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COMMUTATIVE GELFAND THEORY FOR REAL BANACH ALGEBRAS: REPRESENTATIONS AS SECTIONS OF BUNDLES

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ABSTRACT. We are concerned here with the development of a more general real case of the classical theorem of Gelfand ([5], 3.1.20), which represents a complex commutative unital Banach algebra as an algebra of continuous functions defined on a compact Hausdorff space.

In § 1 we point out that when looking at real algebras there is not always a one-to-one correspondence between the maximal ideals of the algebra B, denoted $\mathcal{M}_B$, and the set of unital (real) algebra homomorphisms from B into $\mathbb{C}$, denoted by Φ_B. This simple point and subsequent observations lead to a theory of representations of real commutative unital Banach algebras where elements are represented as sections of a bundle of real fields associated with the algebra (Theorem 3.5). After establishing this representation theorem, we look into the question of when a real commutative Banach algebra is already complex. There is a natural topological obstruction which we delineate. Theorem 4.8 gives equivalent conditions which determine whether such an algebra is already complex.

Finally, in § 5 we abstractly characterize those section algebras which appear as the target algebras for our Gelfand transform. We dub these algebras "almost complex C^*-algebras" and provide a natural classification scheme.

1. **Preliminary results.** Let B be a commutative Banach algebra over $\mathbb{R}$ with identity, 1. Let $\mathcal{M}_B$ denote the set of maximal ideals of B and let Φ_B be the set of unital $\mathbb{R}$-algebra homomorphisms $\Phi: B \longrightarrow \mathbb{C}$. For each $\varphi \in \Phi_B$, $\ker\varphi \in \mathcal{M}_B$ and we let $\epsilon: \Phi_B \longrightarrow \mathcal{M}_B$ be the canonical map $\epsilon(\varphi) = \ker\varphi$. We state the following well-known result and provide a new *real* proof of part (b).

PROPOSITION 1.1. *For each* $\varphi \in \Phi_B$,

(a) range $\varphi = \mathbb{R}$ *or* $\mathbb{C}$.

(b) $\|\varphi\| = \sup_{a\neq 0} \frac{|\varphi(a)|}{\|a\|} = 1$.

(c) range $\varphi = \mathbb{R} \Leftrightarrow \ker\varphi$ *has codimension 1 in B.*

(d) range $\varphi = \mathbb{C} \Leftrightarrow \ker\varphi$ *has codimension 2 in B.*

PROOF. (a) Range φ is a real unital subalgebra of $\mathbb{C}$.

(b) Since $\varphi(1) = 1$, $\|\varphi\| \geq 1$. Suppose that for some $a \in B$, $\|a\| < 1$ we have $|\varphi(a)| = 1$. We choose θ real so that $e^{i\theta}\varphi(a) = 1$ and set

$$b = \sum_{n=0}^{\infty} a^n \cos n\theta,$$

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$$c = \sum_{n=0}^{\infty} a^n \sin n\theta .$$

Both series converge in B since $\|a\| < 1$. Elementary trigonometric identities lead to

$$1 = b - ab\cos\theta + ac\sin\theta$$

and

$$0 = -ab\sin\theta + c - ac\cos\theta .$$

Applying φ to these equations, we obtain

$$1 = \varphi(b) - \varphi(a)\varphi(b)\cos\theta + \varphi(a)\varphi(c)\sin\theta$$
$$0 = -\varphi(a)\varphi(b)\sin\theta + \varphi(c) - \varphi(a)\varphi(c)\cos\theta$$

We multiply the second equation by i and add it to the first equation to obtain:

$$1 = \varphi(b) - \varphi(a)\varphi(b)e^{i\theta} + i\varphi(c) - i\varphi(a)\varphi(c)e^{i\theta} .$$

Since $\varphi(a)e^{i\theta} = 1$, we get

$$1 = \varphi(b) - \varphi(b) + i\big(\varphi(c) - \varphi(c)\big) = 0 \,!$$

Hence, no such a exists and we have $\|\varphi\| = 1$.

(c) and (d) are standard linear algebra results. ∎

We also remind the reader of the following well-known results since they are basic to all that we do.

PROPOSITION 1.2. *(a) Every maximal ideal I of B has codimension 1 or 2.*

(b) For each $\varphi \in \Phi_B$, $\ker\varphi$ *is a maximal ideal.*

(c) For each maximal ideal I of codimension 1 there is a unique $\varphi \in \Phi_B$ with $\ker\varphi = I$. *Moreover* $\operatorname{range}\varphi = \mathbb{R}$.

(d) For each maximal ideal J of codimension 2 there are exactly two elements φ_1, $\varphi_2 \in \Phi_B$ with $J = \ker\varphi_1 = \ker\varphi_2$. Moreover, $\varphi_2 = \bar{\varphi}_1$ and $\operatorname{range}\varphi_1 = \operatorname{range}\varphi_2 = \mathbb{C}$.

PROOF. (a) B/I is a commutative normed division algebra over $\mathbb{R}$, and so by the Gelfand-Mazur Theorem either $B/I \cong \mathbb{R}$ or $B/I \cong \mathbb{C}$.

(b) Since (by Proposition 1(a)) $B/\ker\varphi \cong \mathbb{R}$ or $\mathbb{C}$ is a field, $\ker\varphi$ is a maximal ideal.

(c) In this case, $B/I \cong \mathbb{R}$ as algebras over $\mathbb{R}$ and this isomorphism is unique since $\mathbb{R}$ has no nontrivial $\mathbb{R}$-algebra automorphisms. Thus the composition $\varphi\colon B \longrightarrow B/I \cong \mathbb{R}$ is the unique element of Φ_B with kernel I.

(d) In this case, $B/J \cong \mathbb{C}$ as algebras over $\mathbb{R}$ and since $\mathbb{C}$ has exactly one nontrivial $\mathbb{R}$-algebra automorphism given by conjugation we see that there are exactly two elements; $\varphi, \bar{\varphi} \in \Phi_B$ with kernel J. ∎

We give Φ_B the topology of pointwise convergence on B, i.e., we consider Φ_B as a subspace of $\prod_{b\in B}\mathbb{C}$ with the product topology. Then, as is well-known, Φ_B is a compact Hausdorff space. We call this the *B-topology* on Φ_B. We give $\mathcal{M}_B$ the quotient topology arising from the map $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$. That is, $\mathcal{M}_B$ has the strongest topology which makes the map ϵ continuous [2], Chapter VI.

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PROPOSITION 1.3. *The space $\mathcal{M}_B$ is compact and Hausdorff, and the map ϵ is both open and closed.*

PROOF. Let $\sigma\colon \Phi_B \longrightarrow \Phi_B$ be the homeomorphism $\sigma(\varphi) = \bar{\varphi}$. Then $\sigma^2 = e$, the identity, and we have an action of $\mathbb{Z}_2 = \{\sigma, e\}$ on Φ_B. The equivalence relation determined by orbits is easily seen to be open (and closed). That is, if $U \subseteq \Phi_B$ is open (closed), then the saturation of U namely $U \cup \sigma(U)$ is clearly open (closed). By [2], Chapter VI.4.2, we see that the map $\Phi_B \longrightarrow \Phi_B/\mathbb{Z}_2$ is both open and closed. Now, the diagram

$$\begin{tikzcd} \Phi_B \arrow[r, "\epsilon"] \arrow[d] & \mathcal{M}_B \\ \Phi_B/\mathbb{Z}_2 \arrow[ur] & \end{tikzcd}$$

commutes and the bijection $\mathcal{M}_B \leftrightarrow \Phi_B/\mathbb{Z}_2$ is clearly a homeomorphism so that ϵ is also open and closed. Clearly, $\mathcal{M}_B$ is compact. To see that $\mathcal{M}_B$ is also Hausdorff is an easy exercise using the finiteness of the group $\mathbb{Z}_2$. ∎

As a corollary to Propositions 1 and 2 we prove the following which is usually proved by first complexifying the real Banach algebra B, see [5], [1]. We let $C(\Phi_B)$ denote the continuous complex-valued functions on the compact Hausdorff space, Φ_B endowed with the usual supremum norm.

THEOREM 1.4. *Let B be a commutative Banach algebra over $\mathbb{R}$ with identity, 1.*

(a) The mapping $\Lambda\colon B \longrightarrow C(\Phi_B)$ given by $\hat{b}(\varphi) = \varphi(b)$ for each $\varphi \in \Phi_B$ and each $b \in B$ is a norm-decreasing real algebra homomorphism which is one-to-one if and only if B is semisimple.

(b) For $b \in B$, $\lambda \in \mathbb{R}$, the element $b - \lambda 1$ is singular if and only if $\lambda = \varphi(b)$ for some $\varphi \in \Phi_B$.

PROOF. (a) Clearly, Λ is an algebra homomorphism, and $\hat{b} = 0 \Leftrightarrow b$ is in the kernel of every $\varphi \in \Phi_B$. That is, $\hat{b} = 0 \Leftrightarrow b \in \bigcap_{I \in \mathcal{M}_B} I \Leftrightarrow b$ is in the Jacobson radical of B. Thus, Λ is one-to-one $\Leftrightarrow B$ is semisimple. Now, since each $\varphi \in \Phi_B$ has norm one, we get for $b \in B$

$$\|\hat{b}\|_\infty = \sup_{\varphi \in \Phi_B} |\hat{b}(\varphi)| = \sup_{\varphi \in \Phi_B} |\varphi(b)| \leq \|b\|$$

so that Λ (the *Gelfand transform*) is norm-decreasing.

(b) The element $b - \lambda 1$ is singular (not invertible) $\Leftrightarrow b - \lambda 1$ lies is some maximal ideal $\Leftrightarrow \varphi(b - \lambda 1) = 0$ for some $\varphi \in \Phi_B \Leftrightarrow \lambda = \varphi(b)$ for some $\varphi \in \Phi_B$. ∎

In the case of complex Banach algebras one can, of course, replace the space Φ_B with the maximal ideal space $\mathcal{M}_B$ since the two spaces are homeomorphic. The main objective of this work is to provide a Gelfand theorem for commutative real Banach algebras using the maximal ideal space $\mathcal{M}_B$ as the common domain of an algebra of functions. We cannot resort to complexifying the algebra since this changes $\mathcal{M}_B$ (enlarges it). Thus we are forced to deal with the (often two-to-one) map $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$.

2. **The canonical $\mathbb{Z}_2$-bundle associated to a real Banach algebra.** The most obvious bundle associated to a real commutative Banach algebra B with identity is the fibred space $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$. From Proposition 1.3 we know that $\mathcal{M}_B$ is just the quotient of Φ_B under the (not necessarily free) action of $\mathbb{Z}_2 = \{e, \sigma\}$ on Φ_B where $\sigma(\varphi) = \bar{\varphi}$ for $\varphi \in \Phi_B$. In order to describe this more fully, we need a little notation. Let

$$\begin{aligned}
\Phi_B^{\mathbb{R}} &= \{\varphi \in \Phi_B \mid \varphi(B) = \mathbb{R}\},\\
\Phi_B^{\mathbb{C}} &= \{\varphi \in \Phi_B \mid \varphi(B) = \mathbb{C}\},\\
\mathcal{M}_B^{\mathbb{R}} &= \epsilon(\Phi_B^{\mathbb{R}}) = \{I \in \mathcal{M}_B \mid B/I \cong \mathbb{R}\},\\
\mathcal{M}_B^{\mathbb{C}} &= \epsilon(\Phi_B^{\mathbb{C}}) = \{I \in \mathcal{M}_B \mid B/I \cong \mathbb{C}\},
\end{aligned}$$

Clearly, $\Phi_B^{\mathbb{R}}$ and $\mathcal{M}_B^{\mathbb{R}}$ are closed sets and so $\Phi_B^{\mathbb{C}}$ and $\mathcal{M}_B^{\mathbb{C}}$ are open sets. Moreover, $\epsilon_{\mathbb{R}} := \epsilon|_{\Phi_B^{\mathbb{R}}}\colon \Phi_B^{\mathbb{R}} \longrightarrow \mathcal{M}_B^{\mathbb{R}}$ is clearly a homeomorphism of compact sets. By the following lemma, $\epsilon_{\mathbb{C}} := \epsilon|_{\Phi_B^{\mathbb{C}}}\colon \Phi_B^{\mathbb{C}} \longrightarrow \mathcal{M}_B^{\mathbb{C}}$ is a locally trivial $\mathbb{Z}_2$-bundle.

LEMMA 2.1. *Let G be a finite group acting freely on the Hausdorff space Y, and let $X = Y/G$. Then X is Hausdorff and the natural map $p\colon Y \longrightarrow X$ makes Y a locally trivial G-bundle over X.*

PROOF. Exactly as in Proposition 1.3 we see that p is open and closed and X is Hausdorff. Let $G = \{g_1, \ldots, g_n\}$ where $g_1 = e$. Let $x \in X$ and choose $y \in Y$ so that $p(y) = x$. Then, $y = g_1y$, g_2y, $\ldots$, g_ny are all the distinct preimages of x in Y, since the action is free. Let $O_1, O_2, \ldots, O_n$ be pairwise disjoint open sets containing g_1y, g_2y, $\ldots$, g_ny respectively. Let

$$\mathcal{U}_1 = O_1 \cap g_2^{-1}O_2 \cap \cdots \cap g_n^{-1}O_n$$

which is a neighbourhood of $y = g_1y$. Let $\mathcal{U}_k = g_k\mathcal{U}_1$ for $k = 1, 2, \ldots, n$. Then, for $k \neq j$, $\mathcal{U}_k \cap \mathcal{U}_j \subseteq O_k \cap O_j = \emptyset$. In particular, p is 1:1 on $\mathcal{U}_1$. Thus, letting $N = p(\mathcal{U}_1)$ we have a neighbourhood of x so that $p|_{\mathcal{U}_1}\colon \mathcal{U}_1 \longrightarrow N$ is a homeomorphism and therefore its inverse $s\colon N \longrightarrow \mathcal{U}_1$ is a continuous local section. Thus, $p\colon Y \longrightarrow X$ is a locally trivial G-bundle. ∎

This lemma is certainly well-known, but we lacked a reference so we provided the obvious proof.

Thus, one might hope that $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$ might have local sections everywhere since both $\epsilon_{\mathbb{R}}\colon \Phi_B^{\mathbb{R}} \longrightarrow \mathcal{M}_B^{\mathbb{R}}$ and $\epsilon_{\mathbb{C}}\colon \Phi_B^{\mathbb{C}} \longrightarrow \mathcal{M}_B^{\mathbb{C}}$ have local sections. However we will see in § 5, an example where this fails. These bundles are functorial in the following sense.

PROSPOSITION 2.2. *Let $\theta\colon B_1 \longrightarrow B_2$ be a unital homomorphism of commutative real Banach algebras with identity. Then, we have morphisms θ^* of:*

1) $\mathbb{Z}_2$-spaces $\begin{array}{ccc} \Phi_{B_1} & \longleftarrow & \Phi_{B_2} \\ \downarrow & \theta^* & \downarrow \\ \mathcal{M}_{B_1} & \longleftarrow & \mathcal{M}_{B_2} \end{array}$ *and*

2) $\mathbb{Z}_2$-bundles $\begin{array}{ccc} \Phi_{B_1}^{\mathbb{C}} & \longleftarrow & \Phi_{B_2}^{\mathbb{C}} \\ \downarrow & \theta^* & \downarrow \\ \mathcal{M}_{B_1}^{\mathbb{C}} & \longleftarrow & \mathcal{M}_{B_2}^{\mathbb{C}} \end{array}$ *where here we must assume that either θ is onto or $\Phi_{B_1}^{\mathbb{C}} = \Phi_{B_1}$. Moreover, this defines contravariant functors between the appropriate categories.*

PROOF. Clear. ∎

COROLLARY 2.3. *Let $\theta : B_1 \longrightarrow B_2$ be an isomorphism of commutative real Banach algebras with identity. Then* θ^*: $\begin{array}{ccc} \Phi_{B_1}^{\mathbb{C}} & \longleftrightarrow & \Phi_{B_2}^{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{B_1}^{\mathbb{C}} & \longleftarrow & \mathcal{M}_{B_2}^{\mathbb{C}} \end{array}$ *is an isomorphism of $\mathbb{Z}_2$-bundles.*

COROLLARY 2.4. *If B is a real commutative Banach algebra with identity, then defining $w_1(B) \in \check{H}^1(\mathcal{M}_B^{\mathbb{C}}, \mathbb{Z}_2)$ to be the Stieffel-Whitney class of the $\mathbb{Z}_2$-bundle $\epsilon_{\mathbb{C}} : \Phi_B^{\mathbb{C}} \longrightarrow \mathcal{M}_B^{\mathbb{C}}$ yields an isomorphism invariant of the algebra.*

PROOF. Elements of $\check{H}^1(X, \mathbb{Z}_2)$ are in natural one-to-one correspondence with isomorphism classes of locally trivial $\mathbb{Z}_2$-bundles over X. ∎

3. **The bundle of real fields associated to a real Banach algebra.** As usual in representing algebras as sections we form the set

$$E_B = \biguplus_{I \in \mathcal{M}_B} B/I$$

where $\biguplus$ denotes disjoint union. Of course, each B/I is a field and an algebra over $\mathbb{R}$, and we have an obvious map $p: E_B \longrightarrow \mathcal{M}_B$. Moreover, we have the usual algebra homomorphism $\Lambda: B \longrightarrow S(E_B)$ where $S(E_B)$ denotes the algebra of all sections (i.e., maps $s: \mathcal{M}_B \longrightarrow E_B$ with $p \circ s = \mathrm{id}$) of the fibred set $p: E_B \longrightarrow \mathcal{M}_B$. Here, of course, $\hat{b}(I) = b + I$ for each $b \in B$.

The problem, then, is to topologize E_B in a reasonable way so that we get a Gelfand transform $\Lambda: B \longrightarrow \Gamma(E_B)$, where $\Gamma(E_B)$ is the algebra of continuous sections which is a Banach algebra in supremum norm. Moreover, we don't want the bundle $p: E_B \longrightarrow \mathcal{M}_B$ to lose the topological information provided by the $\mathbb{Z}_2$-space $\epsilon: \Phi_B \longrightarrow \mathcal{M}_B$. Naturally, we would like to be able to classify the algebras $\Gamma(E_B)$ in some reasonable way (*à la* commutative C^*-algebras), so that we can feel we at least understand our target Banach algebras in this "Gelfand Theory".

The first idea for topologizing E_B from the general theory would be the quotient topology from the map $B \times \mathcal{M}_B \longrightarrow E_B$ given by: $(b, I) \longmapsto b + I$. Unfortunately, this appears to lose the structure of the fibration $\Phi_B \longrightarrow \mathcal{M}_B$. An obvious way to remedy this would be to use the map $B \times \Phi_B \longrightarrow E_B$ given by $(b, \varphi) \longmapsto b + \ker\varphi$. This runs into other problems since the norm on $B/\ker\varphi$ is not necessarily the usual norm on $\mathbb{C}$ for $\varphi \in \Phi_B^{\mathbb{C}}$. For example, let $B = \mathbb{C}$ with the norm $\|a + ib\| = |a| + |b|$. Then, B is a real Banach algebra with one maximal ideal, $\{0\}$, and two elements in Φ_B, namely id and $\overline{\mathrm{id}}$, but the norm on $B = B/\{0\}$ is *not* the usual norm on $\mathbb{C}$. Of course, in general the norm on $B/\ker\varphi \cong \mathbb{C}$ is equivalent to the usual norm on $\mathbb{C}$, but the equivalence need not be uniform as φ varies in $\Phi_B^{\mathbb{C}}$. For example, for $n = 1, 2, \ldots$ let $B_n = \mathbb{C}$ with the norm $\|a + ib\|_n = |a| + n|b|$. Let B denote the c_0-direct sum of the algebras B_n with identity adjoined. Then, $\mathcal{M}_B = \{1, 2, \ldots, \infty\}$; $\mathcal{M}_B^{\mathbb{C}} = \{1, 2, \ldots\}$; and $\mathcal{M}_B^{\mathbb{R}} = \{\infty\}$. Since for any self-respecting "Gelfand Theory" we would want the norm on our target algebra, $\Gamma(E_B)$, to be (at least equivalent to) the spectral radius, we must also abandon this approach. Thus, we are forced into replacing B with $\mathbb{C}$ and the following definition of the topology on E_B.

DEFINITION 3.1. We let $(\mathbb{C} \times \Phi_B)'$, denote the union of the sets $\mathbb{C} \times \Phi_B^{\mathbb{C}}$ and $\mathbb{R} \times \Phi_B^{\mathbb{R}}$ with the topology it inherits as a subspace of $\mathbb{C} \times \Phi_B$. We define a surjection $g\colon (\mathbb{C} \times \Phi_B)' \longrightarrow E_B$ *via* $g(z, \varphi) = b + \ker \varphi$ where b is chosen so that $\varphi(b) = z$. Then, g is well-defined since any two such b's differ by an element of $\ker \varphi$. We give E_B the quotient topology (identification topology) given by the map g, [2], Chapter VI. There is also a natural action of $\mathbb{Z}_2 = \{\sigma, e\}$ on $(\mathbb{C} \times \Phi_B)'$, namely:

$$\sigma(z, \varphi) = (\bar{z}, \bar{\varphi}).$$

LEMMA 3.2. *There is a natural homeomorphism* $(\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2 \cong E_B$ *which makes the following diagram commute:*

$$\begin{array}{ccc} & (\mathbb{C} \times \Phi_B)' & \\ \swarrow & & \searrow^{g} \\ (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2 & \cong & E_B. \end{array}$$

Thus, g is both open and closed.

PROOF. Since the quotient map $(\mathbb{C} \times \Phi_B)' \longrightarrow (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2$ is both open and closed by the argument of Proposition 1.3, it suffices by [2], Chapter VI.3.2, to exhibit a bijection between $(\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2$ and E_B which makes the diagram commute. This bijection follows from the easily checked fact that

$$g(z_1, \varphi_1) = g(z_2, \varphi_2) \Leftrightarrow \begin{cases} \text{either} & (z_1, \varphi_1) = (z_2, \varphi_2) \\ \text{or} & (z_1, \varphi_1) = (\bar{z}_2, \bar{\varphi}_2). \end{cases}$$

∎

REMARK. In the case that $\Phi_B \longrightarrow \mathcal{M}_B$ is a principal $\mathbb{Z}_2$-bundle (which is the case $\Leftrightarrow$ the $\mathbb{Z}_2$ action is free on $\Phi_B \Leftrightarrow \Phi_B = \Phi_B^{\mathbb{C}}$), the fibred space $(\mathbb{C} \times \Phi_B)/\mathbb{Z}_2 \longrightarrow \mathcal{M}_B$ defined by $[(z, \varphi)] \longmapsto \ker \varphi$ is exactly the fibre bundle over $\mathcal{M}_B$ with fibre $\mathbb{C}$ associated to the principal $\mathbb{Z}_2$ bundle $\Phi_B \longrightarrow \mathcal{M}_B$ where, of course, $\mathbb{Z}_2$ acts on $\mathbb{C}$ in the obvious way ([3], 4.5).

In general, we have identified $\mathcal{M}_B \cong \Phi_B/\mathbb{Z}_2$ and $E_B \cong (\mathbb{C} \times \Phi_B)'/\mathbb{Z}_2$, and the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C} \times \Phi_B)' & \xrightarrow{g} & E_B \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p} \\ \Phi_B & \xrightarrow[\epsilon]{} & \mathcal{M}_B \end{array}$$

where π_2 denotes the projection onto the second coordinate.

PROPOSITION 3.3. *(1) For fixed $I \in \mathcal{M}_B$, the subspace topology on $B/I \subseteq E_B$ is just the norm topology on B/I.*

(2) $p: E_B \longrightarrow \mathcal{M}_B$ is continuous and open (so $\mathcal{M}_B$ has the quotient topology from E_B).

(3) If $b \in B$ then $\hat{b}: \mathcal{M}_B \longrightarrow E_B$ is continuous.

(4) If $s: \mathcal{M}_B \longrightarrow E_B$ is a continuous section then $I \longmapsto |s(I)|: \mathcal{M}_B \longrightarrow \mathbb{R}^+$ is continuous, where $|\cdot|$ denotes the norm B/I receives from its embedding into $\mathbb{C}$.

PROOF. (1) Let $I = \ker\varphi$ for some $\varphi \in \Phi_B$. Then,

$$g^{-1}(B/I) = \begin{cases} \mathbb{R} \times \{\varphi\} \\ \text{or} \\ \mathbb{C} \times \{\varphi\} \cup \mathbb{C} \times \{\bar{\varphi}\} \end{cases}$$

which is closed in $(\mathbb{C} \times \Phi_B)'$ and so B/I is closed in E_B. Thus by [2], Chapter VI.2.1, the subspace topology on $B/I \subseteq E_B$ is just the quotient topology induced by the restriction of $g: g^{-1}(B/I) \longrightarrow B/I$. From the form of $g^{-1}(B/I)$ one easily sees that either $\mathbb{R} \times \{\varphi\} \longrightarrow B/I$ or $\mathbb{C} \times \{\varphi\} \longrightarrow B/I$ is a homeomorphism and so B/I has the stated topology.

(2) Now, considering product neighbourhoods of $\mathbb{C} \times \Phi_B$ intersected with $(\mathbb{C} \times \Phi_B)'$ as a basis one can carefully show that the map $\pi_2: (\mathbb{C} \times \Phi_B)' \longrightarrow \Phi_B$ is open. One must be careful here, as the map is not usually closed even when restricted to bounded sets unless $\Phi_B^{\mathbb{C}}$ is closed in Φ_B.

We consider the commutative diagram:

$$\begin{array}{ccc} (\mathbb{C} \times \Phi_B)' & \xrightarrow{g} & E_B \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p} \\ \Phi_B & \xrightarrow[\epsilon]{} & \mathcal{M}_B \end{array}$$

where π_2 and ϵ are continuous and open and g is a quotient map. Let $O \subseteq \mathcal{M}_B$ be open. Then $g^{-1}\big(p^{-1}(O)\big) = (p \circ g)^{-1}(O) = (\epsilon \circ \pi_2)^{-1}(O) = \pi_2^{-1}\big(\epsilon^{-1}(O)\big)$ which is open. Since g is a quotient map, $p^{-1}(O)$ is open and hence p is continuous. By similar considerations, p is open.

(3) If $b \in B$, then by definition of the B-topology on Φ_B, the evaluation map $\varphi \longmapsto \big(\varphi(b), \varphi\big) \in (\mathbb{C} \times \Phi_B)'$ is continuous. Composing this with the quotient map $(\mathbb{C} \times \Phi_B)' \longrightarrow E_B$, we see that $\varphi \longmapsto b + \ker\varphi$ is continuous from $\Phi_B \longrightarrow E_B$. Consider the commutative diagram:

$$\begin{array}{ccc} \Phi_B & \xrightarrow{\epsilon} & \mathcal{M}_B \\ {\scriptstyle \varphi \mapsto b+\ker\varphi}\downarrow & \swarrow{\scriptstyle \hat{b}} & \\ E_B & & \end{array}$$

Since ϵ is a quotient mapping, we see that $\hat{b}$ is continuous by [2]. Chapter VI.3.2.

(4) Now, if $s: \mathcal{M}_B \longrightarrow E_B$ is a continuous section then $s(I) = b + I = b + \ker\varphi$ for some $b \in B$ and $\varphi \in \Phi_B$. Then, by definition of $|\cdot|$, we have $|s(I)| = |\varphi(b)|$ and this is clearly well-defined. Thus, it suffices to see that this absolute-value map is continuous

on E_B. Consider the commutative diagram:

$$\begin{tikzcd} (\mathbb{C} \times \Phi_B)' \arrow[r, "|\pi_1(\cdot)|"] \arrow[d, "g"'] & \mathbb{R}^+ \\ E_B \arrow[ur, "|\cdot|"'] & \end{tikzcd}$$

Since the horizontal map is clearly continuous, we see that $|\cdot|$ is continuous on E_B by [2], Chapter VI.3.2. ∎

PROPOSITION 3.4. *$\Gamma(E_B)$ is a real commutative Banach algebra with identity given the norm $\|s\| = \sup_{I \in \mathcal{M}_B} |s(I)|$.*

PROOF. The first subtle part here, is to show that $\Gamma(E_B)$ is an algebra! To see this, we let $[(\mathbb{C} \times \Phi_B)']^{(2)} \subseteq [(\mathbb{C} \times \Phi_B)' \times (\mathbb{C} \times \Phi_B)']$ be the set

$$\Big\{\big((z_1, \varphi), (z_2, \varphi)\big) \mid (z_i, \varphi) \in (\mathbb{C} \times \Phi_B)' \text{ for } i = 1, 2\Big\}.$$

Then, we have the obviously continuous multiplication map

$$[(\mathbb{C} \times \Phi_B)']^{(2)} \longrightarrow (\mathbb{C} \times \Phi_B)'$$

defined by $\big((z_1, \varphi), (z_2, \varphi)\big) \longmapsto (z_1, z_2, \varphi)$. Moreover, $[(\mathbb{C} \times \Phi_B)']^{(2)}$ is clearly closed and thus by [2], Chapter VI.2.1 and Chapter VI.7.3, the map

$$g \times g\colon [(\mathbb{C} \times \Phi_B)'^{(2)} \longrightarrow E_B^{(2)}$$

is an identification. Now, $E_B^{(2)} = \{(b_1{+}I, b_2{+}I) \mid b_i \in B \text{ and } I \in \mathcal{M}_B\}$ and the continuous multiplication $[(\mathbb{C} \times \Phi_B)']^{(2)} \longrightarrow (\mathbb{C} \times \Phi_B)'$ induces (by [2], Chapter VI.3.2) a *continuous* multiplication $E_B^{(2)} \longrightarrow E_B$. One easily checks that this is the obvious multiplication:

$$(b_1 + I, b_2 + I) \longmapsto (b_1 b_2 + I).$$

Now, if $s_1, s_2\colon \mathcal{M}_B \longrightarrow E_B$ are continuous sections for $p\colon E_B \longrightarrow \mathcal{M}_B$ then $s_1 \times s_2\colon \mathcal{M}_B \longrightarrow E_B \times E_B$ is continuous and has range in $(E_B)^{(2)}$. Since composing with multiplication $E_B^{(2)} \longrightarrow E_B$ gives us the product section, $s_1 s_2$, we see that $s_1 s_2$ is continuous. Similarly, $s_1 + s_2$ and λs_1 are continuous for any $\lambda \in \mathbb{R}$. Thus, $\Gamma(E_B)$ is a commutative algebra over $\mathbb{R}$. Since $\mathcal{M}_B$ is compact, $\|s\| = \sup_{I \in \mathcal{M}_B} |s(I)|$ is easily seen to define a norm on $\Gamma(E_B)$ by 3.3.(4), making it a normed algebra.

The second slightly subtle point is to see that $\Gamma(E_B)$ is complete. This would be obvious if the bundle $p\colon E_B \longrightarrow \mathcal{M}_B$ were locally trivial. As this is not generally the case, we must look more closely at how things are defined. Let $\{s_n\}$ be a Cauchy sequence in $\Gamma(E_B)$, then for each $I \in \mathcal{M}_B$, $\{s_n(I)\}$ is a Cauchy sequence in $(B/I, |\cdot|)$ and hence converges to an element $s(I) \in B/I$. It suffices to see that the section $s\colon \mathcal{M}_B \longrightarrow E_B$ is continuous. So, let $I_o \in \mathcal{M}_B$ and let $s(I_o) = b_0 + \ker \varphi_o$ where $\ker \varphi_o = I_o$. Let $z_o = \varphi_o(b_o)$ and so $g(z_o, \varphi_o) = s(I_o) \in E_B$. Then a basic neighbourhood of $s(I_o)$ in E_B is of the form $g\big((O_\epsilon \times N)'\big)$ where O_ϵ is an ϵ neighbourhood of z_o in $\mathbb{C}$, N is a neighbourhood of φ_o in Φ_B and

$$(O_\epsilon \times N)' = (O_\epsilon \times N) \cap (\mathbb{C} \times \Phi_B)'.$$

Then, $g\big((O_\epsilon \times N)'\big) = \{b + \ker\varphi \mid \varphi \in N \text{ and } |\varphi(b) - z_o| < \epsilon\}$. Now, choose s_n so that $|s(I) - s_n(I)| < \frac{\epsilon}{2}$ for all I. Since s_n is continuous we can find a neighbourhood U of I_o in $\mathcal{M}_B$ so that $s_n(U) \subseteq g\big((O_{\epsilon/2} \times N)'\big)$. We claim that $s(U) \subseteq g\big((O_\epsilon \times N)'\big)$ and so s is continuous at I_o. If $I \in U$ then $s_n(I) = b_n + I = b_n + \ker\varphi$ where $\varphi \in N$ and $|\varphi(b_n) - z_o| < \frac{\epsilon}{2}$. Thus, $s(I) = b + I = b + \ker\varphi$ where $\varphi \in N$. Moreover,

$$\begin{aligned} |\varphi(b) - z_o| &\leq |\varphi(b) - \varphi(b_n)| + |\varphi(b_n) - z_o| \\ &= |s(I) - s_n(I)| + |\varphi(b_n) - z_o| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

■

We are now ready to prove our "Gelfand Theorem".

THEOREM 3.5. *Let B be a commutative Banach algebra over $\mathbb{R}$ with identity and let $p\colon E_B \longrightarrow \mathcal{M}_B$ be the associated bundle of real fields. Then*

(1) $\Gamma(E_B)$ is a commutative Banach algebra over $\mathbb{R}$ with identity (in the supremum norm).

(2) $\Lambda\colon B \longrightarrow \Gamma(E_B)$ is a norm-decreasing algebra homomorphism with kernel, Rad B.

(3) For $b \in B$, $\|\hat{b}\| = \nu(b) := \lim_{n\to\infty} \|b^n\|^{\frac{1}{n}}$.

PROOF. Part (1) is just Proposition 3.4.

For part (2) we observe that Λ is clearly an algebra homomorphism and

$$\|\hat{b}\| = \sup_{I \in \mathcal{M}_B} |b + I| \leq \sup_{I \in \mathcal{M}_B} \|b + I\| \leq \|b\|.$$

Moreover, $\hat{b} = 0 \Leftrightarrow b \in \bigcap_{I \in \mathcal{M}_B} I = \operatorname{Rad} B$.

To see part (3), we have

$$\|\hat{b}\| = \sup_{I \in \mathcal{M}_B} |b + I| = \sup_{\varphi \in \Phi_B} |\varphi(b)| = \nu(b)$$

by [5], Chapter 1.6.4]. ■

4. **Almost complex real Banach algebras.**

DEFINITION 4.1. Let B be a commutative Banach algebra over $\mathbb{R}$. Then B is called *strictly real* if $\Phi_B^{\mathbb{R}} = \Phi_B$ (or, $\mathcal{M}_B^{\mathbb{R}} = \mathcal{M}_B$), see [4]. At the other extreme, B is called *almost complex* if $\Phi_B^{\mathbb{C}} = \Phi_B$ (or, $\mathcal{M}_B^{\mathbb{C}} = \mathcal{M}_B$). We say that the real algebra is *complex* (or, of *complex type*) if it is possible to extend the scalar multiplication to complex scalars so that the algebra becomes a complex (normed) algebra [4]. If B has an identity, then B is complex if and only if there is an element $j \in B$ with $j^2 = -1$ ([4], 6.3).

Now, in order to determine which almost complex Banach algebras are, in fact, complex, we need to formulate the condition of complete regularity. This is usually done for complex Banach algebras in terms of Φ_B. Here, we follow our general philosophy for real Banach algebras and formulate the definition in terms of $\mathcal{M}_B$. In fact, this is necessary: consider $\mathbb{C}$ as a real Banach algebra, then $\Phi_{\mathbb{C}}$ consists of two points which are both closed, but there is no element of $\mathbb{C}$ which is zero on one point and nonzero on the other.

DEFINITION 4.2. Let B be a commutative Banach algebra over $\mathbb{R}$. Then B is called *completely regular* if for each closed set $F \subseteq \mathcal{M}_B$ and each $I \in (\mathcal{M}_B \setminus F)$ we can find a $b \in B$ with

(1) $\hat{b}(I) \neq 0$ (i.e., $b \notin I$),
(2) $\hat{b}(J) = 0$ for all $J \in F$ (i.e., $b \in \bigcap_{J \in F} J$).

PROPOSITION 4.3. *Let B be a complex commutative Banach algebra. Then, B is completely regular as a complex Banach algebra if and only if B is completely regular as a real Banach algebra.*

PROOF. Immediate. ∎

The property of completely regular algebras that we wish to exploit is the existence of "partitions of unity". The simplest proof of this requires a smidgen of hull-kernel theory which we now present.

DEFINITION 4.4. Let B be a commutative Banach algebra over $\mathbb{R}$. Let $I \subseteq B$ be an ideal. Then, the *hull of I* is defined to be:

$$h(I) = \{J \in \mathcal{M}_B \mid I \subseteq J\}.$$

If $S \subseteq \mathcal{M}_B$ is any subset, then the kernel of S is defined to be:

$$k(S) = \bigcap_{J \in S} J.$$

PROPOSITION 4.5. *Let B be a commutative Banach algebra over $\mathbb{R}$.*
(1) If $I_1, \ldots, I_k$ are ideals in B, then $h(I_1 + \cdots + I_k) = \bigcap_{i=1}^{k} h(I_i)$.
(2) If B is completely regular and $F \subseteq \mathcal{M}_B$ is closed then $h\big(k(F)\big) = F$.

PROOF. Straightforward. ∎

THEOREM 4.6. *Let B be a completely regular, commutative Banach algebra over $\mathbb{R}$ with identity. If $U_1, \ldots, U_k$ is an open cover of $\mathcal{M}_B$ then there exist $b_1, \ldots, b_k \in B$ so that*
(1) $b_1 + b_2 + \cdots + b_k = 1$
(2) $\hat{b}_i(U_i^c) \equiv 0$ for each $i = 1, \ldots, k$.

PROOF. For each $i = 1, \ldots, k$ let $I_i = k(U_i^c) = \bigcap_{\substack{J \notin U_k \\ J \in \mathcal{M}_B}} J$. Then, $I = I_1 + \cdots + I_k$ is an ideal and so

$$h(I) = \bigcap_{i=1}^{k} h(I_i) = \bigcap_{i=1}^{k} h\Big(k\big(U_i^c\big)\Big) = \bigcap_{i=1}^{k} U_i^c = \emptyset.$$

Since every proper ideal is contained in some maximal ideal, we must have $I = B$. Thus $1 = b_1 + \cdots + b_k$ with $b_i \in I_i = k(U_i^c) = \bigcap_{J \in U_i^c} J$. That is, for $J \in U_i^c$, $b_i \in J$ and so $\hat{b}_i(J) = b_i + J = 0$. ∎

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LEMMA 4.7. *Let B be a real commutative Banach algebra with identity. Then, B is complex if and only if there is an element $b \in B$ so that* $\mathrm{Im}\big(\varphi(b)\big) \neq 0$ *for each* $\varphi \in \Phi_B$.

PROOF. If B is complex, let $b = i \cdot 1$, so that $\varphi(b) = \pm i$ for each $\varphi \in \Phi_B$.

On the other hand if b is such an element then we can assume that $\nu(b) < 1$. Clearly, b is invertible, and so is $b + t1$ for every real number t. Thus, $b+1$ and $b-1$ are connected in the group of invertibles by the continuous path $t \longmapsto b + t1$ for $t \in [-1, 1]$. Now, $\nu\big((b+1)-1\big) < 1$ and so the straight line path joining $b+1$ to 1 lies entirely in the group of invertibles by the usual Neumann series. Similarly, $b - 1$ is connected to -1. Thus, -1 can be connected to 1 by a path of invertibles and so -1 is an exponential. That is, $-1 = e^x$ for some $x \in B$ and so $j = e^{\frac{1}{2}x}$ satisfies $j^2 = -1$ and hence B is complex. ∎

THEOREM 4.8. *Let B be an almost complex real commutative Banach algebra with identity. Consider the following conditions:*

(1) B is complex,

(2) $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$ is a trivial $\mathbb{Z}_2$-bundle,

(3) $w_1(B) = 0$ in $\check{H}^1(\mathcal{M}_B, \mathbb{Z}_2)$,

(4) $p\colon E_B \longrightarrow \mathcal{M}_B$ is a trivial $\mathbb{C}$-bundle with structure group $\mathbb{Z}_2$,

(5) there is an element $b \in B$ with $\mathrm{Im}\big(\varphi(b)\big) \neq 0$ *for each $\varphi \in \Phi_B$.*

Then, (1)⇔(5)⇒(2)⇔(3) ⇔(4). If B is completely regular, then (2)⇒(1) so that all of the conditions are equivalent.

PROOF. The implications (2)⇔(3)⇔(4) are standard bundle-theoretic fare, [3].

(1)⇔(5): This is the content of 4.7.

(5)⇒(2): Given b so that $\mathrm{Im}\big(\varphi(b)\big) \neq 0$ for all $\varphi \in \Phi_B$, let $\Phi_B^+ = \{\varphi \in \Phi_B \mid \mathrm{Im}\,\varphi(b) > 0\}$ and let $\Phi_B^- = \{\varphi \in \Phi_B \mid \mathrm{Im}\,\varphi(b) < 0\}$. These are disjoint closed and open sets covering Φ_B with $\sigma(\Phi_B^\pm) = \Phi_B^\mp$ so that $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$ is trivial.

Now, suppose that B is completely regular and that $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$ is trivial. Let Φ_B^+ be a compact subset of Φ_B so that the restriction of ϵ, $\epsilon_+\colon \Phi_B^+ \longrightarrow \mathcal{M}_B$ is a homeomorphism, and $\Phi_B = \Phi_B^+ \cup \sigma(\Phi_B^+)$ disjointly. Fix $\varphi \in \Phi_B^+$ and choose $a \in B$ so that $\varphi(a) = i$. Let U be a neighbourhood of φ in Φ_B^+ so that $|\psi(a) - i| < \frac{1}{2}$ for all $\psi \in U$. Let $I = k\big(\epsilon_+(\bar{U})\big)$, a closed ideal in B. Then,

$$\mathcal{M}_{B/I} = \{J \in \mathcal{M}_B \mid I \subseteq J\} = h(I) = hk\big(\epsilon_+(\bar{U})\big) = \epsilon_+(\bar{U})$$

and so one easily identifies $\Phi_{B/I}$ as $\bar{U} \cup \sigma(\bar{U})$. Now, Lemma 4.7 applies to the element $a + I$ in the algebra B/I, and so there is an element $k + I \in B/I$ so that $\psi(k) = \pm i$ for all $\psi \in \bar{U}$ and without loss of generality $\varphi(k) = +i$. By shrinking if necessary, we get a neighbourhood V of φ and $k \in B$ so that $\psi(k) = i$ for all $\psi \in V$. Now, cover Φ_B^+ with neighbourhoods $V_1, V_2, \ldots V_n$ and choose elements $k_1, k_2, \ldots, k_n$ so that $\varphi(k_j) = i$ for all $\varphi \in V_j$. Let $b_1, b_2, \ldots, b_n$ be the partition of unity relative to the open cover $\epsilon_+(V_1), \ldots, \epsilon_+(V_n)$ of $\mathcal{M}_B$ provided by Theorem 4.6. Let $c = \sum_{j=1}^n b_j k_j$. Then, for

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$\varphi \in \Phi_B^+$ we have

$$\varphi(c) = \sum_{j=1}^{n} \varphi(b_j)\varphi(k_j) = \sum_{\varphi(b_j)\neq 0} \varphi(b_j)\varphi(k_j) = \sum_{\varphi(b_j)\neq 0} \varphi(b_j)\cdot i$$
$$= \sum_{j=1}^{n} \varphi(b_j)\cdot i = \varphi(1)\cdot i = i.$$

If $\varphi \in \Phi_B^- := \sigma(\Phi_B^+)$ then $\varphi(c) = -i$ and so by Lemma 4.7, B is complex. ∎

REMARK. It was condition (5) which tipped us off that we were in a bundle situation! This was long before we realized it was equivalent to condition (1).

COROLLARY 4.9. *Any almost complex real commutative Banach algebra with identity is* locally *complex in the following sense: for each $I \in \mathcal{M}_B$, there is a neighbourhood U of I so that for $J := k(\bar{U})$, B/J is complex.*

PROOF. See the first part of the proof of (2)⇒(1) in the previous theorem; complete regularity was not used. ∎

5. **Almost complex C^*-algebras.**

DEFINITION 5.1. An almost complex real commutative Banach algebra B is said to be an *almost complex (hermitian) Banach $*$-algebra* if there is an automorphism, $*$, of period 2 so that $\varphi(b^*) = \overline{\varphi(b)}$ for all $b \in B$, and all $\varphi \in \Phi_B$. (We note that $*$ is uniquely determined if B is semisimple.) If, in addition, B satisfies $\|b^*b\| = \|b\|^2$ for all $b \in B$ then we call B an *almost complex C^*-algebra.*

EXAMPLES 5.2. (i) Let $E \longrightarrow X$ be any principal $\mathbb{Z}_2$-bundle with E a compact Hausdorff space. Thus, $\mathbb{Z}_2 = \{\sigma, e\}$ acts freely on E, and $X = E/\mathbb{Z}_2$. Let

$$B_E = \{f : E \longrightarrow \mathbb{C} \mid f \text{ is continuous and } f\big(\sigma(t)\big) = \overline{f(t)} \text{ for all } t \in E\}.$$

Then B_E is a real commutative Banach algebra with identity under pointwise operations and the supremum norm. One easily checks that $f^*(t) = \overline{f(t)}$ is a well-defined automorphism of period 2. One checks that $\Phi_{B_E} \cong E$ and that $\mathcal{M}_{B_E} \cong X$ and that the natural choices for these homeomorphisms makes the following $\begin{array}{ccc} \Phi_{B_E} & \cong & E \\ \downarrow & & \downarrow \\ \mathcal{M}_{B_E} & \cong & X \end{array}$ an isomorphism of $\mathbb{Z}_2$-bundles. Thus, the map $*$ makes B_E into an almost complex C^*-algebra whose canonical $\mathbb{Z}_2$ bundle is naturally isomorphic to the given bundle.

(ii) Let $p : T \longrightarrow S^1$ be the usual double cover of S^1, $p(z) = z^2$. Then, $\sigma : T \longrightarrow T$ is given by $\sigma(z) = -z$ so that $T/\mathbb{Z}_2 = S^1$. Define

$$B = \{f : T \longrightarrow \mathbb{C} \mid f \text{ is } C^1 \text{ and } f(-z) = \overline{f(z)}\}.$$

We give B the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Then, B is a real commutative Banach algebra with identity and $f^*(z) = \overline{f(z)}$ makes B into an almost complex (hermitian) Banach $*$-algebra which is clearly not an almost complex C^*-algebra.

(iii) Let B be the usual (complex) disc algebra considered as a real Banach algebra, then B is an almost complex Banach algebra which cannot be made into an almost complex Banach $*$-algebra.

We are aiming for the following theorem.

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THEOREM 5.3. *Let B be a commutative almost complex C^*-algebra with identity. Then the Gelfand transform $\Lambda\colon B \longrightarrow \Gamma(E_B)$ is an isometric $*$-isomorphism.*

We prove this in a series of easily digested lemmas.

LEMMA 5.4. *If B is a commutative almost complex C^*-algebra, then $*$ is isometric and $\|b\| = \nu(b)$ for all $b \in B$.*

PROOF. $\|b\|^2 = \|b^*b\| \leq \|b^*\|\,\|b\|$ so that $\|b\| \leq \|b^*\|$. Thus, taking $*$ gives the other inequality.

Now, $\|b^n\|^{\frac{1}{n}} = \|(b^n)^*b^n\|^{\frac{1}{2n}} = \|(b^*b)^n\|^{\frac{1}{2n}}$ and if $n = 2^k$ then $\|(b^*b)^n\| = \|b^*b\|^n$ since $(b^*b)^* = b^*b$. Thus, for $n = 2^k$ we get $\|b^n\|^{\frac{1}{n}} = \|b^*b\|^{\frac{1}{2}} = \|b\|$. Thus $\nu(b) = \lim_{k\to\infty} \|b^{2^k}\|^{\frac{1}{2^k}} = \|b\|$. ∎

DEFINITION 5.5. If B is a real commutative Banach algebra with identity and $b \in B$, then we let $\mathrm{sp}(b) = \{\varphi(b) \mid \varphi \in \Phi_B\}$ and call this the *spectrum of* b. By Theorem 3.1.6 of [5], this agrees with the usual terminology.

LEMMA 5.6. *If B is a commutative almost complex C^*-algebra with identity and if $b \in B$ has real spectrum, then there is an isometric monomorphism: $C_{\mathbb{R}}\big((\mathrm{sp}(b)\big) \longrightarrow B$ defined by $x \longmapsto b$ and $1 \longmapsto 1$ whose* range *is the Banach $*$ subalgebra of b generated by b and 1. Moreover, $\varphi\big(f(b)\big) = f\big(\varphi(b)\big)$ for all $f \in C_{\mathbb{R}}\big(\mathrm{sp}(b)\big)$ and $\varphi \in \Phi_B$.*

PROOF. If p is a polynomial with real coefficients then $p(b) \in B$ and

$$\|p(b)\| = \|\widehat{p(b)}\|_\infty = \|p(\hat{b})\|_\infty = \sup_{\varphi\in\Phi_B} \big|\varphi\big(p(\hat{b})\big)\big| = \sup_{\varphi\in\Phi_B} \big|p\big(\varphi(b)\big)\big|$$
$$= \|p\|_{\infty,\mathrm{sp}(b)}.$$

Thus, we can extend by continuity to get the required map. ∎

LEMMA 5.7. *Let B be a commutative almost complex C^*-algebra with identity. Then B is completely regular.*

PROOF. Let $I_1 \neq I_2 \in \mathcal{M}_B$. Choose $b \in B$ so that $\hat{b}(I_1) \neq 0$ and $\hat{b}(I_2) = 0$. By looking at b^*b and then scaling we can assume in addition that $\hat{b}(I_1) = 1$ and $\varphi(b) \in [0,\infty)$ for all $\psi \in \Phi_B$. Let f be the continuous function on $\mathbb{R} \supseteq \mathrm{sp}(b)$ so that

$$f(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{4} \\ \text{linear} & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4}. \\ 1 & \text{if } t \geq \frac{3}{4} \end{cases}$$

Then $a = f(b) \in B$, $a = a^*$ and $\|a\| = 1$. Now,

$$\{I \mid \hat{a}(I) = 1\} = \epsilon(\{\varphi \mid \varphi(a) = 1\})$$
$$\supseteq \epsilon\big(\{\varphi \mid \varphi(b) \subseteq (\tfrac{3}{4},\infty)\}\big)$$

is open in $\mathcal{M}_B$. That is, $\hat{a} \equiv 1$ on a neighbourhood of I_1. Similarly, $\hat{a} \equiv 0$ on a neighbourhood of I_2. A simple compactness argument shows that B is completely regular. ∎

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REMARK. In this setting, one can easily show that one can choose partitions of unity (Theorem 4.6) so that the b_k's are nonnegative.

PROOF OF THE THEOREM. By Lemma 5.4, the transform $\Lambda\colon B \longrightarrow \Gamma(E_B)$ is isometric. We now show that the map is onto. First, we observe that we have an involution, $*$, defined on $\Gamma(E_B)$ making the map Λ a $*$-map. Namely for $s \in \Gamma(E_B)$ we define $s^*(I)$ to be $\overline{s(I)}$ where we identify $B/I \cong \mathbb{C}$. One easily checks that this is well defined and s^* is continuous (see the remark after Lemma 3.2). Moreover, with this defintion it is clear that Λ is a $*$-map. Now, if $s = s^*$ in $\Gamma(E_B)$ then given $\epsilon > 0$ we can find real numbers $t_1, \ldots, t_n$ and an open cover $U_1, \ldots, U_n$ of $\mathcal{M}_B$ so that $|s(I) - t_k 1| < \epsilon$ for all $I \in U_k$ for each $k = 1, \ldots, n$. Let $b_1, \ldots, b_n$ be a nonnegative partition of unity in B subordinate to this open cover. Then $\forall I \in \mathcal{M}_B$ we have

$$\begin{aligned} \Big|s(I) - \sum_{k=1}^{n} b_k(I)t_k\Big| &= \Big|\sum_{I \in U_k} b_k(I)s(I) - \sum_{I \in U_k} b_k(I)t_k\Big| \\ &= \Big|\sum_{I \in U_k} b_k(I)\big(s(I) - t_k\big)\Big| \\ &\leq \sum_{I \in U_k} |b_k(I)|\epsilon \leq \epsilon. \end{aligned}$$

Thus, $\|s - \sum_{k=1}^{n} b_k t_k\| \leq \epsilon$ and since $\hat{B}$ is closed in $\Gamma(E_B)$ we have a $s \in \hat{B}$.

Now suppose $s^* = -s$ is in $\Gamma(E_B)$. By Corollary 4.9, for each $I \in \mathcal{M}_B$, we can find a neighbourhood U of I and an element $a \in B$ so that $\widehat{a^2} = -1$ on U. Now, this implies $\widehat{a^*}(J) = -\hat{a}(J)$ for all $J \in U$. Thus, $\hat{a} \cdot s \in \Gamma(E_B)$ is self-adjoint when restriced to U. By the above, we can find $b \in B$ with $\hat{b} = \hat{a} \cdot s$ *on* U. Then $-\widehat{ab} = -\hat{a}\hat{b} = -\hat{a}^2 \cdot s = s$ on U. A partition of unity argument shows that we can find $b \in B$ with $\hat{b} = s$ on all of $\mathcal{M}_B$. Finally, if $s \in \Gamma(E_B)$ then

$$s = \frac{1}{2}(s + s^*) + \frac{1}{2}(s - s^*)$$

and we have elements $a, b \in B$ with $\hat{a} = \frac{1}{2}(s+s^*)$ and $\hat{b} = \frac{1}{2}(s-s^*)$ and so $(a+b)\hat{} = s$. ∎

REMARK. Combining 2.2, 4.8, 5.2 and 5.3 we can show the following.

THEOREM 5.8. *The map $B \longmapsto w_1(B)$ induces a bijection from isomorphism classes of unital commutative almost complex C^*-algebras with $\mathcal{M}_B = X$ to the abelian group $\check{H}^1(X, \mathbb{Z}_2)$.*

REMARK. By a trivial variation on example 5.2(i) we can construct a real commutative Banach algebra B with identity so that $\epsilon\colon \Phi_B \longrightarrow \mathcal{M}_B$ is isomorphic to *any* (compact, Hausdorff) $\mathbb{Z}_2$-space $E \longrightarrow X$. The next example shows that such "bundles" do not always have local sections in neighbourhoods of points $x_0 \in X$ which only have one pre-image in E but which are limits of "doubly-covered" points $\{x_n\}$.

Let $E_n \longrightarrow X_n$ for each $n = 1, 2, 3, \ldots$ be a nontrivial $\mathbb{Z}_2$-bundle. Let E and X denote the one-point compactifications of the disjoint unions of the E_n's and the X_n's, respectively. Then, the natural map $E \longrightarrow X$ makes E a $\mathbb{Z}_2$-space over X. Clearly, there are no local sections in a neighbourhood of ∞ in X.

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