SOME GENERALIZED THEOREMS ON CONNECTIVITY

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The "k-dense" subgraphs of a connected graph G are connected and contain neighbours of all but at most k-1 points. We consider necessary and sufficient conditions that a point be in Γ_k, the union of the minimal k-dense subgraphs. It is shown that Γ_k contains all the "$[m, k]$-isthmuses" and "$[m, k]$-articulators"—minimal subgraphs which disconnect the graph into at least $k+1$ disjoint graphs—and that an $[m, k]$-isthmus or $[m, k]$-articulator of Γ_k disconnects G. We define "central points," "degree" of a point, and "chromatic number" and examine the relationship of these concepts to connectivity. Many theorems contain theorems previously proved **(1)** as special cases.

1. Definitions. The concepts *points*, *graph*, and *subgraph* will be used here in precisely the same sense as in a previous paper **(1)**, in which were also defined the *union*, *intersection*, and *difference* of two subgraphs, together with *neighbours*, *path* of *length k*, *diameter* of a graph, *connected* points and graphs, *m-connected* and *completely connected* graphs, *articulator*, a subgraph which *disconnects G*, and the *partition* of a *disconnected* graph. Unless otherwise specified, a connected graph G will have a finite number "n" of points, and the null graph will be assumed disconnected. If G' and G'' are subgraphs, $G'(G'')$ will denote the subgraph determined by all points in G'-G'' which have neighbours in G''. If G'' is a single point p, we denote this subgraph by $G'(p)$. The number of points in $G(p)$ is the degree of p, and the set of all points in G which have a given degree forms a *degree class.*

The *distance* between a point p_1 and a subgraph S contained in G-p_1 is the smallest positive integer q such that for some point p_2 in S there is a path of length q connecting p_1 and p_2. A subgraph G' *disconnects* two subgraphs which are contained in separate graphs of the partition of G-G'. An $[m, k]$-*isthmus* ($[m, k]$-*articulator*) is a completely connected (not completely connected) subgraph G' which disconnects G and has precisely m points, such that G' contains no proper disconnecting subgraph, and the partition of G-G' consists of at least $k+1$ graphs. The generic term *isthmus* will refer to a subgraph which, for any m, is an $[m, 1]$-isthmus. Ω_k will denote the union of all subgraphs which, for any m, are $[m, k]$-articulators or $[m, k]$-isthmuses of G. In Figure I, the points $\{2, 5\}$ determine a $[2, 1]$-isthmus, while $\{2, 4\}$ determine a $[2, 2]$-articulator.

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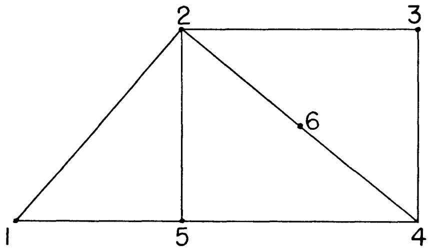

FIGURE 1

2. k-dense subgraphs. If k is any positive integer, a k-dense subgraph G' is a connected subgraph such that there are at most $k - 1$ points of $G - G'$ which have no neighbours in G'. In Figure 2, the points "1" and "4" are 1-dense while "2" and "3" are 2-dense.

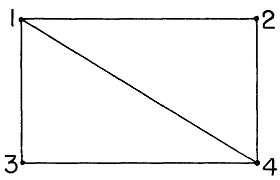

FIGURE 2

Clearly if G' is k-dense, then G' is p-dense for $p > k$. Another obvious result is

LEMMA 2.1. *A connected subgraph which contains a k-dense subgraph is k-dense.*

Let S_k denote the set of all k-dense subgraphs having at least k points, plus the null graph. A trivial consequence of 2.1 is

THEOREM 2.2. *The subgraphs in S_k form a lattice under the relation of set inclusion.*

If G is m-connected for $m < n$ and contains a point p_1, $G(p_1)$ must have at least m points which implies

THEOREM 2.3. *If G is m-connected for $m < n$, every connected subgraph is $(n\text{-}m)$-dense.*

A k-dense subgraph which properly contains no other k-dense subgraph is said to be D_k-minimal. The reader will easily verify:

THEOREM 2.4. *If a point is D_k-minimal, then it is contained in every $[m, k]$-isthmus and $[m, k]$-articulator.*

If $n > 1$ and a point p is not a $[1, 1]$-isthmus, $G - p$ is 1-dense and hence k-dense. If p is a $[1, 1]$-isthmus and the partition of $G - p$ contains precisely

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$q + 1$ graphs, q of which contain fewer than k points among them, then the remaining graph in the partition is k-dense, which cannot be the case if every q graphs in the partition contain at least k points. This proves:

THEOREM 2.5. *For $n > 1$, the intersection of all the k-dense subgraphs is precisely the subgraph G' of $[1, 1]$-isthmuses with the property that for any $[1, q]$-isthmus G'' in G' such that G'' is not a $[1, q + 1]$-isthmus, any set of q graphs in the partition of $G - G''$ contain at least k points among them.*

We shall now show how the assumption that G is k-connected or that a point is D_k-minimal restricts the distances between points of G. The *associated number* of a point p_1 is the greatest positive integer d such that for some point p_2 in G, d is the distance between p_1 and p_2. A point having minimum associated number is called a *central point* of G.

LEMMA 2.6. *Suppose G is k-connected. Let p be a point in G with associated number d, and define $G^r(p)$, $r = 1, \ldots, d$, to be the set of all points distant r from p, while $G^\circ(p) \equiv p$. It follows that:*[1]

1. $$G = \sum_{r \leqslant d} G^r(p).$$
2. *For $1 \leqslant r < d$ and $s > 0$, $G^r(p)$ disconnects $G^{r+s}(p)$ from all $G^i(p)$ for which $i \leqslant r - 1$.*
3. *For each r, $1 \leqslant r < d$, $G^r(p)$ contains at least k points.*

For any point $p_1 \neq p$, the distance of p_1 from p *is* $\leqslant d$ and thus p_1 is in some $G^i(p)$ for $i \leqslant d$, which proves Part 1. Let $\{p_0 = p^1, \ldots, p^q = p'\}$ determine a path connecting p_0 in $G^i(p)$, for $i < r$, to p' in $G^{r+s}(p)$. If p^i is distant $< r$ and p^{i+1} distant $> r$ from p, then there exists a path $\{p = q^1, \ldots, q^t = p^i\}$ such that $t \leqslant r$ and $\{q^1, \ldots, q^t, p^{i+1}\}$ is a path of length $\leqslant r$ connecting p and p^{i+1}, which is a contradiction. Therefore by induction it follows that no path with p_0 as initial point can contain a point in $G^i(p)$ for $i > r$ if it contains no point in $G^r(p)$—which proves Part 2. Since G is k-connected and $G^r(p)$ disconnects G for $1 \leqslant r < d$, $G^r(p)$ must contain at least k points.

THEOREM 2.7. *If G is k-connected, S is an r-dense subgraph for $r \leqslant mk + 1$, m being a non-negative integer, and p is a point in $G - S$, then the distance from p to S is $\leqslant m + 1$.*

By 2.6, if the distance from p to S is $\geqslant m + 2$, there exist disjoint subgraphs $G^i(p)$, $i = 0, \ldots, m$, in which no point has a neighbour in S and m of which have at least k points each, contrary to the assumption that S is r-dense for $r \leqslant mk + 1$. When S consists of a single point and $m = 1$, 2.7 implies

[1]The summation sign will be used to denote the union of all the subgraphs designated as summands.

THEOREM 2.8. *If G contains a point which is D_1-minimal, the set of central points is precisely the set of D_1-minimal points, and if G contains no point which is D_1-minimal, but G is k-connected and contains a point p which is D_r-minimal for $r \leqslant k + 1$, then p is a central point of G.*

THEOREM 2.9. *If G has an isthmus or articulator G', and also a point p_1 with associated number d, and a point p_2 distant d from G', then there is a graph G'' in the partition of $G - G'$ such that all the central points of G are contained in the union of G' and G''.*

Suppose p_2 and a point p_3 are in two disconnected graphs of the partition of $G - G'$. Then the distance from p_2 to p_3 is at least $d + 1$, and p_3 is not a central point.

3. Union of the D_k-minimal subgraphs. The symbol "Γ_k" will denote the union of all the D_k-minimal subgraphs of G.

LEMMA 3.1. *For given k, an $[m, k]$-isthmus or $[m, k]$-articulator of Γ_k disconnects G.*

Let Γ' be an $[m, k]$-isthmus or $[m, k]$-articulator of Γ_k and suppose $G - \Gamma'$ is connected. Every point in Γ' has a neighbour in $G - \Gamma'$, for Γ' can have no proper subgraph disconnecting Γ_k, and thus $G - \Gamma'$ is 1-dense and contains a D_k-minimal subgraph G'. But $G - \Gamma' - (G - \Gamma_k) = \Gamma_k - \Gamma'$ is not connected, and its partition consists of at least $k + 1$ graphs, so that it contains no D_k-minimal subgraph. Thus G' must contain some point of $G - \Gamma_k$, which is a contradiction.

If G is m-connected, no $[m, k]$-articulator or $[m, k]$-isthmus of Γ_k can have a proper subgraph which disconnects G. Thus we have proved:

THEOREM 3.2. *If G is m-connected, an $[m, k]$-articulator or $[m, k]$-isthmus of Γ_k is respectively an $[m, 1]$-articulator or $[m, 1]$-isthmus of G.*

THEOREM 3.3. *If S is any subgraph determined by k points and containing no D_k-minimal subgraph, then $\Gamma_k(S)$ disconnects G.*

If $G - \Gamma_k(S)$ is connected, it is k-dense and contains a D_k-minimal subgraph in which some point not in S must have a neighbour in S, which is impossible.

THEOREM 3.4. *If G is m-connected and $k \leqslant m$, then Γ_k contains at least $m - k + 1$ points.*

If there exists S satisfying the hypothesis of 3.3, then $\Gamma_k(S)$ contains at least m points as does Γ_k. If every subgraph of k points contains a D_k-minimal subgraph of G, $G - \Gamma_k$ contains fewer than k points, and so Γ_k contains more than $n - k$ points, that is, at least $m - k + 1$ points.

We now consider general conditions under which articulators and isthmuses of G are contained in Γ_k.

THEOREM 3.5. *If G' is an articulator or isthmus such that the partition of $G - G'$ consists of precisely q graphs, every $q - 1$ of which contain at least k points among them, then G' is contained in Γ_k.*

If p is a point in G', then $G - G' + p$ is 1-dense and contains a D_k-minimal subgraph which cannot be contained in $G - G'$. An immediate consequence is

THEOREM 3.6. *Ω_k is contained in Γ_k.*

In proving the following theorem, we use the results, proved in **(1)**, which assert that if G is not completely connected, it contains a disconnecting subgraph, and a proper subgraph which disconnects G contains an articulator or an isthmus.

THEOREM 3.7. *If G is not completely connected and for given k, every articulator or isthmus is respectively, for some m, an $[m, k]$-articulator or $[m, k]$-isthmus, then Γ_k is contained in Ω_k.*

Suppose p is a point in Γ_k. If p is D_k-minimal, p is contained in every $[m, k]$-isthmus and $[m, k]$-articulator by 2.4, and since G is not completely connected, Ω_k is not null. If p is not D_k-minimal, let Γ' be a D_k-minimal subgraph containing p. Then either $\Gamma' - p$ is not connected or there is a point p_1 in $G - \Gamma'$ such that p_1 is a neighbour of p and of no other point in Γ'. In the first of these two cases, $G - \Gamma' + p$ disconnects G, and therefore contains an $[m, k]$-isthmus or $[m, k]$-articulator which must contain p since there are at most $k - 1$ points in $G - \Gamma'$ which have no neighbours in Γ'. In the second case, $G - \Gamma' - p_1 + p$ disconnects G and contains an $[m, k]$-articulator or $[m, k]$-isthmus which must contain p by the same reasoning. For the case $k = 1$, 3.6 and 3.7 jointly imply:

THEOREM 3.8. *If G is not completely connected, then $\Omega_1 = \Gamma_1$.*

We now proceed to obtain necessary conditions and sufficient conditions that a point be contained in $G - \Gamma_k$.

THEOREM 3.9. *If p is a point which is not D_k-minimal and is not a $[1, 1]$-isthmus, and whenever G' is a minimal subgraph with the property that $G' + p$ disconnects two points of $G(p)$, G' also disconnects p from at least k points of $G - G'$, then p is contained in $G - \Gamma_k$.*

Suppose a point p satisfies the conditions of the theorem and there is a D_k-minimal subgraph Γ' containing p and a point which is a neighbour of p. Then either $\Gamma' - p$ is not connected (Case I), or there is a point p_1 in G which is a neighbour of p and of no other point in Γ' (Case II). In Case I, $G - \Gamma' + p$ disconnects two points of the intersection of $G(p)$ with Γ', and since p is not a $[1, 1]$-isthmus, $G - \Gamma'$ contains a minimal subgraph G', such that $G' + p$ disconnects these two points. But if G' disconnects p from k points of $G - G'$, then Γ' is not D_k-minimal. Thus $\Gamma' - p$ must be connected, and we consider Case II. In this case, $G - \Gamma' - p_1 + p$ disconnects p_1 in $G(p)$ from another point in the intersection of $G(p)$ with Γ'. Then $G - \Gamma'$ contains a minimal subgraph G' such that $G' + p$ disconnects these two points, but by the same reasoning as in Case I, G' cannot disconnect p from k points of $G - G'$ if Γ' is D_k-minimal. Thus p is contained in $G - \Gamma_k$.

THEOREM 3.10. *A point p is contained in $G - \Gamma_k$ only if whenever G' is a minimal subgraph with the property that $G' + p$ disconnects at least $k + 1$ points of $G(p)$ from one another, G' also disconnects p from at least k points of $G - G'$.*

If G' is a minimal subgraph with the property that $G' + p$ disconnects $k + 1$ points of $G(p)$, then every point of G' is connected to at least one of these $k + 1$ points by a path which, except for its initial point, contains only points of $G - G'$. Otherwise G' would contain a proper subgraph with this property. Accordingly, $G - G'$ contains a k-dense subgraph if G' does not disconnect p from at least k points in $G - G'$. But $G - G' - p$ cannot contain a k-dense subgraph, since it contains $k + 1$ points no pair of which is connected. Accordingly p belongs to a D_k-minimal subgraph. For the case $k = 1$, 3.9 and 3.10 yield

THEOREM 3.11. *A point p which is not a* [1, 1]*-isthmus and is not D_1-minimal is contained in $G - \Gamma_1$ if, and only if, whenever G' is a minimal subgraph with the property that $G' + p$ disconnects two points of $G(p)$, G' also disconnects G.*

In a previous paper **(1)**, it was shown that every connected graph contains at least two proper 1-dense subgraphs. This result will now be used to prove.

THEOREM 3.12. *If $G - \Gamma_k$ contains fewer than k points and the D_k-minimal subgraphs are mutually disjoint, then no D_k-minimal subgraph can have more than k points.*

Suppose $S_1, \ldots, S_r$ are the D_k-minimal subgraphs containing more than one point, and let S_r contain $q > k$ points. Then there is a point p_r' in S_r such that $S_r' = S_r - p_r'$ is a proper connected subgraph containing $q - 1 \geqslant k$ points. Then each of the S_i $(i = 1, \ldots, r - 1)$ being k-dense, must contain a point p_i having a neighbour in S_r'. There exists a proper subgraph $S_i' = S_i - p_i'$ containing p_i and all but one of the points, p_i', of S_i, since S_i contains two proper connected subgraphs. Consider the subgraph

$$S = \sum_{i=1}^{r} S_i'.$$

S is connected since each of the S_i' is connected and all the S_i' for $i < r$ contain neighbours of points in the connected subgraph S_r'. Since every D_k-minimal subgraph containing only one point must have a neighbour in S_r' and since each of the points p_i' has a neighbour in S_i', S contains neighbours of all the points in Γ_k which are not in S. Since $G - \Gamma_k$ contains fewer than k points, S is a k-dense subgraph containing no D_k-minimal subgraph, which is impossible.

In Figure 3 is shown an example of a graph in which all the conditions of 3.12 are satisfied for $k = 2$. The points numbered "1, 4, 6" are each D_2-minimal, as is the pair $\{2, 3\}$. The point numbered "5" is the only point in $G - \Gamma_2$.

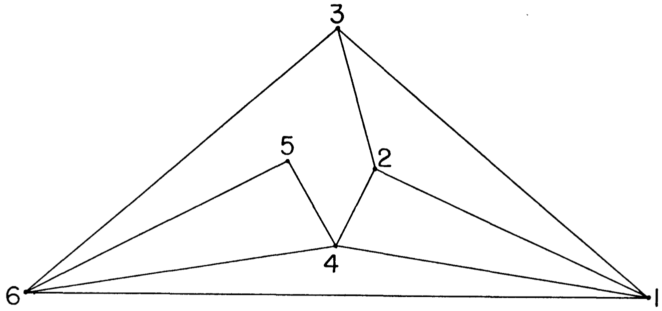

FIGURE 3

4. Colour Classes. A *colour class* is a subgraph no two points of which are neighbours. We shall study the relationship of this concept to connectivity, a relationship also investigated by Dirac (**3**; **4**). A *k-colouring* of G is a set of k colour classes such that each point of G lies in one, and only one, class of the set. The *chromatic number* of a subgraph G', which we shall denote by $c(G')$, is the smallest integer k such that there exists a k-colouring of G'. The graph in Figure 3 has chromatic number 3, a 3-colouring being formed from the colour classes $\{1, 5\}$, $\{3, 4\}$, $\{2, 6\}$.

THEOREM 4.1. *If G is m-connected for $m < n$:*

1. $c(g) \geqslant n/(n - m)$.
2. *If for any positive integer k, G has diameter $\geqslant 2k + 1$, then $c(G) \leqslant n - 2(mk - m + 1)$.*
3. *If G has diameter $d = k + 1$ for $k \geqslant 3$, no point of G can have degree $> n - m(k - 3) - 3$.*
4. *The diameter d of G satisfies the condition:*

$$d - 4 \leqslant (n - m - 3)/m.$$

By 2.3, there can be no set of more than $n - m$ points in which no pair are neighbours which implies Part 1.

To prove Part 2, let p be a point in G with associated number $d \geqslant 2k + 1$. By 2.6, there is sequence of subgraphs $G^r(p)$, $0 \leqslant r \leqslant d$, such that $G^i(p)$, $i \geqslant 0$, has no neighbour in $G^{i+2}(p)$, while from Part 3 of 2.6, each $G^i(p)$, $1 \leqslant i \leqslant 2k$, contains at least m points. Thus we can form m colour classes, each containing one point from $G^1(p), G^3(p), \ldots, G^{2k-1}(p)$, and m other colour classes, each containing a point from $G^2(p), \ldots, G^{2k}(p)$. The point p can be assigned the same colour as a point in $G^2(p)$, while a point in $G^{2k+1}(p)$

can be assigned the same colour as a point in $G^{2k-1}(p)$. If an additional colour is given to each of the $n - 2(mk + 1)$ points of G not already coloured, we obtain an $[n - 2\ (mk - m + 1)]$ – colouring of G.

If p' is a point in $G^r(p)$ for $2 \leqslant r \leqslant k - 1$, p′ has no neighbours in $G^s(p)$ for $s < r - 1$ and $s > r + 1$. Thus, including p', there are at least $m(k - 3) + 3$ points in G which are not neighbours of p', which accordingly is of degree at most $n - m(k - 3) - 3$. If p' is in $G^r(p)$ for $r = 0, 1$, then p' has no neighbours in $G^s(p)$ for $s \geqslant 3$, and thus p' is of degree at most $n - m(k - 2) - 2 \leqslant n - m(k - 3) - 3$. A similar argument applies when p' is in $G^k(p)$ or $G^{k+1}\ (p)$.

Part 4, for the case $d \geqslant 4$, follows from Part 3 and the observation, from 2.3, that if G is m-connected, no point has degree $< m$. If $d < 4$, the theorem is obviously true unless $m \geqslant n - 2$, in which case $d \leqslant 2$. Then $d - 4 \leqslant - 2 \leqslant - 2/m \leqslant (n - m - 3)/m$ since $m \leqslant$ n $- 1$.

THEOREM 4.2. *If G contains a subgraph G′ with the property that* $c(G') = m$, *the partition of* $G - G'$ *consists of* $q > 1$ *connected graphs* $G_i (i = 1, \ldots, q)$, *and* $r = \max\ \{c(G_i)\}$, *than* $c(G) \leqslant m + r$.

The reader can readily show that there exists an r-colouring of $G - G'$ which, combined with an m-colouring of G', gives an $(m + r)$-colouring of G.

THEOREM 4.3. *For k an integer* $\geqslant 1$, *if G contains at least* 2 *k degree classes,* $c(G) \leqslant n - k + 1$.

Let k degree classes $D_i (i = 1, \ldots, k)$ be arranged in order of decreasing degree, so that the degree of a point in D_j exceeds the degree of a point in D_{j+m} by at least $2m$. From each D_i choose one point p_i $(i = 1, \ldots, k)$. We can choose p_{12} in $G\ (p_1) - G\ (p_2) - p_2$. Assuming we have chosen $j - 1$ distinct points $p_{12}, \ldots, p_{1j}$ in such a way that p_{1r} is in

$$G(p_1) - G(p_r) - \sum_{i \leqslant r} p_i \qquad (r = 2, \ldots, j),$$

then since p_1 has at least $2j$ neighbours which are not neighbours of p_{j+1} and we have already chosen at most $2j - 1$ of these, we can choose $p_{1,\ j+1}$, distinct from p_{1r} for $r \leqslant j$, in

$$G(p_1) - G(p_{j+1}) - \sum_{i \leqslant j+1} p_i.$$

In this way we can form $k - 1$ colour classes $S_i = p_i + p_{1i} (i = 2, \ldots, k)$ each having two points. We obtain an $(n - k + 1)$ - colouring of G by colouring each of the remaining $n - 2(k - 1)$ points with a separate colour.

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