Can. J. Math., Vol. XXVIII, No. 6, 1976, pp. 1340–1344

UNIQUELY COLOURABLE GRAPHS WITH LARGE GIRTH

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Tutte [**1**], writing under a pseudonym, was the first to prove that a graph with a large chromatic number need not contain a triangle. The result was rediscovered by Zykov [**5**] and Mycielski [**4**]. Erdös [**2**] proved the much stronger result that for every $k \geqq 2$ and g there exist a k-chromatic graph whose girth is at least g.

A graph is said to be *uniquely k-colourable* if it is k-colourable and any k-colouring of the vertices gives the same colour classes. Harary, Hedetniemi and Robinson [**3**] proved that a uniquely k-colourable graph need not contain a complete graph of order k. The aim of this note is to prove a common extension of these results.

The vertex set of a graph H is denoted by $V(H)$. C^l is a *cycle* of length l and P^l is a *path* of length l. Two edges are *independent* if they are not adjacent. In the proof of our theorem we shall make use of some straightforward inequalities involving binomial coefficients. For convenience we list them here.

If $0 < b < a$ then Stirling's formula gives

$$(1) \quad \binom{a}{b} < \left(\frac{ea}{b}\right)^b.$$

If $0 \leqq x < b$ and $b + x < a$ then by expanding the coefficients we obtain

$$(2) \quad \binom{a-x}{b}\binom{a}{b}^{-1} \leqq \left(\frac{a-b}{a}\right)^x < e^{-bx/a}$$

and

$$(3) \quad \binom{a-x}{b-x}\binom{a}{b}^{-1} \leqq \left(\frac{b}{a}\right)^x.$$

THEOREM. *For all $k \geqq 2$ and $g \geqq 3$ there is a uniquely k-colourable graph whose girth is at least g.*

Proof. Let $V_1, V_2, \ldots, V_k$ be disjoint n-sets. Let $\mathcal{G}$ be the set of all k-partite graphs with vertex sets $V_1, V_2, \ldots, V_k$, containing $m = \left[\binom{k}{2} n^{1+\epsilon}\right]$ edges,

Received June 4, 1976.

https://doi.org/10.4153/CJM-1976-133-5 Published online by Cambridge University Press

where $0 < \epsilon < 1/4g$. Note that

$$|\mathscr{G}| = \binom{\binom{k}{2}n^2}{m}.$$

We shall find it convenient to adopt the language of probability theory. From now on by a graph we mean a member of $\mathscr{G}$, and each graph is supposed to occur with the same probability. In all subsequent inequalities n is supposed to be sufficiently large. The proof is based on estimating three different subsets of $\mathscr{G}$.

(*i*) The expected number of cycles C^l ($l \geqq 3$ is fixed) in a graph $G \in \mathscr{G}$ is at most

$$N_l = \binom{kn}{l}\frac{l!}{2l}\binom{\binom{k}{2}n^2 - l}{m-l}\binom{\binom{k}{2}n^2}{m}^{-1},$$

since there are $\binom{kn}{l}\frac{l!}{2l}$ ways of choosing a cycle C^l with $V(C^l) \subset V = \bigcup_1^k V_i$, and a cycle C^l is contained in 0 or

$$\binom{\binom{k}{2}n^2 - l}{m-l}$$

of the graphs belonging to $\mathscr{G}$.

Note that by (3)

$$N_l < \frac{(kn)^l}{2l}m^l\left(\binom{k}{2}n^2\right)^{-l} < (kn)^l n^{-l(1-\epsilon)} = k^l n^{l\epsilon},$$

so

$$\sum_{l=3}^{g-1} N_l < n^{-\epsilon/2} n^{g\epsilon}.$$

This implies, in particular, that if $\mathscr{G}_1$ denotes the set of all graphs with at most $f = [n^{g\epsilon}]$ cycles of length less than g then $|\mathscr{G}_1| \geqq (1 - n^{-\epsilon/2})\,|\mathscr{G}|$.

(*ii*) Let us estimate now the number of graphs in $\mathscr{G}$ that do not contain two cycles of length less than g having a vertex in common. Suppose $G \in \mathscr{G}$ does contain two such cycles. Then it also contains a cycle C^{l_1}, $l_1 < g$, and a path P^{l_2}, $l_2 < g$, joining two vertices of C^{l_1} such that C^{l_1} and P^{l_2} have no edge in common. The expected number of such pairs is at most

$$N(l_1, l_2) = l_1(kn)^{l_1}(kn)^{l_2-1}\binom{\binom{k}{2}n^2 - l_1 - l_2}{m - l_1 - l_2}\binom{\binom{k}{2}n^2}{m}^{-1}.$$

https://doi.org/10.4153/CJM-1976-133-5 Published online by Cambridge University Press

Applying (3) we see that

$$N(l_1, l_2) < l_1 k^{l_1+l_2} n^{\epsilon(l_1+l_2)} n^{-1}.$$

Since $2g\epsilon < \frac{1}{2}$ we obtain that

$$\sum_{\substack{3 \leqq l_1 < g \\ 1 \leqq l_2 < g}} N(l_1, l_2) < n^{-1/2}.$$

In particular, if $\mathscr{G}_2$ is the set of all graphs in $\mathscr{G}_1$ that do not contain two cycles of length less than g with a vertex in common, then

(4) $|\mathscr{G}_2| > (1 - n^{-\epsilon/3}) |\mathscr{G}|.$

(*iii*) Let us show now that most graphs $G \in \mathscr{G}$ have the property that if we omit a set E_0 of at most $n^{g\epsilon}$ *independent* edges, then the obtained graph G' is uniquely k-colourable.

Suppose $G \in \mathscr{G}$ and $G - E_0$ has a k-colouring whose colour classes are not $V_1, V_2, \ldots, V_k$. Let W be a colour class of maximal cardinality different from the V_i's. We distinguish two cases according to the sizes of the intersections $W \cap V_i$. Let $0 < \eta < 1/2k$.

(a) There are at least two colour classes, V_i and V_j, say, such that

$$|W \cap V_i| > \eta n \quad \text{and} \quad |W \cap V_j| > \eta n.$$

This implies that there exist sets $W_i, W_j, W_i \supset V_i, W_j \supset V_j, |W_i| = |W_j| = t = [\eta n]$ such that G has $q < n^{g\epsilon}$ edges joining W_i to W_j. Denote by $M(q)$ the expected number of pairs W_i, W_j joined by exactly q edges, $q < n^{g\epsilon}$. Our aim is to show that $M = \sum_{q \leqq n^{g\epsilon}} M(q)$ is small. We have

$$M(q) = \binom{k}{2}\binom{n}{t}^2\binom{t^2}{q}\binom{\binom{k}{2}n^2 - t^2}{m-q}\binom{\binom{k}{2}n^2}{m}^{-1}$$

Replacing $\binom{\binom{k}{2}n^2 - t^2}{m-q}$ with $\binom{\binom{k}{2}n^2 - t^2}{m}$ and applying (2) we get

$$M(q) < k^2 (e/\eta)^{2\eta n}\left(\frac{e\eta^2 n^2}{q}\right)^q e^{-1/2\eta^2 n^{1+\epsilon}} < n^{(1+\epsilon)q} e^{-1/3\eta^2 n^{1+\epsilon}}.$$

Hence

(5) $M < \exp(-n^{1+\epsilon/2} + 2n^{g\epsilon}\log n) < e^{-n}.$

(b) $|W \cap V_j| < \eta n$ unless $j = i$. Let $a_l = |W_l| = |W \cap V_l|$ for $l \neq i$ and let $W_i \subset W \cap V_i$, $|W_i| = n - \sum_{l \neq i} a_l = n - A$. Then $0 < A < k\eta n$ and G contains

$$q \leqq \min\{A, n^{g\epsilon}\}$$

edges joining the W_l's.

https://doi.org/10.4153/CJM-1976-133-5 Published online by Cambridge University Press

Denote by $L(i, (a_j), q)$ the expected number of subgraphs with vertex classes $W_1, W_2, \ldots, W_k$ and exactly q edges ($|W_l| = a_l$, $l \neq i$ and $|W_i| = n - A = n - \sum_{l \neq i} a_l$). Then

$$L(i, (a_j), q)$$
$$< \left(\prod_{j \neq i} \binom{n}{a_j}\right) \binom{n}{A} \binom{An}{q} \binom{\binom{k}{2}n^2 - A(n-A)}{m-q} \binom{\binom{k}{2}n^2}{m}^{-1}$$
$$< n^{2A}\left(\frac{eAn}{q}\right)^q e^{-A/2n^{\epsilon}}.$$

In the estimate above we used that

$$\binom{\binom{k}{2}n^2 - An + A^2}{m-q}\binom{\binom{k}{2}n^2}{m}^{-1} < \binom{\binom{k}{2}n^2 - An/2}{m}\binom{\binom{k}{2}n^2}{m}^{-1} < e^{-An^{\epsilon}/2}.$$

Denoting by $L(A)$ the sum of all $L(i, (a_j), q)$'s for which $\sum_{j \neq i} a_j = A$ we have

$$L(A) < k\, A^k\, (en)^{2A} \left(\frac{eAn}{q}\right)^q e^{-An^{\epsilon}/2}$$
$$< \left(\frac{eAn}{q}\right)^q \exp\left(-\frac{A}{2}n^{\epsilon} + 3A \log n\right) < n^{3q} e^{-An^{\epsilon}/3}.$$

Now if $1 \leqq A \leqq n^{g\epsilon}$ then $q \leqq A$ so

$$L(A) < n^{3A} e^{-A/3n\epsilon} < e^{-1/4n^{\epsilon}}.$$

Similarly, if $A \geqq n^{g\epsilon}$ and $g \leqq n^{g\epsilon}$ then

$$\log L(A) < 3n^{g\epsilon} \log n - \tfrac{1}{3}n^{(g+1)\epsilon} < -n^{g\epsilon},$$

so

$$L(A) < e^{-n^{g\epsilon}}.$$

Consequently if $L = \sum_A L(A)$ where the summation is over all values of A, $1 \leqq A \leqq \eta n$, then

(6) $L < e^{-n^{\epsilon}/2}$.

In particular, if $\mathscr{G}_3$ denotes the set of graphs in $\mathscr{G}$ which are such that no matter which $f = [n^{g\epsilon}]$ independent edges of it we omit, the resulting graph is still uniquely k-colourable, then (5) and (6) imply

(7) $|\mathscr{G}_3| > (1 - e^{-n^{\epsilon}/6})|\mathscr{G}|$.

Armed with inequalities (4) and (7) the theorem follows easily. Put $\mathscr{G}_4 = \mathscr{G}_2 \cap \mathscr{G}_3$. Then (4) and (7) give

$$|\mathscr{G}_4| \geqq (1 - n^{\epsilon/4})|\mathscr{G}|.$$

Let $G \in \mathscr{G}_4$. Then, since $G \in \mathscr{G}_2$, we can omit a set of $f = [n^{g\epsilon}]$ independent edges such that the resulting graph G^* has girth at least g. Since $G \in \mathscr{G}_4$ this graph G^* is uniquely k-colourable.

Call a graph G of at least $k + 1$ vertices *critically uniquely k-colourable* if it is uniquely k-colourable but no proper subgraph of it is. Clearly G cannot have $k + 1$ vertices and it is not trivial that for a given $k \geqq 3$ there are critically uniquely k-colourable graphs with arbitrarily many vertices. However, if G is uniquely k-colourable and its girth is at least $g > k$ then its minimal uniquely k-colourable subgraph (which must be critically uniquely k-colourable) must have at least $(k - 2)^{(g-2)/2}$ or, trivially, at least g vertices. Thus we have the following corollary of our theorem.

COROLLARY. *For every $k \geqq 3$ and n there is a critically uniquely k-colourable graph with at least n vertices.*

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