $J \subset \operatorname{int} T$. Assume T does not contain critical points of f. Let $H^{\pm}$ be the connected components of $T \backslash J$. Set

$$D(T, J) = \frac{\lambda(J)\lambda(T)}{\lambda(H^-)\lambda(H^+)},$$

$$B(f, T, J) = \frac{D(fT, fJ)}{D(T, J)}.$$

Further in this section the sense of T, J and $H^{\pm}$ is the same as above.

Let $\theta > 0$. Denote by $\mathscr{A}^{\theta}$ the set of maps $f \in \mathscr{A}$ such that $B(f, T, J) \geq \theta$ for any T and J. Note that the maps $f \in \mathcal{O}$ with negative Schwarzian derivative lie in $\mathscr{A}^1$ [2].

MINIMUM PRINCIPLE (SMOOTH VERSION) [2]. *Let* $f \in \mathscr{A}^{\theta}$, $T = [a, b]$, $x \in \operatorname{int} T$. Then

$$|f'(x)| \geq \theta^3 \min\{|f'(a)|, |f'(b)|\}.$$

THE SECOND DISTORTION LEMMA (SMOOTH VERSION) (cf § 1.5). *Let* $f \in \mathscr{A}^{\theta}$. *Then there exists a function* $\gamma_{\theta} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that if*

$$\lambda(fH^+)/\lambda(fJ) \geq \alpha \quad \text{and} \quad \lambda(fH^-)/\lambda(fJ) \geq \alpha$$

$$\text{then} \quad \lambda(H^{\pm})/\lambda(J) \geq \gamma_{\theta}(\alpha).$$

Proof. One may assume $\lambda(T) = \lambda(fJ) = 1$, $\lambda(fH^{\pm})/\lambda(fJ) = \alpha$. Then $\lambda(fT) = 1 + 2\alpha$ and

$$\frac{1+2\alpha}{\alpha^2} = \frac{\lambda(fJ)\lambda(fT)}{\lambda(fH^+)\lambda(fH^-)} \geq \theta \frac{\lambda(J)\lambda(T)}{\lambda(H^+)\lambda(H^-)} \geq \theta \frac{\lambda(J)}{\lambda(H^{\pm})}. \qquad \square$$

Remark. One may show that the Köebe Property (see § 1.5) also holds in class $\mathscr{A}^{\theta}$.

THE THIRD DISTORTION LEMMA [2]. *Let* $f \in \mathscr{A}$. *There exists* $C = C(f)$ *such that if* $f^m \mid T$ *has no critical points and*

$$\sum_{i=0}^{m-1} \lambda(f^i T) \leq s$$

then $f^m \mid T \in \mathscr{A}^{\theta}$ *where* $\theta = \exp(-Cs^2)$.

So we have the uniform estimate of the distortion of all iterates f^m on intervals T satisfying the assumptions of the last Lemma. Further the main problem will be to verify these assumptions for appropriate intervals.

2.3. *Collections of homtervals*

Let $J_{m_0}, J_{m_1}, \ldots$ be the sequence of all nearest to critical point c homtervals. Assume that for some $k < l$ we have

(C1) $m_{k+1} - m_k = m_{k+2} - m_{k+1} = \cdots = m_l - m_{l-1} = a$;

(C2) $f^a | [J_{m_k}, J_{m_l}]$ is monotone.

In such a case we say that $\{J_{m_i}\}_{i=k}^{l}$ is *the collection of homtervals*. In particular, a collection may consist of a unique homterval. Observe that the sequence $\{J_{m_i}\}_{i=0}^{\infty}$ is divided into the union of maximal collections.

Provided $\{J_{m_i}\}_{i=k}^{l}$ is the maximal collection and $s = m_i$ for some $i \in [k, l]$, the value of min $(i-k, l-i)$ we call *the depth of* J_s (or s) and denote by $dp(J_s) = dp(s)$. In particular, for a collection $\{J_s\}$ consisting of the unique homterval we have $dp(s) = 0$.

Let H_t be some interval on which f^t is monotone, $J \subset \operatorname{int} H_t$, $H_t^{\pm}$ be the components of $H_t \backslash J$. Set $M_t = f^t H_t$, $M_t^{\pm} = f^t H_t^{\pm}$. Provided J_t lies near a critical point c, denote by M_t^- that interval which lies farther from c than J_t.

We want to apply the Distortion Lemmas to $f^t | H_t$. To this end we must estimate intersection multiplicity of intervals $f^i H_t$ $(i = 0, 1, \ldots, t)$. It is this problem that leads to the above introduced concept of the collection of homtervals.

LEMMA 2.1. *Let J_t be the nearest to c homterval. Suppose M_t does not contain any interval J_s for $s < t$. Then*

(a) $\sum_{i=0}^{t} \lambda(f^i H_t) \le (2dp(t) + 7)\lambda(M)$;

(b) *if additionally $M_t^+ \subset (J_t, J_n]$ where J_n is the nearest to c homterval following J_t, then*

$$\sum_{i=0}^{t} \lambda(f^i H_t) \le 9\lambda(M).$$

Proof. Let us define the orientation in a neighbourhood of c such that $J_t < c$. We want to estimate the intersection multiplicity of the intervals $f^i H$ $(i = 0, 1, \ldots, t)$ where $H = H_t^{\gamma}$, $\gamma = \pm 1$. Let $x \in \bigcap_{k=1}^{\varkappa} f^{i_k} H$. We may assume that intervals $f^{i_k} H$ for $k = 1, \ldots, r$ lie on the one side of x and for $k = r+1, \ldots, \varkappa$ they lie on the other side of x. We may assume also that

$$[J_{i_1}, x] \supset [J_{i_2}, x] \supset \cdots [J_{i_r}, x].$$

Then it follows from the assumption of the lemma that $i_1 < i_2 < \cdots < i_r \le t$.

Note also that if $J_m \subset [J_{i_1}, x]$ for $m \le i_r$ then $m = i_k$ for some $k = 1, \ldots, r$. Indeed, $f^{m-i_1}[J_{i_1}, x] \not\subset [J_{i_1}, x]$ since otherwise the orbit of J tends to a cycle. Besides, the assumption of the lemma implies that $f^{m-i_1}[J_{i_1}, x] \not\supset J_{i_1}$. Hence $x \in f^{m-i_1}[J_{i_1}, x] \subset f^m H$ and we are done.

Now let us estimate r. Applying f^{t-i_r} we may assume also that $i_r = t$. First let $\gamma = -1$. Suppose $r \ge 2$. Then set $K = [J_{i_1}, J_t]$, $p = t - i_1$. Since $f^p K \subset M_t^-$, we conclude that $f^p K$ lies on the same side of J_t as K. But $f^p K \not\subset K$ because otherwise the orbit of J tends to a cycle. Consequently $f^pK \supset K$ and hence $M_t^- \supset J_{i_1}$. This contradicts the assumption of the Lemma.

Thus $r \le 1$. Of course, the same estimate holds for $\varkappa - r$ and hence $\varkappa \le 2$. Consequently

$$\sum_{i=0}^{t} \lambda(f^i H_t^-) \le 2\lambda(M). \tag{2.1}$$

(ii) Further, let $\gamma = +1$, $H = H_t^+$. Suppose $r \ge 3$. Set $a = t - i_{r-1}$. Then f^{a+1} is monotone on $K \equiv [J_{i_1}, J_{i_{r-1}}] \subset f^{i_1}H$. Hence $f^aK \not\ni c$. Besides $J_{i_1+a} \not\subset [J_t, c]$ since $i_1 + a < t$. Therefore $J_{i_1+a} < J_t$. On the other hand, the assumption of the lemma implies that $f^aK \not\supset J_{i_1}$. Consequently

$$f^aK = [J_{i_1+a}, J_t] \subset (J_{i_1}, J_t].$$

The interval f^aK contains at least $r-1$ homtervals J_s with $s < t$, namely J_{i_m+a} $(m = 1, \ldots, r-1)$. On the other hand, the interval $(J_{i_1}, J_t]$ contains exactly $r-1$ such homtervals $J_{i_2}, \ldots, J_{i_r} = J_t$. Consequently $i_m + a = i_{m+1}$ $(m = 1, \ldots, r-1)$.

Further, set $i_k = t + (k-r)a$ $(k = r, r+1, \ldots)$. As $f^{t-i_1} = f^{(r-1)a}$ is monotone on $[J_{i_1}, J_t]$, f^a is monotone on

$$f^{(r-2)a}[J_{i_1}, J_t] = [J_{i_{r-1}}, J_{i_{2r-2}}].$$

Consequently, f^a is monotone on the interval $L = [J_{i_1}, J_{i_{2r-2}}]$ containing the set of homtervals $\{J_{i_m}\}_{m=1}^{2r-2}$. In particular $L \not\ni c$.

Now let us prove by induction that J_{i_k} is the $(i_{k+1}-1)$-nearest to c homterval for $k = 1, \ldots, 2r-3$.

Assume $J_m \subset (J_{i_1}, \tau(J_{i_1}))$ for $m < i_2$. Clearly $J_m \not\subset (J_{i_1}, \tau(J_t)]$. Hence $J_m \subset \tau(J_{i_l}, J_{i_{l+1}})$ for some $l \in [1, \ldots, r-1]$. Applying f^a we obtain

$$J_{m+a} \subset (J_{i_{l+1}}, J_{i_{l+2}}). \tag{2.2}$$

If $l < r-1$, then (2.2) is impossible since $m + a < t$ and there are no homtervals with such indices in $[J_{i_1}, J_t]$ except J_{i_k}. If $l = r-1$, then (2.2) is impossible since J_t is the nearest homterval. So J_{i_1} is the $(i_2 - 1)$-nearest homterval.

Let $J_{i_{k-1}}$ is the $(i_k - 1)$-nearest homterval. Assume $J_{i_k+b} \subset (J_{i_k}, \tau(J_{i_k}))$ for some $b \in (0, a)$. Then $(J_{i_{k-1}+b}, J_{i_k+b})$ contains $J_{i_{k-1}}$ or $\tau(J_{i_{k-1}})$. Applying f^{a-b} we see that $(J_{i_k}, J_{i_{k+1}}) \supset J_{i_{k-1}+a-b}$ which contradicts the induction conjecture.

So $\{J_{i_k}\}_{k=1}^{2r-2}$ is the set of all nearest to c hometervals J_m with indices $m \in [i_1, i_{2r-2}]$. Consequently, $\{J_{i_k}\}_{k=1}^{2r-2}$ is the collection of homtervals.

Thus $r - 2 \le dp(t)$. The same estimate holds for $\varkappa - r$. Hence $\varkappa \le 2dp(t) + 4$. Finally we obtain

$$\sum_{i=0}^{t} \lambda(f^i H_t^+) \le (2dp(t) + 4)\lambda(M). \tag{2.3}$$

Estimates (2.1), (2.3) and $\sum_{i=0}^{t} \lambda(J_i) \le \lambda(M)$ imply (a).

To prove (b) observe that as we have proved $M_t^+ \supset \{J_{t+a}, \ldots, J_{t+(r-2)a}\}$. Hence under the condition of (b) we have $r \le 3$ and $\varkappa \le 6$. Now (b) follows. □

LEMMA 2.2. *Let $\{J_{m_i}\}_{i=k}^{l}$ be the collection of homtervals, $m_{i+1} - m_i = a$, $T = [J_{m_k}, J_{m_{l-1}}]$, $G = [J_{m_k}, \tau(J_{m_k})]$. Then the intervals $G, fT, \ldots, f^{a-1}T$ are pairwise disjoint.*

Proof. Clearly, it is sufficient to show that $f^jT \cap G = \varnothing$ for $j = 1, \ldots, a-1$. Suppose $f^jx \in G$ for some $x \in T, j \in [1, a-1]$. Since J_{m_k} is the $(m_k + a - 1)$-nearest homterval, $x \in T \backslash J_{m_k}$ and $f^j[J_{m_k}, x]$ intersects J_{m_k} or $\tau(J_{m_k})$. Consequently, there exist points $y \in [J_{m_k}, x]$, $z \in J_{m_k} \cup \tau(J_{m_k})$ such that $f^jy = z$. Finally we have

$$f^{a-j}z = f^a y \in G$$

which is impossible as we already mentioned. □

COROLLARY. *In the notations of Lemma* 2.2.

$$\sum_{i=0}^{a-1} \lambda(f^iT) \leq \lambda(M).$$

Due to this Corollary we may apply to $f^a | T$ the Distortion Theorems and the Minimum Principle.

Let J_n and J_{n+l} lie near critical points c_1, c_2 correspondingly. We say that $f^l | J_n$ *preserves the orientation if* for $x, y \in J_n$, $y \in (x, c_1)$ we have $f^ly \in (f^lx, c_2)$.

To analyse unimodal decompositions we need two more lemmas.

LEMMA 2.3. *Let* $n_1 < n_2 < \cdots < n_k$, $\{J_{n_i}\}_{i=1}^k$ *be the sequence of the* n_k*-nearest homtervals (to different critical points* c_i*). Then*

(a) $dp(n_j) \leq dp(n_k) + 1$;

(b) *if* $f^{n_k - n_j} | J_{n_j}$ *reverses the orientation then* $dp(n_j) = 0$.

Proof. Assume that $r = dp(n_j) > 0$. Let $\{J_{n_j+ka}\}_{k=-r}^r$ be the collection of homtervals, $L_s = [J_{n_j-sa}, J_{n_j+sa}]$. Since J_{n_j} is the n_k-nearest homterval, we have $n_k < n_j + a$. Consequently, $f^{n_k - n_j} | L_r$ does not contain c_k (since $f^a | L_r$ is monotone).

Now, if $f^{n_k - n_j} | J_{n_j}$ reverses the orientation then $J_{n_k - a}$ lies nearer to c_k than J_{n_k}. This contradiction proves (b).

Let $f^{n_k - n_j} | J_{n_j}$ preserve the orientation. Consider the interval $K = f^{n_k - n_j} L_{r-1}$. Then homtervals $J_{n_k + sa}$ lie in K for $|s| \leq r - 1$ and approach monotonously c_k with increasing of s. Besides, $f^a | K$ is monotone. Finally, $J_{n_k + sa}$ is the $(n_k + (s+1)a - 1)$-nearest homterval for $|s| \leq r - 1$. This follows easily from the observation that K lies nearer to c_k than other intervals f^iL_{r-1} for $i = 0, \ldots, a-1$ (by Lemma 2.2).

So $\{J_{n_k + sa}\}_{s=-r+1}^{r-1}$ is the collection of homtervals and hence $dp(n_k) \geq r - 1$. □

By *k-collection* of J_n we call the collection of J_n in the orbit $\{J_m\}_{m=k}^\infty$ (here $n \geq k$). Denote by $dp_k(n) \equiv dp_k(J_n)$ the depth of J_n in the k-collection.

LEMMA 2.4. *If* $k \leq n$ *then* $dp_k(n) \leq dp(n)$.

Proof. Let J_s be the homterval $(n-1)$-nearest to c. If $k \leq s$ then the k-collection of J_n is the part of 0-collection of J_n and the desired estimate follows.

Otherwise it is easy to show that $dp(n) = 0$. □

2.4. *The estimate of the distorition of unimodal factors*

Looking through the proof of Main Theorem for $f \in \mathcal{O}$ we see that Lemma 1.2 (§ 1.6) is the only result which is not generalized directly to the smooth case. Here we establish the modified version of Lemma 1.2.

Consider the unimodal decomposition $\{G_i, n_i\}_{i=0}^k$. Recall that by Q_i^- we denote that component of $G_i \backslash J_{n_i}$ which does not contain c_i and by Q_i^+ the other component.

LEMMA 2.5. *There exist functions $\sigma_r(\alpha) > 0$ such that provided $dp(n_k) \leqq r$ the following implication holds*

$$\lambda(Q_{i+1}^-)/\lambda(J_{n_{i+1}}) \geq \alpha \Rightarrow \lambda(Q_i^\pm)/\lambda(J_{n_i}) \geqq \sigma_r(\alpha).$$

Proof. Following the proof of Lemma 1.2 let us consider the interval $\tilde{T}_i = f^\varkappa Q_i^- \cup J_{n_i+\varkappa} \cup \tilde{R}_i$ where $\tilde{R}_i = R_i \cap f^{-\nu_i} G_{i+1}$ (for notations see § 1.4). Then $f^{n_{i+1}-n_i-\varkappa} \tilde{T}_i = G_{i+1}$.

By Lemmas 2.3 and 2.4 $dp_{n_i+\varkappa}(n_{i+1}) \leq r+1$ $(i = 0, \ldots, k-1)$. As G_{i+1} does not contain intervals J_l for $l < n_{i+1}$, Lemma 2.1(a) implies

$$\sum_{m=0}^{n_{i+1}-n_i-\varkappa} \lambda(f^m \tilde{T}_i) \leq (2r+9)\lambda(M).$$

So using the Third and the Second Distortion Lemmas (smooth version) we obtain the estimate from below for $\lambda(f^\varkappa Q_i^-)/\lambda(J_{n_i+\varkappa})$. Application of the First Distortion Lemma (see [1]) completes the proof. □

LEMMA 2.6. *Let J_s, J_n, J_t be the successive nearest to c homtervals, $M_n \subset [J_s, J_t]$. Then there exists the function $\zeta(\alpha) > 0$ such that*

$$\lambda(M_n^\pm)/\lambda(J_n) \geq \alpha \Rightarrow \lambda(H_n^\pm)/\lambda(J) \geq \zeta(\alpha)$$

Proof. This is the immediate consequence of the Distortion Lemmas and Lemma 2.1(b). □

2.5. *Absence of non-solenoidal homtervals for $f \in \mathscr{A}$*

Further we modify the argument of [1].

Let J_n be some nearest to c homterval with $dp(n) \leq 1$, J_s be the $(n-1)$-nearest to c homterval. Fix a large ξ. Let us construct the unimodal decomposition $\{G_i, n_i\}_{i=0}^k$, $n_k = n$ is the same way as in § 1.3.

Now repeating the argument of § 1.8, using Lemma 2.5 instead of 1.2, we obtain the estimate

$$\lambda(J_n) \geq \xi \lambda(J_s) \tag{2.4}$$

for n, s sufficiently large, $dp(n) \leq 1$.

Further, let $\{J_{m_i}\}_{i=0}^\infty$ be all nearest to c homtervals. Let us consider the collection of homtervals $\{J_{m_i}\}_{i=q}^p$, $m_{i+1} - m_i = a$. Set $T = [J_{m_q}, J_{m_{p-1}}]$. Due to Lemma 2.2 the Minimum Principle holds for $f^a | T$:

$$|(f^a)'(x)| \geq \eta \min\{|(f^a)'(x_1)|, |(f^a)'(x_2)|\}, \tag{2.5}$$

where $x \in [x_1, x_2] \subset T$, η does not depend on the collection under consideration.

But (2.4) implies that for some $x_1 \in J_{m_q}$, $x_2 \in J_{m_{p-1}}$ we have

$$|(f^a)'(x_i)| \geq \xi.$$

Consequently $|(f^a)'(x)| \geq \xi\eta$ for $x \in (J_{m_q}, J_{m_{p-1}})$. Hence $\lambda(J_{m_{i+1}}) \geq \xi\eta\lambda(J_{m_i})$ for $i = q$, $q+1, \ldots, p-1$. But we may a priori choose ξ such that $\xi\eta > 1$. Then $\lambda(J_{m_{i+1}}) > \lambda(J_{m_i})$ $(i = q, \ldots, p-1)$.

So $\lambda(J_{m_{i+1}}) > \lambda(J_{m_i})$ for all sufficiently large i. Certainly this is impossible.

2.6. *Absence of solenoidal homtervals for $f \in \mathcal{A}$*

As in [1] we use induction in d. The base of induction ($d=0$) is non-trivial now: it follows from the Denjoy theorem.

Let $J_m \to c$ be the sequence of all nearest to c homtervals. As in § 1.10 consider two cases.

(i) *There exists arbitrary large s such that*

$$\lambda(J_{m_s}) \geq \lambda(J_{m_{s\pm 1}}). \tag{2.6}$$

Looking through the argument of § 1.10 using Lemma 2.5 instead of Lemma 1.2, we see that $dp(s)$ must be large for large s satisfying (2.6).

Consider the collection of homtervals $\{J_{m_i}\}_{i=q}^{p}$ containing J_s. We want to show that

$$\lambda[J_{m_{s-2}}, J_{m_{s-1}})/\lambda(J_{m_{s-1}}) \geq \rho > 0, \tag{2.7}$$

where ρ is independent of s. For this end consider the maximal unimodal decomposition $\{G_i, n_i\}_{i=0}^{k}$ of $f^{m_{s+1}-m_{s-1}}|J_{m_{s-1}}$ constructed as in §1.3, $n_k = m_{s+1} - m_{s-1}$. Lemma 1.4 and the definition of a collection imply $k \leq 2d$, $dp_{m_{s-1}}(m_{s+1}) \leq 2$.

But $\lambda(Q_k^-)/\lambda(J_{m_{s+1}}) \geq 1/2$ by (2.6). Using Lemna 2.5 we obtain the estimate from below for $\lambda(Q_0^-)/\lambda(J_{m_{s-1}})$. Since $Q_0^- \subset [J_{m_{s-2}}, J_{m_{s-1}})$, (2.7) follows.

Further, consider the maximal interval $H_n \supset J$ on which f^n is monotone, $M_n = f^n H_n$. Then

$$M_{m_{s-1}} \supset [J_{m_{s-2}}, J_{m_s}]. \tag{2.8}$$

Indeed, by (2.6) and the Minimum Principle

$$\lambda(J_{m_{q+2}})/\lambda(J_{m_{q+1}}) \leq \eta^{-1} \quad \text{or} \quad \lambda(J_{m_p})/\lambda(J_{m_{p-1}}) \leq \eta^{-1}. \tag{2.9}$$

If the first estimate holds then by the standard argument $\operatorname{ord}(m_{q+1})$ is large. Consequently, by Lemma 1.5 $M_{m_{q+1}} \supset [J_{m_q}, J_{m_{q+2}}]$. Applying $f^{s-m_{q+1}-1}$ to this inclusion we obtain (2.8).

If the second estimate (2.9) holds then $\operatorname{ord}(m_p)$ is large. So there exists the unmodial decomposition $\{G_i, n_i\}_{i=0}^{k}$ of high order such that $n_k = m_p$. Using Lemma 1.4 and the definition of a collection it is easy to verify that all numbers m_j for $j = q+1, \ldots, p$ belong to the set $\{n_i\}_{i=1}^{k}$. So Lemma 1.5 implies (2.8) again.

By (2.8) there exists $\tilde{H}_{m_{s-1}} \subset H_{m_{s-1}}$ such that $f^{m_{s-1}}\tilde{H}_{m_{s-1}} = [J_{m_{s-2}}, J_{m_s}]$. Then Lemma 2.6 and estimates (2.6), (2.7) imply $\lambda(\tilde{H}_{m_{s-1}} \backslash J) \geq \zeta(\rho)\lambda(J)$. But this is impossible for large s.

(ii) *The sequence $\{\lambda(J_{m_i})\}_{i=1}^{\infty}$ is monotone for sufficiently large i (for each critical point c).* Set $F_i = [J_{m_i}, \tau(J_{m_i})]$. Looking through the argument of § 1.10 (and using Lemma 2.5 instead of 1.2) we see that if $dp(J_{m_i}) \leq 1$ then

$$f^{m_{i+1}-m_i} F_i \subset F_{i+1}. \tag{2.10}$$

So if $\{J_{m_i}\}_{i=q}^{p}$ is the collection of homtervals, $m_{i+1} - m_i = a$, then $f^a F_{p-1} \subset F_p$. Hence for each $i \in [q, p]$ we have

$$\begin{aligned} f^{m_{i+1}-m_i} F_i &= f^a(F_{p-1} \cup [J_{m_i}, J_{m_{p-1}}) \cup \tau[J_{m_i}, J_{m_{p-1}})) \\ &\subset F_p \cup [J_{m_{i+1}}, J_{m_p}) \subset F_{i+1}. \end{aligned}$$

Thus (2.10) holds for sufficiently large i. But this implies that c is a periodic point (see [1]) and we arrive at a contradiction.

2.7. *Concluding remark*

We have completed the proof of the Main Theorem under the following assumption (see § 1.2)

ASSUMPTION A. *There are no wandering intervals containing singular points.*

In § 1.11 we pointed out how to modify the argument to obtain the proof without this assumption. A significantly simpler approach was proposed by the referee. Now we describe it.

Let J be a wandering interval containing a singular point a, J_m be the maximal wandering interval containing $f^m J$. Taking an appropriate iterate of J one may assume that the intervals J_m do not contain singular points $(m = 1, 2, \dots)$.

At first suppose $a \in \partial M$. Let $[a, b] \subset M$ be a connected component of M. Then let us consider an interval $[\tilde{a}, b] \supset [a, b]$ and extend f onto $[\tilde{a}, b]$ in such a way that $\tilde{a}$ becomes preperiodic and there are no new extrema on $\tilde{J} = [a, J]$. Hence $\tilde{J}$ is a non-wandering interval of the constructed d-model map. If a is an extremum then by chainging f on J one may turn a into a preperiodic point.

By finite number of such surgeries one will construct a new map $\tilde{f}$ having a wandering interval and satisfying Assumption A.

Added in proof. Martens, de Melo and van Strien by modifying the argument of the present paper have removed the restriction (U2) on critical points ('Julia-Fatou-Sullivan theory for real one-dimensional dynamics', preprint). So, every C^∞-map with non-flat critical points has no wandering intervals. They also have proved the finiteness theorem for limit cycles (cf. Remark in § 2.1).

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