We can prove these results[1] by verifying the several elements first of X^2 and then of XX^2.

Now the roots $\Delta_n = 0$ are the latent roots of X, which are known to be cube roots of those of X^3, by a theorem due to Sylvester. And since $X^3 = 1$ its latent roots are all equal to unity. Hence the only factors of Δ_n are

$$(1 - \lambda),\ (\omega - \lambda),\ (\omega^2 - \lambda)$$

where $1 + \omega + \omega^2 = 0$. Thus for certain integral values p, q, r we must have

$$\Delta_n = c\,(1 - \lambda)^p\,(\omega - \lambda)^q\,(\omega^2 - \lambda)^r.$$

Since Δ_n is real, $q = r$, and since $\Delta_n = 1$ if $\lambda = 1$, then $c = 1$.

Finally equating the coefficient of λ^{n-1} on both sides we obtain

$$s_n = p - q,$$

where s_n is the sum of the leading diagonal terms of the determinant with λ suppressed throughout. But it has been shewn that

$$s_{3k-1} = -1,\ s_{3k} = 0,\ s_{3k+1} = 1$$

from which we deduce the actual values of p and q and thereby establish results. ....(4)

Similar results may be deduced by starting with an array where alternate *rows* have negative signs.

H. W. TURNBULL.

Principal radii of curvature at a point on an ellipsoid.

The values of the principal radii of curvature ρ_1, ρ_2 at a point on an ellipsoid are α^2/p, β^2/p where α^2, β^2 are the squares of the semi-axes of the central section parallel to the tangent plane at the point, and p is the distance from the centre to the tangent plane.

Probably the neatest proof of this theorem is that given in C. Smith's textbook (7th ed. p. 228). The following proof has a certain interest, as it assumes less than the other.

Choose the origin O, at the point, take OZ along the inward drawn normal, and OX, OY parallel to the axes of any section by a

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[1] "On the matrix square and cube roots of unity." *Journal of the London Math. Soc.*, **2** (1927) November.

plane parallel to the tangent plane at O. With these axes the equation of the surface has the form

$$2z = ax^2 + by^2 + 2gxz + 2fyz + cz^2.$$

For points on the surface in the neighbourhood of the origin, z is comparable with x^2 and y^2, from which it follows that the paraboloid

$$2z = ax^2 + by^2$$

has the same principal radii at the origin as the ellipsoid.

$$\therefore \quad \rho_1 = \frac{1}{a}, \ \rho_2 = \frac{1}{b}.$$

Now when a conicoid $F(x, y, z) = 0$ is referred to parallel axes through the centre (ξ, η, ζ), the terms of first degree disappear, so that

$$\frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial \eta} = \frac{\partial F}{\partial \zeta} = 0,$$

and the constant term is $F(\xi, \eta, \zeta)$ where

$$2F(\xi, \eta, \zeta) = \xi \frac{\partial F}{\partial \xi} + \eta \frac{\partial F}{\partial \eta} + \zeta \frac{\partial F}{\partial \zeta} + t \frac{\partial F}{\partial t} = t \frac{\partial F}{\partial t}.$$

In this case

$$F(\xi, \eta, \zeta) \equiv a\xi^2 + b\eta^2 + 2g\xi\zeta + 2f\eta\zeta + c\zeta^2 - 2\zeta.$$

$$\therefore \quad \frac{\partial F}{\partial t} = -2\zeta, \text{ when } \zeta = p.$$

Hence the equation of the ellipsoid referred to parallel axes through the centre is

$$p = ax^2 + by^2 + 2gxz + 2fyz + cz^2.$$

Therefore α, β are the semi-axes of the ellipse

$$p = ax^2 + by^2, \ z = 0$$

so that $\quad \alpha^2 = \dfrac{p}{a}, \ \beta^2 = \dfrac{p}{b}$

and $\quad \therefore \ \rho_1 = \dfrac{\alpha^2}{p}, \ \rho_2 = \dfrac{\beta^2}{p}.$

NEIL M'ARTHUR.