ON COMMUTATIVE NOETHERIAN RINGS WHICH SATISFY THE RADICAL FORMULA

by KA HIN LEUNG and SHING HING MAN

(Received 14 February 1996)

Abstract. In this paper, we show that a commutative Noetherian ring which satisfies the radical formula must be of dimension at most one. From this we give a characterization of commutative Noetherian rings that satisfy the radical formula.

1. Introduction It is well known that the set of nilpotent elements of a commutative ring forms an ideal and is equal to the intersection of all the prime ideals. The above notion has been generalized by R. L. McCasland to modules. Unfortunately, not every module satisfies McCasland's radical formula. This paper looks at commutative Noetherian rings which satisfy McCasland's radical formula.

In this paper, all the rings are commutative with 1 and all the modules are unitary and not necessarily finitely generated. Let M be a module over the ring R. A submodule P of M is called a *prime submodule* of M if

(i) $P \neq M$,
(ii) whenever $r \in R$ and $m \in M \setminus P$ with $rm \in P$, then $rM \subseteq P$.

It is clear that if P is a prime submodule of M, then $\mathrm{Ann}_R(M/P)$, the annihilator of M/P over R, is a prime ideal. We say that P is a $\mathfrak{P}$-prime submodule of M if P is a prime submodule of M with $\mathfrak{P} = \mathrm{Ann}_R(M/P)$. It is clear that the prime submodules of the R-module R are precisely the prime ideals of R. Prime submodules have been studied in **[1]** and **[7]**.

Let N be a submodule of M with $N \neq M$. The radical of N in M, denoted by $M\text{-rad}_R\, N$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then we put $M\text{-rad}_R\, N = M$. The envelope of N in M, denoted by $E_M(N)$, is defined to be the set

$$\{rm : r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some natural number } n \geq 1\}.$$

It is clear that $\langle E_M(N)\rangle$, the submodule generated by $E_M(N)$, is contained in $M\text{-rad}_R\, N$. As in **[6]**, we say that N satisfies the radical formula (N s.t.r.f.) in M if $M\text{-rad}_R\, N = \langle E_M(N)\rangle$. M s.t.r.f. if every submodule of M s.t.r.f. in M. A ring R s.t.r.f. if every R-module s.t.r.f.

The question of what kinds of modules s.t.r.f. has been considered in **[2]**–**[6]**. The main objective of this paper is to classify all the Noetherian rings which s.t.r.f. Prior to this paper, all known Noetherian rings which s.t.r.f. are of dimension at most one. It suggests that only those Noetherian rings of dimension at most one can s.t.r.f. This will be proved in Section 2. If a Noetherian ring is of dimension zero, it is then Artinian. We shall prove in Section 3 that all Artinian rings s.t.r.f. That leaves us with only Noetherian rings of dimension one. In Section 4, we shall deal with the local case. In Section 5, we prove our main theorem which is stated as follows.

THEOREM 1.1. *Let R be a commutative Noetherian ring and $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ be all the minimal prime ideals of R. R s.t.r.f. if and only if R is Artinian or the following conditions are satisfied.*

(i) $\dim R = 1$ *and, for each* $i = 1, 2, \ldots, n$, $R/\mathfrak{P}_i$ *is a Dedekind domain and* $\mathfrak{P}_i$ *is the only* $\mathfrak{P}_i$*-primary ideal.*

(ii) *For* $k = 1, 2, \ldots, n-1$, $\left(\bigcap_{i=1}^{k} \mathfrak{P}_i\right) + \mathfrak{P}_{k+1} = \bigcap_{i=1}^{k} (\mathfrak{P}_i + \mathfrak{P}_{k+1})$, *if* $n \geq 2$.

(iii) *For all* $1 \leq i < j \leq n$, $R = \mathfrak{P}_i + \mathfrak{P}_j$ *or* $R/(\mathfrak{P}_i + \mathfrak{P}_j)$ *is semi-simple Artinian, if* $n \geq 2$.

2. Some preliminary results. We first fix the following notation for the rest of this paper.

(i) Unless stated otherwise, R denotes a (commutative) Noetherian ring.
(ii) $\dim R$ denotes the (Krull) dimension of R.
(iii) For any ideal I of R, $\operatorname{rad} I$ denotes the usual radical of I. In particular, $\operatorname{rad} 0$ is the set of nilpotent elements of R.
(iv) The elements in $R^2 = R \oplus R$ will be written as (a, b), where $a, b \in R$.
(v) Let M be an R-module. We use $\operatorname{Ann}_R M$ to denote the annihilator of M over R.

Let us recall some basic results that will be used later. The following Proposition 2.1 is known. (i)–(iii) are parts of [**6**, Theorem 1] and (iv) is just [**3**, Proposition 2.4].

PROPOSITION 2.1. *Let R be a ring, not necessarily Noetherian.*

(i) *R s.t.r.f. provided M-$\operatorname{rad}_R(0) \subseteq \langle E_M(0) \rangle$, for every R-module M.*
(ii) *Suppose that M is an R-module which s.t.r.f. and I is an ideal of R. Then M/IM s.t.r.f. as an R/I-module. Consequently, if R s.t.r.f., then the quotient ring R/I s.t.r.f., for any ideal I of R.*
(iii) *Suppose that M is an R-module and N is a submodule of M. Then every prime submodule of M/N is of the form P/N, for some prime submodule P of M contains N. Furthermore, N s.t.r.f. in M if and only if the zero submodule of M/N s.t.r.f. in M/N.*
(iv) *Suppose R is Noetherian and M is a finitely generated R-module. Then M s.t.r.f. if and only if $M_{\mathfrak{M}}$ s.t.r.f. as an $R_{\mathfrak{M}}$-module, for every maximal ideal $\mathfrak{M}$ of R.*

By definition, a ring R s.t.r.f. if every R-module s.t.r.f. However, as suggested in [**3**, Theorem 3.3], the key is to study when R^2 s.t.r.f. as an R-module. As we shall see later, it turns out that R s.t.r.f. if and only if R^2 s.t.r.f. as an R-module.

THEOREM 2.2. *Suppose that $(R, \mathfrak{M})$ is a local ring, not necessarily Noetherian, and R^2 s.t.r.f. as an R-module, Let $a, b \in \mathfrak{M}$ and I be an ideal of R such that*

$$(*) \qquad b \notin Ra + (\operatorname{Ann}_R x^n) \cap (Ra + Rb) \quad \text{for any } x \in (\operatorname{rad} Ra) \setminus I \text{ and } n \in \mathbb{N}.$$

Then $a \in I + \mathfrak{M}^2$.

Proof. Let $J = Ra + Rb$. It is easy to verify that a prime submodule of R^2 contains $J(a, b)$ if and only if it contains (a, b); (see the proof of [**2**, Theorem 11]). Thus, R^2-$\operatorname{rad}_R J(a, b) = R^2$-$\operatorname{rad}_R R(a, b)$. As R^2 s.t.r.f., we have R^2-$\operatorname{rad}_R R(a, b) = \langle E_{R^2}(J(a, b)) \rangle$. Hence $(a, b) \in \langle E_{R^2}(J(a, b)) \rangle$. By the definition of $E_{R^2}(J(a, b))$, there exist

(a) $r_1, r_2, \ldots, r_k \in R \setminus \{0\}$,
(b) $(c_1, d_1), (c_2, d_2), \ldots, (c_k, d_k) \in R^2 \setminus \{(0,0)\}$, and
(c) non-negative integers $n_1, n_2, \ldots, n_k$,

such that

(i) $(a,b) = \sum_{i=1}^{k} r_i(c_i, d_i)$, and
(ii) for each $1 \leq i \leq k, r_i^{n_i}(c_i, d_i) = f_i(a,b)$, for some $f_i \in J$.

By (i), we have $a = \sum_{i=1}^{k} r_i c_i$. We are done if we can show that each $r_i c_i \in I + \mathfrak{M}^2$. Let $1 \leq i \leq k$ be given. If r_i is a unit, then from (ii) we have $r_i c_i \in Ja \subseteq \mathfrak{M}^2$. If $r_i \in I$, then $r_i c_i \in I$. Suppose that $r_i \in \mathfrak{M} \setminus I$. We now show that $c_i \in \mathfrak{M}$. Suppose not. From (ii) above, we would have $r_i \in \operatorname{rad} Ra$ and $r_i^{n_i}(ad_i - bc_i) = 0$. Hence $b \in Ra + (\operatorname{Ann}_R r_i^{n_i}) \cap (Ra + Rb)$, which is a contradiction. Therefore, we must have $c_i \in \mathfrak{M}$ and hence $r_i c_i \in \mathfrak{M}^2$. □

Note that for the remainder of this section, R is not necessarily local.

Corollary 2.3. *Suppose that R^2 s.t.r.f. as an R-module, $\dim R \geq 1$ and $\mathfrak{P}$ is a minimal prime ideal of R. Then $\mathfrak{P}$ is the only $\mathfrak{P}$-primary ideal of R and $R/\mathfrak{P}$ is a Dedekind domain. In particular, R is a Dedekind domain if R is a domain.*

Proof. We first assume R is local with maximal ideal $\mathfrak{M}$ and R is $\mathfrak{P}$-primary. To prove our desired result, we only need to show R is a DVR.

As $\dim R \geq 1$, $\mathfrak{M} \neq \mathfrak{P}$. Thus, we can choose $a \in \mathfrak{M} \setminus (\mathfrak{M}^2 + \mathfrak{P})$. If $\mathfrak{M} \neq Ra$, then we can choose $b \in \mathfrak{M} \setminus Ra$. Let $I = \mathfrak{P}$. As $\mathfrak{P}$ is the set of all zero-divisors of R, the condition (*) of Theorem 2.2 is now satisfied. By Theorem 2.2, we get $a \in \mathfrak{M}^2 + \mathfrak{P}$. This contradicts our choice of a. Theorem $\mathfrak{M} = Ra$ and hence R is a DVR.

We now go back to the general case. Let I' be a $\mathfrak{P}$-primary ideal. By Proposition 2.1(ii), $R/I' \oplus R/I'$ s.t.r.f. as an R/I'-module. In view of Proposition 2.1 (iv) and the result proved earlier, we see that R/I' is a Dedekind domain. In particular, $I' = \mathfrak{P}$ and $\dim R/I' = \dim R = 1$. □

In [**2**], Jenkins and Smith proved that any Dedekind domain s.t.r.f. (see [**2**, Theorem 9]). In the same paper, they also give a partial characterization of Noetherian domains which s.t.r.f. (See [**2**, Corollary 13].) In view of Corollary 2.3, we see that Dedekind domains are the only Noetherian domains which s.t.r.f. This answers a question raised in [**2**].

The next result is immediate from Corollary 2.3.

Corollary 2.4. *Suppose that R^2 s.t.r.f. as an R-module. Then $\dim R \leq 1$.*

Next, we prove a key result which allows us to reduce to the case when $\operatorname{rad} 0 = 0$. This result can also be viewed as a partial converse to Proposition 2.1(ii).

Proposition 2.5. *Let R be a ring, not necessarily Noetherian. Suppose that*

(i) *$R/\operatorname{rad} 0$ s.t.r.f. as a ring,*
(ii) *there exist maximal ideals $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n$ and natural numbers $k_1, k_2, \ldots, k_n$ with*

$$\operatorname{rad} 0 \cap \mathfrak{M}_1^{k_1} \cap \mathfrak{M}_2^{k_2} \cap \ldots \cap \mathfrak{M}_n^{k_n} = 0.$$

Then R s.t.r.f.

Proof. We may assume that all the k_i's are equal to a common value k, say. Let M be an R-module. To show R s.t.r.f., by Proposition 2.1(i) it suffices to prove $M\text{-rad}_R 0 \subseteq \langle E_M(0)\rangle$. Clearly, we have $(\text{rad}\,0)M \subseteq \langle E_M(0)\rangle$. Let $m \in M\text{-rad}_R 0$ be given. Then, by Proposition 2.1(iii), $m + (\text{rad}\,0)M \in M/((\text{rad}\,0)M)\text{-rad}_{R/\text{rad}\,0}\,0$. As $R/\text{rad}\,0$ s.t.r.f., we have

$$m + (\text{rad}\,0)M = \sum_i r_i m_i + (\text{rad}\,0)M,$$

where $r_i \in R$, $m_i \in M$ and $r_i^{n_i} m_i \in (\text{rad}\,0)M$, for some natural number n_i. Hence $m = y + \sum_i r_i m_i$, for some $y \in (\text{rad}\,0)M$. We are done if we can show that each $r_i m_i$ is in $\langle E_M(0)\rangle$.

First, observe that if $r \in \mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_n$ and $m \in M$, with $r^t m \in (\text{rad}\,0)M$ for some natural number t, then by (ii) we have $r^{t+k} m = 0$. Hence $rm \in E_M(0)$.

Thus, if $r_i \in \mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_n$ then, by the above observation, $r_i m_i \in E_M(0)$. Suppose $r_i \notin \mathfrak{M}_j$, for some $1 \leq j \leq n$. Without loss of generality, we may assume $r_i \in \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \ldots \cap \mathfrak{M}_l$, and $r_i \notin \mathfrak{M}_{l+1} \cup \ldots \cup \mathfrak{M}_n$. Then $R = Rr_i^{n_i} + \mathfrak{M}_{l+1} \cap \ldots \cap \mathfrak{M}_n$. Write $1 = sr_i^{n_i} + x$, for some $s \in R$ and $x \in \mathfrak{M}_{l+1} \cap \ldots \cap \mathfrak{M}_n$. Then $r_i m_i = sr_i^{n_i+1} m_i + r_i x m_i$. As $r_i^{n_i} m_i \in (\text{rad}\,0)M$, $sr_i^{n_i+1} m_i$ and $(r_i x)^{n_i} m_i$ are also in $(\text{rad}\,0)M$. In particular, $sr_i^{n_i+1} m_i \in \langle E_M(0)\rangle$. On the other hand $r_i x \in \mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_n$. By an earlier observation, $r_i x m_i \in E_M(0)$ also. This proves $r_i m_i \in \langle E_M(0)\rangle$. □

Remark. Note that condition (ii) in Proposition 2.5 cannot be dropped. To see this, we let D be a Dedekind domain and $R = D[x]/(x^2)$. Clearly (x) is the minimal prime ideal of R, R is (x)-primary and $R/(x)$ s.t.r.f. However, it follows from Corollary 2.3 that R does not s.t.r.f.

Proposition 2.6. *Suppose that* $\dim R = 1$ *and every minimal ideal* $\mathfrak{P}$ *of* R *is the only* $\mathfrak{P}$*-primary ideal in* R*. Then condition* (ii) *of Proposition* 2.5 *is satisfied.*

Proof. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be all the minimal prime ideals of R. As R is Noetherian, a reduced primary decomposition of the zero ideal can be written as follows:

$$0 = J_1 \cap J_2 \cap \ldots \cap J_r \cap I_1 \cap I_2 \ldots \cap I_n,$$

where each J_i is a $\mathfrak{P}_i$-primary ideal and I_i is an $\mathfrak{M}_i$-primary ideal, for some maximal ideal $\mathfrak{M}_i$. By assumption $J_i = \mathfrak{P}_i$. Therefore, we get

$$0 = \text{rad}\,0 \cap I_1 \cap I_2 \ldots \cap I_n.$$

As I_i is $\mathfrak{M}_i$-primary, we have $\mathfrak{M}^{k_i} \subseteq I_i$, for some large enough natural number k_i. Our desired result now follows. □

To end this section, we generalize the last statement of Corollary 2.3.

Corollary 2.7. *Suppose that* $\dim R = 1$ *and there exists a unique minimal ideal* $\mathfrak{P}$ *in* R*. Then* R *s.t.r.f. if and only if* $R/\mathfrak{P}$ *is a Dedekind domain and* $\mathfrak{P}$ *is the only* $\mathfrak{P}$*-primary ideal in* R*.*

Proof. We only need to prove sufficiency. This follows from Proposition 2.5 and 2.6. □

3. Local-global principle for rings s.t.r.f. In the first half of this section we shall show that an R-module M s.t.r.f. provided every finitely generated R-submodule of M s.t.r.f. From this, we deduce that a ring R s.t.r.f. if and only if $R_{\mathfrak{M}}$ s.t.r.f. for any maximal ideal $\mathfrak{M}$. As a consequence, we prove also that every Artinian ring s.t.r.f.

DEFINITION. Let $\mathfrak{P}$ be a prime ideal of R, where R is not necessarily Noetherian. Suppose M is an R-module. We define $M(\mathfrak{P}) = \{m \in M : sm \in \mathfrak{P}M \text{ for some } s \in R \setminus \mathfrak{P}\}$.

Next, we recall a result which was proved implicitly in both [**1**] and [**7**].

LEMMA 3.1. *Let M be an R-module, where R is not necessarily Noetherian. Then $M\text{-rad}_R\, 0 = \cap M(\mathfrak{P})$, where the intersection is taken over all the prime ideals of R. Furthermore, if M is finitely generated, then the above intersection can be taken over $\{\mathfrak{P} \in \operatorname{spec}(R) : \operatorname{Ann}_R(M) \subseteq \mathfrak{P}\}$.*

Proof. By [**1**, Proposition 1.1], if $M(\mathfrak{P}) \neq M$, then $M(\mathfrak{P})$ is a $\mathfrak{P}$-prime submodule of M. Also [**1**, Proposition 1.1] tells us that if N is a $\mathfrak{P}$-prime submodule of M, then $M(\mathfrak{P}) \subseteq N$. The first assertion follows easily from the above remarks. By [**1**, Corollary 1.2], if M is finitely generated, then $M(\mathfrak{P})$ is a prime submodule of M for every prime ideal $\mathfrak{P} \supseteq \operatorname{Ann}_R(M)$. Note that if $\mathfrak{P}$ is a prime ideal such that $\operatorname{Ann}_R(M)$ is not contained in $\mathfrak{P}$, then $M = M(\mathfrak{P})$. The last assertion follows. □

COROLLARY 3.2. (cf [**2**, Lemma 3]). *Suppose* $\dim R \leq 1$ *and M is an R-module. Let*

$N_1 = \cap M(\mathfrak{P})$, *where the intersection is taken over all the minimal prime ideals of R,*
$N_2 = \cap \mathfrak{M}M$, *where the intersection is taken over all the maximal ideals of R.*
Then $M\text{-rad}_R\, 0 = N_1 \cap N_2$.

Proof. Note that $M(\mathfrak{M}) = \mathfrak{M}M$ or M, for every maximal ideal $\mathfrak{M}$. The required result follows immediately from Lemma 3.1. □

The next lemma is a generalization of [**2**, Lemma 5].

LEMMA 3.3. *Suppose* $\dim R \leq 1$, *M is an R-module and N is a submodule of M with $N \neq M$. Then $M\text{-rad}_R\, N = \cup L\text{-rad}_R\, N$, where the union is taken over all submodules L of M which contains N, and L/N is finitely generated.*

Proof. By Proposition 2.1(iii), we may assume that $N = 0$. We shall follow the approach used in the proof of [**2**, Lemma 5]. By [**2**, Lemma 4], $L\text{-rad}_R\, 0 \subseteq M\text{-rad}_R\, 0$, for any finitely generated submodule L of M. We now show that the inclusion holds the other way.

Let $m \in M\text{-rad}_R\, 0$ be given. Let $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ (finitely many) be all the minimal prime ideals of R. By Corollary 3.2, for each $i = 1, 2, \ldots, n$, there exists $s_i \in R \setminus \mathfrak{P}_i$ with $s_i m \in \mathfrak{B}_i M$. For each $i = 1, 2, \ldots, n$, there are only finitely many (possibly none) maximal ideals of R which contain both s_i and $\mathfrak{P}_i$. Let $\mathfrak{M}_{i1}, \ldots, \mathfrak{M}_{in_i}$ be all the maximal ideals which contain s_i and $\mathfrak{P}_i$. By Corollary 3.2, for each $1 \leq i \leq n$, $m \in \mathfrak{M}_{ij}M$, for $j = 1, 2, \ldots, n_i$. Together with $s_i m \in \mathfrak{P}_i M$ for $i = 1, 2, \ldots n$, we see that there exists a finitely generated submodule L of M such that

https://doi.org/10.1017/S0017089500032225 Published online by Cambridge University Press

(i) $s_i m \in \mathfrak{P}_i L$ for $i = 1, 2, \ldots n$,
(ii) for each $1 \leq i \leq n$, $m \in \mathfrak{M}_{ij} L$ for $j = 1, 2, \ldots, n_i$.

Let $\mathfrak{Q}$ be a maximal ideal such that $\mathfrak{Q} \notin \{\mathfrak{M}_{i1}, \ldots, \mathfrak{M}_{in_i}\}$, for $i = 1, 2, \ldots, n$. Without loss of generality, we may assume $\mathfrak{P}_1 \subseteq \mathfrak{Q}$. Then $Rs_1 + \mathfrak{Q} = R$ and hence $Rm = Rs_1 m + \mathfrak{Q}m$. Since $s_1 m \in \mathfrak{P}_1 L \subseteq \mathfrak{Q}L$ and $m \in L$, we have $m \in \mathfrak{Q}L$. By Corollary 3.2, $m \in L\text{-rad}_R 0$. Hence $M\text{-rad}_R 0 \subseteq \cup L\text{-rad}_R 0$. □

We are now ready to prove the Local-Global principle for rings s.t.r.f.

THEOREM 3.4. *The following statements are equivalent.*

(i) *R s.t.r.f.*
(ii) *$R_{\mathfrak{M}}$ s.t.r.f., for any maximal ideal $\mathfrak{M}$ of R.*
(iii) *Every finitely generated $R_{\mathfrak{M}}$-module s.t.r.f., for any maximal ideal $\mathfrak{M}$ of R.*
(iv) *Every finitely generated R-module s.t.r.f.*

Proof. (i)$\Rightarrow$(iv) is obvious. (iv)$\Rightarrow$(i) follows from Corollary 2.4 and Lemma 3.3. Similarly, we have (ii)$\Leftrightarrow$(iii). By Proposition 2.1(iv), (iii)$\Leftrightarrow$(iv). □

Later, it will be shown that statement (iv) of Theorem 3.4 can be replaced by R^2 s.t.r.f. as R-module.

THEOREM 3.5. *Any Artinian ring s.t.r.f.*

Proof. By Theorem 3.4, we may assume that R is local Artinian with maximal ideal $\mathfrak{M}$. As $\text{rad}\, 0 = \mathfrak{M}$, $R/\text{rad}\, 0$ is a field and therefore s.t.r.f. On the other hand, since $\mathfrak{M}^n = 0$ for some positive integer n, R s.t.r.f. by Proposition 2.5. □

4. The s.t.r.f. condition on one dimensional local rings. Throughout this section, we shall assume that R is a one dimensional local ring. We shall give a necessary condition for R to s.t.r.f.

Let $\mathfrak{M}$ be the maximal ideal of R and $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ all the minimal prime ideals of R. In Corollary 2.7, we have already dealt with the case when $n = 1$. We may therefore assume $n \geq 2$. For $i = 1, \ldots, n$, we define $I_i = \bigcap_{\substack{k=1\\k\neq i}}^{n} \mathfrak{P}_k$. If $n \geq 3$, we also define $I_{ij} = \bigcap_{\substack{k=1\\k\neq i\\k\neq j}}^{n} \mathfrak{P}_k$, for all $1 \leq i, j \leq n$ with $i \neq j$. The notation above is fixed throughout this section.

LEMMA 4.1. *Suppose $n \geq 3$ and $\mathfrak{M} = I_{ij} + \mathfrak{P}_i$, for all $1 \leq i, j \leq n$ with $i \neq j$. Then the following conditions are equivalent.*

(i) *$\mathfrak{M} = I_i + \mathfrak{P}_i$, for some $1 \leq i \leq n$.*
(ii) *$\mathfrak{M} = I_i + \mathfrak{P}_i$, for each $i = 1, 2, \ldots, n$.*
(iii) *$I_{ij} = I_i + I_j$, for all $1 \leq i, j \leq n$ with $i \neq j$.*
(iv) *$I_{ij} = I_i + I_j$, for some $1 \leq i, j \leq n$ with $i \neq j$.*

https://doi.org/10.1017/S0017089500032225 Published online by Cambridge University Press

Proof. Suppose that $\mathfrak{M} = I_i + \mathfrak{P}_i$, for some $1 \le i \le n$. Let $1 \le j \le n$ with $j \ne i$ be given. By assumption, we have $\mathfrak{M} = I_{ij} + \mathfrak{P}_i$. Let $a_{ij} \in I_{ij}$. Then $a_{ij} = a_i + p_i$, for some $a_i \in I_i, p_i \in \mathfrak{P}_i$. It follows that $p_i = a_{ij} - a_i \in I_j$. Hence $a_{ij} \in I_i + I_j$. Therefore $I_{ij} \subseteq I_i + I_j$. Clearly, $I_{ij} \supseteq I_i + I_j$ and so we have $I_{ij} = I_i + I_j$. By assumption, we have $\mathfrak{M} = I_{ij} + \mathfrak{P}_j$. Substitute for I_{ij} and note that $I_i \subseteq \mathfrak{P}_j$; we get $\mathfrak{M} = I_i + I_j + \mathfrak{P}_j = I_j + \mathfrak{P}_j$.

Using the above argument, we easily get (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(i). Lastly, it is trivial that (iii)$\Rightarrow$(iv). □

Note that the assertion of Lemma 4.1 holds for arbitrary rings, not necessarily Noetherian; also $\mathfrak{M}$ and $\mathfrak{P}_i$'s need not be maximal and prime ideals respectively.

THEOREM 4.2. *Suppose $(R, \mathfrak{M})$ is a one dimensional local ring and $n \ge 2$. If R^2 s.t.r.f. as an R-module, then there exist $x_1, \dots, x_n \in R$ such that for $i = 1, 2, \dots, n$, $I_i = Rx_i + \operatorname{rad} 0$, $\mathfrak{P}_i = \sum_{\substack{k=1\\k\ne i}}^{n} I_k$; and $\mathfrak{M} = \mathfrak{P}_i + I_i = \sum_{k=1}^{n} I_k$.*

Proof. Clearly, we may assume $\operatorname{rad} 0 = 0$. Hence R is reduced and $\bigcap_{i=1}^{n} \mathfrak{P}_i = 0$. Note that, by Corollary 2.3, $R/\mathfrak{P}_i$ is a DVR for $i = 1, 2, \dots, n$.

Let $1 \le i \le n$ be given. As $R/\mathfrak{P}_i$ is a DVR, we may write $\mathfrak{M} = Ry + \mathfrak{P}_i$, for some $y \in \mathfrak{M}$. Note that $I_i \ne 0$ and $I_i \nsubseteq \mathfrak{P}_i$. Hence $I_i + \mathfrak{P}_i = Ry^l + \mathfrak{P}_i$, for some $l \ge 1$. There exist $x_i \in I_i$ and $p_i \in \mathfrak{P}_i$ with $x_i = y^l + p_i$. We now show that x_i generates I_i. Let $a \in I_i$. Then $a = ry^l + q_i$, for some $r \in R$ and $q_i \in \mathfrak{P}_i$. It follows that $rx_i - a = rp_i - q_i \in I_i \cap \mathfrak{P}_i = 0$. Hence $a = rx_i$. Therefore $I_i = Rx_i$.

Suppose $n = 2$. In this case, $I_1 = \mathfrak{P}_2$ and $I_2 = \mathfrak{P}_1$. It remains to show that $\mathfrak{M} = \mathfrak{P}_1 + \mathfrak{P}_2$. As $R/\mathfrak{P}_2$ is a DVR, we have $\mathfrak{M} = Ra + \mathfrak{P}_2$, where $a \in \mathfrak{M} \setminus \mathfrak{P}_2$. As $n \ge 2$, R is not a DVR and $\mathfrak{M} \ne Ra$. Choose $b \in \mathfrak{M} \setminus Ra$. Let $x \in (\operatorname{rad} Ra) \setminus (\mathfrak{P}_1 + \mathfrak{P}_2)$. As $\mathfrak{P}_1 \cup \mathfrak{P}_2$ contains all zero divisors of R, we have $\operatorname{Ann}_R x^n = 0$, for all $n \in \mathbb{N}$. By Theorem 2.2, $a \in \mathfrak{M}^2 + \mathfrak{P}_1 + \mathfrak{P}_2$. It follows that $\mathfrak{M} = \mathfrak{M}^2 + \mathfrak{P}_1 + \mathfrak{P}_2$. By Nakayama's lemma, we get $\mathfrak{M} = \mathfrak{P}_1 + \mathfrak{P}_2$.

Suppose $n \ge 3$. For each $i = 1, 2, \dots, n$, R/I_i is a one dimensional reduced local ring. By Proposition 2.1(ii), we also know that each $R/I_i \oplus R/I_i$ s.t.r.f. as an R/I_i-module. By applying the induction hypothesis to each $R/I_i \oplus R/I_i$, we get

(i) for $i = 1, 2, \dots, n$, $\mathfrak{M} = \sum_{\substack{k=1\\k\ne i}}^{n} I_{ik}$,

(ii) for all $1 \le i, j \le n$ with $i \ne j$, $\mathfrak{P}_i = I_j + \sum_{\substack{k=1\\k\ne i}}^{n} I_{jk}$ and $I_{ij} = Rx_{ij} + I_j$, for some $x_{ij} \in \mathfrak{M}$.

Clearly, $I_{ik} \subseteq \mathfrak{P}_j$ if i, j, k are all distinct. By this observation, (i) gives

$$\mathfrak{M} = I_{ij} + \mathfrak{P}_j = Rx_{ij} + I_j + \mathfrak{P}_j, \quad \text{for all } i \ne j. \tag{1}$$

Suppose $\mathfrak{M} \ne I_n + \mathfrak{P}_n$. By Lemma 4.1,

$$\mathfrak{M} \ne I_i + \mathfrak{P}_i, \quad \text{for all } i, \tag{2}$$

$$I_{ij} \ne I_i + I_j, \quad \text{for all } i \ne j. \tag{3}$$

https://doi.org/10.1017/S0017089500032225 Published online by Cambridge University Press

From (ii) and (3), we get $x_{12} \in I_{12} \setminus (I_1 + I_2)$. Let $x \in (\operatorname{rad} Rx_{12}) \setminus (I_1 + I_2)$. As R is reduced, $\operatorname{Ann}_R x^n = \operatorname{Ann}_R x$, for any $n \in \mathbb{N}$. Note that $x \in I_{12} \setminus (\mathfrak{P}_1 \cup \mathfrak{P}_2)$ as $\operatorname{rad} Rx_{12} \subseteq I_{12}$, $I_{12} \cap \mathfrak{P}_1 = I_2$ and $I_{12} \cap \mathfrak{P}_2 = I_1$. Therefore, $\operatorname{Ann}_R x \subset \mathfrak{P}_1 \cap \mathfrak{P}_2$. Clearly, $Rx_{12} + Rx_{23} \subseteq \mathfrak{P}_4 \cap \mathfrak{P}_5 \cap \ldots \cap \mathfrak{P}_n$. Hence

$$Rx_{12} + (\operatorname{Ann}_R x) \cap (Rx_{12} + Rx_{23}) \subseteq Rx_{12} + I_3.$$

Suppose $x_{23} \in Rx_{12} + (\operatorname{Ann}_R x) \cap (Rx_{12} + Rx_{23})$. Then $x_{23} \in Rx_{12} + I_3$. By (1), $\mathfrak{M} = Rx_{23} + I_3 + \mathfrak{P}_3 \subseteq Rx_{12} + I_3 + \mathfrak{P}_3 = \mathfrak{P}_3 + I_3$, and hence $\mathfrak{M} = \mathfrak{P}_3 + I_3$, which contradicts (2). Therefore $x_{23} \notin Rx_{12} + (\operatorname{Ann}_R x) \cap (Rx_{12} + Rx_{23})$. Now, Theorem 2.2 gives $x_{12} \in \mathfrak{M}^2 + I_1 + I_2$. By (1), $\mathfrak{M} = Rx_{12} + I_2 + \mathfrak{P}_2 \subseteq \mathfrak{M}^2 + I_1 + I_2 + \mathfrak{P}_2 = \mathfrak{M}^2 + I_2 + \mathfrak{P}_2$. Thus $\mathfrak{M} = \mathfrak{M}^2 + I_2 + \mathfrak{P}_2$. By Nakayama's lemma, we have $\mathfrak{M} = I_2 + \mathfrak{P}_2$, which contradicts (2). Therefore $\mathfrak{M} = I_n + \mathfrak{P}_n$. By Lemma 4.1, $I_{ij} = I_i + I_j$, for all $i \neq j$. The required result follows from (i) and (ii). □

5. Proof of the main result. For the convenience of readers, we recall a result in [**4**].

THEOREM 5.1. *Let $n \geq 2$ and $J_1, \ldots, J_n$ be prime ideals in R. (R need not be local.) Suppose that*

(i) $J_1 \cap J_2 \ldots \cap J_n = 0$,

(ii) *R/J_i is a Dedekind domain, for all i,*

(iii) *for $k = 1, 2, \ldots, n-1$, $J_{k+1} + \bigcap_{i=1}^{k} J_i = \bigcap_{i=1}^{k} (J_{k+1} + J_i)$,*

(iv) *for any $1 \leq i < j \leq n$, $R = J_i + J_j$ or $R/(J_i + J_j)$ is semi-simple Artinian.*

Then R s.t.r.f.

By Theorem 5.1, we now show the converse of Theorem 4.2 is true when R is reduced. Set $J_i = \mathfrak{P}_i$. Then (iii) and (iv) are automatically true as $\mathfrak{P}_i + I_i = \mathfrak{M}$. Finally, observe that $\mathfrak{M}/\mathfrak{P}_i$ is generated by $x_i + \mathfrak{P}_i$. Therefore $R/\mathfrak{P}_i$ is a DVR. Hence, R. s.t.r.f.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.5, we may assume R is not Artinian. Thus, we may assume $\dim R = 1$, by Corollary 2.4. In view of Corollary 2.7, we may also assume $n \geq 2$.

Assume R^2 s.t.r.f. as an R-module. By Corollary 2.3, (i) is satisfied. Under localization at any maximal ideal $\mathfrak{M}$ of R, $\mathfrak{P}_i R_{\mathfrak{M}} = R_{\mathfrak{M}}$ if $\mathfrak{P}_i \nsubseteq \mathfrak{M}$ and $\mathfrak{P}_i R_{\mathfrak{M}}$ remains prime otherwise. By Theorem 4.2, (iii) holds in $R_{\mathfrak{M}}$, and that both sides of the condition (ii) become $\mathfrak{M} R_{\mathfrak{M}}$ if $\mathfrak{M}$ contains $\mathfrak{P}_{k+1}$ and $\mathfrak{P}_i$, for some i with $1 \leq i \leq k$. Otherwise, both sides will be equal to $R_{\mathfrak{M}}$. Hence (ii) and (iii) hold globally.

Suppose (i), (ii) and (iii) hold. By Proposition 2.5 and 2.6, we may then assume $\operatorname{rad} 0 = 0$. Now all the conditions required in Theorem 5.1 are satisfied. Hence R s.t.r.f. □

It is not difficult to see that (ii) and (iii) holding in Theorem 1.1 is equivalent to the condition that for each i, $R = \mathfrak{P}_i + \bigcap_{j \neq i} \mathfrak{P}_j$ or $R/(\mathfrak{P}_i + \bigcap_{j \neq i} \mathfrak{P}_j)$ is semi-simple Artinian. (For a proof, one needs only to consider the localization of R over every maximal ideal.) Furthermore, in the proof of necessity of Theorem 1.1, we only need to assume R^2 s.t.r.f. We have therefore obtained the following result.

https://doi.org/10.1017/S0017089500032225 Published online by Cambridge University Press

COROLLARY 5.2. *A ring R s.t.r.f. if and only if R^2 s.t.r.f. as an R-module.*

We now end this paper with the following examples.

EXAMPLES. (1) So far, all the known examples of one dimensional rings which s.t.r.f. are reduced. We now give an example of a one dimensional ring which is not reduced and s.t.r.f. Let $R = F[X, Y]/(X^2, XY)$, where F is a field and X, Y are indeterminates. It is easily verified that RX is the only minimal prime ideal of R and $R/RX \cong F[Y]$. In fact, RX, $RX + RY$ are all the associated prime ideals and $0 = RX \cap RY^2$ is a primary decomposition of the zero ideal. By Theorem 1.1, R s.t.r.f.

(2) In the global case, we can define I_i as in Section 4. However, it is no longer true that I_i is principal even when $\operatorname{rad} 0 = 0$. Here is a counterexample. Let S be a Dedekind domain with a non-principal maximal ideal $\mathfrak{M}$. Let $R = \{(s_1, s_2) \in S \times S : s_1 - s_2 \in \mathfrak{M}\}$. Clearly, R is a commutative Noetherian ring under the usual componentwise addition and multiplication. Let $\mathfrak{P}_1 = \{(s, 0) : s \in \mathfrak{M}\}$ and $\mathfrak{P}_2 = \{(0, s) : s \in \mathfrak{M}\}$. It is easily verified that

(i) $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are prime ideals of R with $\mathfrak{P}_1 \cap \mathfrak{P}_2 = 0$,
(ii) for $i = 1, 2$, $R/\mathfrak{P}_i \cong S$ and $\mathfrak{P}_i \cong \mathfrak{M}$.

By Theorem 1.1, R s.t.r.f. As $\mathfrak{M}$ is non-principal, neither $\mathfrak{P}_1$ nor $\mathfrak{P}_2$ is principal.

ACKNOWLEDGEMENT. The authors would like to thank Professor P. F. Smith for an example of a primary ring which does not satisfy the radical formula.

REFERENCES

1. S. Abu-Saymeh, On dimensions of finitely generated modules, *Comm. Algebra* **23** (1995), 1131–1144.

2. J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring. *Comm. Algebra* **20** (1992), 3593–3602.

3. S. H. Man, One dimensional domains which satisfy the radical formula are Dedekind domains, *Arch. Math. (Basel)* **66**, (1996), 267–279.

4. S. H. Man, Using pullback to construct rings which satisfy the radical formula, preprint.

5. R. L. McCasland and M. E. Moore, On radicals of submodules of finitely generated modules, *Canad. Math. Bull.* **29** (1986), 37–39.

6. R. L. McCasland and M. E. Moore, On radicals of submodules, *Comm. Algebra* **19** (1991), 1327–1341.

7. R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993), 1041–1062.

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE 119260