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# ON SOME PRODUCTS OF FINITE GROUPS

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*Abstract* A classical result of Baer states that a finite group $G$ which is the product of two normal supersoluble subgroups is supersoluble if and only if $G'$ is nilpotent. In this article, we show that if $G = AB$ is the product of supersoluble (respectively, $w$-supersoluble) subgroups $A$ and $B$, $A$ is normal in $G$ and $B$ permutes with every maximal subgroup of each Sylow subgroup of $A$, then $G$ is supersoluble (respectively, $w$-supersoluble), provided that $G'$ is nilpotent. We also investigate products of subgroups defined above when $A \cap B = 1$ and obtain more general results.

*Keywords:* finite groups; residuals; semidirect products; supersoluble groups; direct product

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## 1. Introduction

All groups considered here will be finite.

A significant number of articles investigating the properties of groups expressible as a product of two supersoluble subgroups were published since the 1957 paper by Baer [2] in which he proved that a normal product $G = AB$ of two supersoluble subgroups

$A$ and $B$ is supersoluble provided that the derived subgroup $G'$ is nilpotent. There has been many generalizations of this theorem. Instead of having normal subgroups, certain permutability conditions were imposed on the factors. The case in which $A$ permutes with every subgroup of $B$ and $B$ permutes with every subgroup of $A$, that is, when $G$ is a *mutually permutable product* of $A$ and $B$, is in fact one of the most interesting cases and has been investigated in detail (see [5] for a thorough review of results in this context and also [1] for general results on products).

In this article, we study a weak form of a normal product arising quite frequently in the structural study of mutually permutable products and appears not to have been investigated in detail.

**Definition 1.1.** *Let $G = AB$ be a product of subgroups $A$ and $B$. We say that $G$ is a weak normal product of $A$ and $B$ if*

*(a) $A$ is normal in $G$.*

*(b) $B$ permutes with all the maximal subgroups of Sylow subgroups of $A$.*

As an important first step in the study of weak normal products $G = AB$ and motivated by the mutually permutable case, we analyse the situation $A \cap B = 1$. In this case, they are semidirect products of $A$ and $B$.

**Definition 1.2.** *Let the group $G = AB$ be the weak normal product of $A$ and $B$ with $A$ normal in $G$. We say that $G$ is a weak direct product of $A$ and $B$ if $A \cap B = 1$. In this case, we write $G = [A]B$.*

We study these products when the factors are supersoluble and widely supersoluble and analyse the behaviour of the residuals associated to these classes of groups. Recall that a widely supersoluble group, or *w-supersoluble* group for short, is defined as a group $G$ such that every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal in $G$ (a subgroup $H$ of a group $G$ is $\mathbb{P}$-subnormal in $G$ whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G$, such that $|H_i{:}H_{i-1}|$ is a prime for every $i = 1, \ldots, n$).

The class of $w$-supersoluble groups, denoted $w\mathfrak{U}$, is a subgroup-closed saturated formation containing the subgroup-closed saturated formation $\mathfrak{U}$ of all supersoluble groups. Moreover $w$-supersoluble groups have a Sylow tower of supersoluble type (see [8, Corollary]).

Our first aim is to show that the saturated formations of all supersoluble groups and w-supersoluble groups are closed under the formation of weak direct products.

**Theorem A.** *Let $G = [A]B$ be a weak direct product of $A$ and $B$. If $A$ and $B$ belong to $\mathfrak{U}$, then $G$ is also supersoluble.*

Theorem A will be very useful in the proofs of Theorem B and Theorem C. Also we obtain the following result as a corollary.

**Corollary A.** *Let $G = [A]B$ be a weak direct product of $A$ and $B$. If $A$ and $B$ belong to $w\mathfrak{U}$, then $G$ is $w$-supersoluble.*

Our second aim is to show that the product of the supersoluble (respectively, w-supersoluble) residuals of the factors of weak direct products is just the supersoluble (respectively, w-supersoluble) residual of the group.

**Theorem B.** *Suppose that $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F} = w\mathfrak{U}$. Let $G = [A]B$ be a weak direct product of $A$ and $B$. Then*

$$G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}.$$

We now analyse the behaviour of weak normal products with respect to the formations of all supersoluble and w-supersoluble groups. Our next result shows that Baer's theorem can be generalized in this new direction:

**Theorem C.** *Let $G = AB$ be a weak normal product of $A$ and $B$. If $G'$ is nilpotent, $A$ is normal in $G$ and $A, B \in \mathfrak{U}$, then $G \in \mathfrak{U}$.*

As a corollary, we obtain the result for $w\mathfrak{U}$-groups.

**Corollary B.** *Let the group $G = AB$ be a weak normal product of $w\mathfrak{U}$-subgroups $A$ and $B$. If $G'$ is nilpotent and $A$ is normal in $G$, then $G$ belongs to $w\mathfrak{U}$.*

Our second objective is to investigate the residuals of weak normal products. Unfortunately, it does not follow that $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$ when $G$ is a weak normal product as the following examples show. Example 1.3(ii) generalizing (i) was communicated to the authors by the referee:

**Example 1.3.**

(i) Let

$$\begin{aligned} A = \langle g_2, g_4, g_5, g_6, g_7 \mid\ & g_2^3 = g_4^3 = g_5^3 = g_6^3 = g_7^3 = 1, \\ & g_4^{g_2} = g_4g_6, g_5^{g_2} = g_5g_7, g_6^{g_2} = g_6, g_7^{g_2} = g_7, \\ & g_5^{g_4} = g_5, g_6^{g_4} = g_6, g_7^{g_4} = g_7, \\ & g_6^{g_5} = g_6, g_7^{g_5} = g_7, \\ & g_7^{g_6} = g_7\rangle. \end{aligned}$$

Let $Q = \langle b\rangle \cong C_4$ act on $A$ via

$$g_2^b = g_2, \qquad g_4^b = g_4g_5, \qquad g_5^b = g_4g_5^2, \qquad g_6^b = g_6g_7, \qquad g_7^b = g_6g_7^2.$$

Let $G = [A]Q$ be the corresponding semidirect product.

Note that $A' = \Phi(A) = \langle g_6, g_7\rangle$. Let $A_0 = \langle g_4, g_5\rangle$. Then $A_0$ is not a normal subgroup of $A$ but is normalized by $Q$. Let $B = A_0\langle b\rangle$, then $\mathrm{Core}_G(B) = 1$. Furthermore, $B$ permutes with the 13 maximal subgroups of $A$. The supersoluble residual of $G$ is $\langle g_4, g_5, g_6, g_7\rangle$, giving a quotient isomorphic to $C_{12}$. Consequently, $G^{\mathfrak{U}} \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. This group corresponds to `SmallGroup(972, 406)` of GAP.

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(ii) Suppose $p$ is a prime number and $n$ is a positive integer, where $n$ is not a multiple of $p$, and the order of $p$ modulo $n$ is 2. Let $\mathbf{F}_{p^2}$ be the Galois field of order $p^2$, and note that $n$ is a factor of $p^2-1$, so there is an element $\beta$ of multiplicative order $n$ in $\mathbf{F}_{p^2}$. Let $V = U_0 \oplus U_1$ be a vector space of dimension 2 over $\mathbf{F}_{p^2}$, where $U_0$ and $U_1$ are one-dimensional $\mathbf{F}_{p^2}$-subspaces of $V$. Take elements $a$ and $b$ in the general linear group $\mathrm{GL}_2(p^2)$, with

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}.$$

Then $V$ can be regarded as an $\mathbf{F}_{p^2}\langle a,b\rangle$-module. Let $G = [V]\langle a,b\rangle$ be the corresponding semidirect product. Consider the following subgroups of $G$:

$$P = \langle a\rangle \simeq \mathbf{C}_p, \quad A = VP, \quad Q = \langle b\rangle \simeq \mathbf{C}_n, \quad B = U_0Q.$$

As in (i), $A$ is normal in $AB = G$, $A' = \Phi(A) = [V,P] = U_1$, $U_0$ is an $\mathbf{F}_pQ$-simple module, $B' = [U_0,Q] = U_0$, the core of $B$ in $G$ is 1, the number of maximal subgroups of $A$ is $(p^3-1)/(p-1) = p^2+p+1$, $A$ is the unique Sylow $p$-subgroup of $G$ and all the maximal subgroups of $A$ permute with $B$, while the supersoluble residual of $G$ is $G^{\mathfrak{U}} = V$, with $G/G^{\mathfrak{U}} \simeq PQ \simeq \mathbf{C}_{pn}$ and $G^{\mathfrak{U}} \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. This construction can be carried out when $p=2$ and $n=3$, giving an Example with $|G| = 2^5 3 = 96$; moreover, there is a maximal subgroup of $A$ which does not permute with the Sylow subgroup $Q$ of $B$.

We will use Theorem C to prove the following result (note that Example 1.3(ii) suggests the permutability hypothesis in Theorem D and Corollary D):

**Theorem D.** *Let the group $G = AB$ be a product of the subgroups $A$ and $B$. Assume that $A$ is a normal subgroup of $G$ and every Sylow subgroup of $B$ permutes with every maximal subgroup of every Sylow subgroup of $A$. If $G'$ is nilpotent, then $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$.*

An immediate consequence is:

**Corollary C.** *Let the group $G = AB$ be a product of the subgroups $A$ and $B$. Assume that $A$ is a normal subgroup of $G$ and every Sylow subgroup of $B$ permutes with every maximal subgroup of every Sylow subgroup of $A$. If $G'$ is nilpotent, then $G^{w\mathfrak{U}} = A^{w\mathfrak{U}}B^{w\mathfrak{U}}$.*

Denote by $\mathfrak{N}$ the class of all nilpotent groups. A nice result of Monakhov [7, Theorem 1] states that if $G = AB$ is the mutually permutable product of the supersoluble subgroups $A$ and $B$, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}$. We prove an analogue of this result for weak normal products.

**Corollary D.** *Let $G = AB$ be a weak normal product of the supersoluble subgroups $A$ and $B$. If $A$ is normal in $G$, we have that $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}$.*

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## 2. Preliminary results

It is easy to see that factor groups of weak normal products are also weak normal products. For weak direct products, we have the following:

**Lemma 2.1.** *Let $G = [A]B$ be a weak direct product of $A$ and $B$.*

*(a) If $N$ is a normal subgroup of $G$ such that $N \leqslant A$ or $N \leqslant B$, then $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$.*

*(b) If $K$ is a subgroup of $B$, then $[A]K$ is a weak direct product of $A$ and $K$.*

**Proof.** (a) Let $H/N$ be a Sylow $p$-subgroup of $AN/N$. Then $H/N = PN/N$, where $P$ is a Sylow $p$-subgroup of $A$. Let $K/N$ be a maximal subgroup of $H/N$. Then $K = K \cap PN = N(P \cap K)$ and $K/N = N(P \cap K)/N$. Thus,

$$p = |PN/N : (P \cap K)N/N| = \frac{|P||N|}{|P \cap N|}\frac{|P \cap K \cap N|}{|P \cap K||N|} = |P : P \cap K|.$$

Hence, $P \cap K$ is a maximal subgroup of $P$. Then $B$ permutes with $P \cap K$, and so $BN/N$ permutes with $K/N$. Therefore, $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$.

(b) Let $K$ be any proper subgroup of $B$ and $H$ be any maximal subgroup of a Sylow subgroup of $A$. By the hypotheses, we have $HB = BH$ and so $H = H(A \cap B) = A \cap HB$. Since $A$ is normal in $G$, it follows that $H$ is normal in $HB$ and so $B$ normalizes $H$. Hence, $K$ permutes with $H$. Therefore, $[A]K$ is a weak direct product of $A$ and $K$. □

Our second lemma contains some of the properties of $\mathbb{P}$-subnormal subgroups.

**Lemma 2.2.** **[8, Lemma 1.4]** *Let $G$ be a soluble group and $H$ and $K$ two subgroups of $G$. The following properties hold:*

*(i) If $H$ is $\mathbb{P}$-subnormal in $G$ and $N$ is normal in $G$, then $HN/N$ is $\mathbb{P}$-subnormal in $G/N$.*

*(ii) If $N$ is normal in $G$ and $HN/N$ is $\mathbb{P}$-subnormal in $G/N$, then $HN$ is $\mathbb{P}$-subnormal in $G$.*

*(iii) If $H$ is $\mathbb{P}$-subnormal in $K$ and $K$ is $\mathbb{P}$-subnormal in $G$, then $H$ is $\mathbb{P}$-subnormal in $G$.*

## 3. Supersoluble and w-supersoluble residuals

We start this section by proving Theorem A.

**Proof of Theorem A.** Assume that the result is false, and let $G$ be a counterexample of minimal order. Clearly, $G$ is soluble and $A$ and $B$ are proper subgroups of $G$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$, then applying Lemma 2.1(a),

$G/N = [A/N](BN/N)$ is a weak direct product of $A/N$ and $BN/N$. By the minimality of $G$, $G/N \in \mathfrak{U}$. Since the class of all supersoluble groups is a saturated formation, there exists a unique minimal normal subgroup $N$ of $G$ contained in $A$, $N$ a $p$-group for some prime $p$, $|N| > p$ and $\Phi(A) = 1$. Since $A$ is supersoluble, $A$ has a normal Sylow subgroup, and since $N$ is the unique minimal normal subgroup of $G$ contained in $A$, it follows that $\mathbf{F}(A)$ is a $p$-group and $\mathbf{F}(A)$ is an elementary abelian Sylow $p$-subgroup of $A$.

Assume that $A$ is not a $p$-group. Then $\mathbf{F}(A)$ is a completely reducible $A$-module, and so $\mathbf{F}(A) = N \times Z$, for some $A$-module $Z$. Let $L$ be a minimal normal subgroup of $A$ contained in $N$. Then $N = L \times D$, for some $A$-module $D$. Then $\mathbf{F}(A) = L \times DZ$, and $DZ$ is a maximal subgroup of $\mathbf{F}(A)$ because $L$ is of prime order. Therefore, $E = DZ$ permutes with $B$. Hence, $DZ = A \cap (DZ)B$, and so $DZ$ is normalized by $B$. Since $DZ$ is also normalized by $A$, it follows that $DZ$ is a normal subgroup of $G$. The minimality of $N$ forces $D = 1$ and so $N$ is of prime order, which is a contradiction. Consequently, $A$ is an elementary abelian $p$-group. Note that $A$ cannot be cyclic since $|N| > p$. Let $1 \neq X$ be a maximal subgroup of $A$. Arguing as above, we have that $X$ is normal in $XB$ so that $X$ is normalised by $B$. Hence $X$ is normal in $G$ because $A$ is abelian. Therefore, $N$ is contained in $X$, and so $N \leq \Phi(A) = 1$, our final contradiction. □

**Proof of Corollary A.** Assume, by way of contradiction, that the result fails, and let $G$ be a counterexample of least order. Clearly, $G$ is soluble and $A$ and $B$ are proper subgroups of $G$. Since the class of all w-supersoluble groups is a saturated formation, we can argue as in Theorem A to conclude that there exists a unique minimal normal subgroup $N$ of $G$ contained in $A$, and $N$ is a $p$-group for some prime $p$. Moreover, $\Phi(A) = 1$, and $A_p = \mathbf{F}(A)$ is the Sylow $p$-subgroup of $A$. By the minimality of $G$, $G/N$ is $w$-supersoluble. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P/N$ is $\mathbb{P}$-subnormal in $G/N$. By Lemma 2.2(ii), $P$ is $\mathbb{P}$-subnormal in $G$. Suppose that for every prime $q \neq p$ dividing $|G|$ and every Sylow $q$-subgroup $B_q$ of $B$, we have that $AB_q$ is a proper subgroup of $G$. Let $A_q$ be a Sylow $q$-subgroup of $A$ such that $G_q = A_q B_q$ is a Sylow $q$-subgroup of $G$. Since $G/N$ is $w$-supersoluble, it follows that $G_q N$ is $\mathbb{P}$-subnormal in $G$. By Lemma 2.1(b), $AB_q$ satisfies the hypotheses of the theorem. Hence, $AB_q$ is $w$-supersoluble by the choice of $G$. Thus, $G_q N \leq AB_q$ is $w$-supersoluble. Consequently, $G_q$ is $\mathbb{P}$-subnormal in $G_q N$, which is $\mathbb{P}$-subnormal in $G$. Applying Lemma 2.2(iii), $G_q$ is $\mathbb{P}$-subnormal in $G$. Therefore, the Sylow subgroups of $G$ are $\mathbb{P}$-subnormal in $G$, and so $G$ is $w$-supersoluble, a contradiction. Thus, we may assume there exists $q \neq p$ such that $G = AB_q$. Let $T = A_p G_q = (A_p A_q) B_q$. Since $A$ is normal in $G$, we have that $A_q$ is normal in $G_q$ and then $A_p A_q$ is normalized by $B_q$. Moreover, $A_p A_q$ is a $w$-supersoluble metanilpotent subgroup of $G$. By [8, Theorem 2.13(1)], $A_p A_q$ is supersoluble. It is clear that $T$ is a weak direct product of the supersoluble subgroups $A_p A_q$ and $B_q$. Applying Theorem A, it follows that $T$ is supersoluble. Therefore, $T$ is $w$-supersoluble. But $G_q N \leq T$, which is $w$-supersoluble. Thus, $G_q$ is $\mathbb{P}$-subnormal in $G_q N$, which is $\mathbb{P}$-subnormal in $G$. Again the application of Lemma 2.2(iii) yields $G_q$ is $\mathbb{P}$-subnormal in $G$. If $G_r$ is a Sylow $r$-subgroup of $G$ for some prime $r \neq p, q$, then $G_r$ is contained in $A$ and so $G_r$ is $\mathbb{P}$-subnormal in $A$. Since $A$ is also $\mathbb{P}$-subnormal in $G$, we have that $G_r$ is $\mathbb{P}$-subnormal in $G$. Consequently, every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal in $G$, and $G$ is w-supersoluble. This final contradiction completes the proof of the corollary. □

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**Proof of Theorem B.** Suppose that the result is not true, and let $G$ be a minimal counterexample. Then

*(i)* $A \in \mathfrak{F}$, $B^{\mathfrak{F}} \neq 1$, $\mathrm{Core}_G(B) = 1$ *and* $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$ *for every minimal normal subgroup* $N$ *of* $G$ *such that* $N \leqslant A$.

Let $N$ be a minimal normal subgroup of $G$ such that $N \leqslant A$ or $N \leqslant B$. Then $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$ by Lemma 2.1(a). The minimal choice of $G$ implies that $G^{\mathfrak{F}}N/N = (A^{\mathfrak{F}}N/N)(B^{\mathfrak{F}}N/N)$, that is, $G^{\mathfrak{F}}N = A^{\mathfrak{F}}B^{\mathfrak{F}}N$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$, $AG^{\mathfrak{F}}/G^{\mathfrak{F}}$ and $BG^{\mathfrak{F}}/G^{\mathfrak{F}}$ also belong to $\mathfrak{F}$ and then $A^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$ and $B^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. If $G^{\mathfrak{F}} \cap N = 1$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}(G^{\mathfrak{F}} \cap N) = A^{\mathfrak{F}}B^{\mathfrak{F}}$, a contradiction. Hence, $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}N$ for every minimal normal subgroup $N$ of $G$ such that $N \leqslant A$ or $N \leqslant B$. If $A^{\mathfrak{F}} \neq 1$, then there exists a minimal normal subgroup $N$ of $G$ contained in $A^{\mathfrak{F}}$ because $A^{\mathfrak{F}}$ is normal in $G$. This contradiction yields $A \in \mathfrak{F}$ and $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$ for every minimal normal subgroup $N$ of $G$ such that $N \leqslant A$ or $N \leqslant B$. If $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$ by Theorem A and Corollary A, contrary to the assumption. Hence, $B^{\mathfrak{F}} \neq 1$. Suppose that $Core_G(B) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $B$, and let $R$ be a minimal normal subgroup of $G$ contained in $A$. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}}N \cap B^{\mathfrak{F}}R \leq B \cap B^{\mathfrak{F}}R = B^{\mathfrak{F}}(B \cap R) = B^{\mathfrak{F}}$, a contradiction. Consequently, we have that $\mathrm{Core}_G(B) = 1$.

*(ii)* $\mathbf{F}(A)$ *is a Sylow* $p$*-subgroup of* $A$*, where* $p$ *is the largest prime dividing* $|A|$ *.*

Since $A \in \mathfrak{F}$, it follows that $A$ is a Sylow tower group of supersoluble type. In particular, $1 \neq \mathbf{O}_p(A)$ is the Sylow $p$-subgroup of $A$, where $p$ is the largest prime dividing $|A|$. If $\mathbf{F}(A)$ is not a $p$-group, then $1 \neq \mathbf{O}_q(A) \leqslant \mathbf{O}_q(G)$. Let $N_1$ be a minimal normal subgroup of $G$ contained in $\mathbf{O}_p(A)$, and let $N_2$ be a minimal normal subgroup of $G$ contained in $\mathbf{O}_q(A)$. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}}N_1 = B^{\mathfrak{F}}N_2$, which is a contradiction since $B^{\mathfrak{F}} \cap N_1 = B^{\mathfrak{F}} \cap N_2 = 1$. Therefore, $\mathbf{F}(A) = \mathbf{O}_p(A)$ is the Sylow $p$-subgroup of $A$.

*(iii)* $G$ *is soluble,* $AK$ *belongs to* $\mathfrak{F}$ *for every proper subgroup* $K$ *of* $B$*; in particular,* $B$ *is a minimal non-supersoluble group and* $B^{\mathfrak{F}}$ *is a* $q$*-subgroup of* $B$ *for some prime* $q$*.*

Suppose that $K$ is a proper subgroup of $B$. By Lemma 2.1, $AK$ satisfies the hypotheses of the theorem, and so $(AK)^{\mathfrak{F}} = K^{\mathfrak{F}}$ by the minimal choice of $G$. Since $(AK^x)^{\mathfrak{F}} = (K^x)^{\mathfrak{F}} = (K^{\mathfrak{F}})^x$ for any $x \in B$, it follows that $A$ normalizes $(K^{\mathfrak{F}})^x$. Thus, $A$ normalizes $\langle (K^{\mathfrak{F}})^x \mid x \in B\rangle$. Then $\langle (K^{\mathfrak{F}})^x \mid x \in B\rangle \lhd G$, contrary to $\mathrm{Core}_G(B) = 1$. Hence, $(K^{\mathfrak{F}})^x = 1$. Consequently, $AK$ belongs to $\mathfrak{F}$. This shows that $B$ is $\mathfrak{F}$-critical, and by [8, Theorem 2.9], we have that $B$ is a minimal non-supersoluble group. By [3, Theorem 10], we have that $B^{\mathfrak{F}}$ is a $q$-group for some prime $q$. In particular, $B$ and then $G$ are soluble.

*(iv)* $G^{\mathfrak{F}} = B^{\mathfrak{F}} \times N$ *is an elementary abelian* $p$ *-group.*

Applying (iii), it follows that $B^{\mathfrak{F}}$ is a $q$-group for some prime $q$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$ by (i), and $N$ is a $p$-group by (ii).

Suppose that $B^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$. Then $G^{\mathfrak{F}}/B^{\mathfrak{F}}$ is an elementary abelian $p$-group. Consequently, the residual $X$ of $G^{\mathfrak{F}}$ associated to the formation of all elementary abelian $p$-groups is a normal subgroup of $G$ contained in $B$. Hence, $X \leq \mathrm{Core}_G(B) = 1$, and $G^{\mathfrak{F}}$ is an elementary abelian $p$-group.

Assume that $p \neq q$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$, and $N$ is a $p$-group by (ii). Hence, $B^{\mathfrak{F}}$ is a Sylow $q$-subgroup of $G^{\mathfrak{F}} = B^{\mathfrak{F}}N$. Applying Frattini's argument, we have that $G = G^{\mathfrak{F}}N_G(B^{\mathfrak{F}}) = NN_G(B^{\mathfrak{F}})$. Since

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$\mathrm{Core}_G(B) = 1$, it follows that $N_G(B^{\mathfrak{F}})$ is a proper subgroup of $G$. Hence, $N$ is not contained in $\Phi(G)$ for each minimal normal subgroup $N$ of $G$ contained in $A$. If $\Phi(A) \neq 1$, a minimal normal subgroup of $G$ must be contained in $\Phi(A) \leq \Phi(G)$, a contradiction. Therefore, $\Phi(A) = 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $N = N_1 \times N_2 \times \cdots \times N_r$ is a direct product of minimal normal subgroups of $A$, and there exists $i \in \{1, 2, \ldots, r\}$ such that $N_i$ is not contained in $\Phi(A)$. Suppose $i = 1$. Let $M$ be a maximal subgroup of $A$ such that $A = N_1M$ and $N_1 \cap M = 1$. Assume first that $A$ is a $p$-group. Then $BM$ is a subgroup of $G$, and $M = BM \cap A$ is a normal subgroup of $BM$. Hence, $M$ is normalized by $B$, and so $M$ is a normal subgroup of $G$. Now $N = N_1(M \cap N)$. But $M \cap N$ is normal in $G$. The minimality of $N$ yields $N = N_1$ and then $|N| = p$. Thus, $G/C_G(N)$ is abelian. Hence, $G^{\mathfrak{F}}$ centralizes $N$, and $B^{\mathfrak{F}}$ is a normal subgroup in $G^{\mathfrak{F}}$, and so $G^{\mathfrak{F}}$ is an elementary abelian $p$-group. This contradiction implies that $A$ is not a $p$-group. Then $T = \mathbf{F}(A)B$ is a proper subgroup of $G$ which is a weak direct product of $\mathbf{F}(A)$ and $B$. By the minimality of $G$, $T^{\mathfrak{F}} = B^{\mathfrak{F}}$. Then $B^{\mathfrak{F}}$ is a normal subgroup of $G^{\mathfrak{F}}$, and so $G^{\mathfrak{F}}$ is an elementary abelian $p$-group, a contradiction which shows that $p = q$. Then $B^{\mathfrak{F}}$ is a subnormal subgroup of $G$. By [6, Lemma A.14.3], $N$ normalizes $B^{\mathfrak{F}}$, and therefore $B^{\mathfrak{F}}$ is a normal subgroup of the elementary abelian $p$-group $G^{\mathfrak{F}}$.

*(v) Final contradiction.* By [6, IV, 5.18], since $B^{\mathfrak{F}}$ is abelian, there exists an $\mathfrak{F}$-projector $K$ of $B$ such that $B = B^{\mathfrak{F}}K$ and $K \cap B^{\mathfrak{F}} = 1$. Consider the subgroup $Z = AK$ of $G$. Applying (iii), $Z$ belongs to $\mathfrak{F}$ and $G = B^{\mathfrak{F}}Z = F(G)Z$. By [6, III, 3.23(b)], there exists a unique $\mathfrak{F}$-projector of $G$ containing $Z$, $E$ say. Hence, $G = B^{\mathfrak{F}}Z = G^{\mathfrak{F}}E$ and $G^{\mathfrak{F}} \cap E = 1$ by (iii) and [6, IV, 5.18]. In particular, $B^{\mathfrak{F}} \cap Z = 1$. Now $|Z||B^{\mathfrak{F}}| = |E||G^{\mathfrak{F}}| = |E||B^{\mathfrak{F}}||N|$. Hence, $|Z| = |E||N|$. This implies $Z = E$ and then $B^{\mathfrak{F}} = G^{\mathfrak{F}}$, a contradiction. □

**Proof of Theorem C.** Assume that the result is false and let $G$ be a minimal counterexample. Then every proper epimorphic image of $G$ is supersoluble, and hence $G$ has exactly one minimal normal subgroup $N$ which is not contained in the Frattini subgroup of $G$. Since $G$ is soluble, it follows that $N$ is abelian, $N = C_G(N) = F(G)$, and there exists a core-free maximal subgroup of $G$ such that $G = NM$ and $N \cap M = 1$. Let $p$ be the prime dividing $|N|$. Then $|N| > p$. Since $1 \neq G'$ is nilpotent, we have that $G' = N$ and $M$ is abelian. However, $\mathbf{O}_p(M) = 1$ by [6, Lemma A.13.6]. Hence, $M$ is a $p'$-group and $N$ is the Sylow $p$-subgroup of $G$. Since $B \neq G$, we have that $N \leq A$. Note that $N = C_A(N) = \mathbf{O}_{p'p}(A)$. Therefore, $A/\mathbf{O}_{p'p}(A) = A/\mathbf{O}_p(A) = A/N$ is abelian of exponent dividing $p-1$ because $A$ is supersoluble. Assume $BN$ is a proper subgroup of $G$. Then, by the minimality of $G$, $BN$ is supersoluble, and so $B_{p'} \cong BN/\mathbf{O}_{p'p}(BN)$ is abelian of exponent dividing $p-1$. Consequently, $M$ is abelian of exponent dividing $p-1$. Since $N$ is an irreducible and faithful module for $M$, we have that $N$ has order $p$ by [6, Theorem B.9.8], a contradiction. Hence, $G = BN$. Now $B \cap N$ is a normal subgroup of $G$ contained in $N$. Thus, $B \cap N = 1$, and $G = BN$ is the weak direct product of $B$ and $N$. By Theorem A, $G$ is supersoluble. This contradiction proves the theorem. □

**Proof of Corollary B.** Note that since $G'$ is nilpotent, $A$ and $B$ are metanilpotent. By [8, Theorem 2.11], $A$ and $B$ are supersoluble. By Theorem C, $G$ is supersoluble, and hence $G \in w\mathfrak{U}$. □

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**Proof of Theorem D.** Suppose the theorem is not true and let $(G, A, B)$ be a counterexample with $|G|+|A|+|B|$ as small as possible. Let $N$ be a minimal normal subgroup of $G$. It is easy to check that $G/N$ satisfies the hypotheses of the theorem. By the minimality of $G$, we have that $G^{\mathfrak{U}}N = A^{\mathfrak{U}}B^{\mathfrak{U}}N$. Hence, $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}(G^{\mathfrak{U}} \cap N)$. Consequently, $\mathrm{Soc}(G)$ is contained in $G^{\mathfrak{U}}$ and $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}N$ for every minimal normal subgroup $N$ of $G$. Since $G^{\mathfrak{U}}$ is contained in $G'$, we have that $G^{\mathfrak{U}}$ is nilpotent.

Note that $A^{\mathfrak{U}}$ is a normal subgroup of $G$. If $A^{\mathfrak{U}} \neq 1$, then there exists a minimal normal subgroup $N$ of $G$ such that $N \leq A^{\mathfrak{U}}$ and so $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}N = A^{\mathfrak{U}}B^{\mathfrak{U}}$, a contradiction. Hence, we may assume that $A$ is supersoluble and that $G^{\mathfrak{U}} = B^{\mathfrak{U}}N$ for every minimal normal subgroup $N$ of $G$. If $B$ were supersoluble, then $G$ would be supersoluble by Theorem C, which is a contradiction. Hence, $B^{\mathfrak{U}} \neq 1$. Furthermore, $B^{\mathfrak{U}}$ cannot contain a normal subgroup of $G$. Hence, $\mathrm{Core}_G(B^{\mathfrak{U}}) = 1$. Let $p$ be the largest prime dividing $|A|$. Since $A$ is a Sylow tower group of supersoluble type, $A$ has a normal Sylow $p$-subgroup, $A_p$ say, which is also normal in $G$. Hence, $G$ has a minimal normal subgroup $N$ of $G$ which is a $p$-group. Since $G^{\mathfrak{U}}$ is nilpotent, we have that $B^{\mathfrak{U}}$ is a subnormal subgroup of $G$. By [6, Lemma A.14.3], $B^{\mathfrak{U}}$ is normalized by $N$. Thus, $B^{\mathfrak{U}}$ is a normal subgroup of $G^{\mathfrak{U}}$, and $G^{\mathfrak{U}}/B^{\mathfrak{U}}$ is an elementary abelian $p$-group. Consequently $B^{\mathfrak{U}}$ contains the residual $X$ of $G^{\mathfrak{U}}$ associated to the formation of all elementary abelian $p$-groups. Since $X$ is a normal subgroup of $G$, it follows that $X \leq \mathrm{Core}_G(B^{\mathfrak{U}}) = 1$. Hence, $G^{\mathfrak{U}}$ is an elementary abelian $p$-group.

Since $\mathrm{Soc}(G)$ is contained in $G^{\mathfrak{U}}$, $\mathbf{O}_{p'}(G) = 1$ and hence $\mathbf{F}(G) = \mathbf{O}_p(G)$. Therefore $G' \leq \mathbf{F}(G)$ is a $p$-group, and $\mathbf{F}(G)$ is the unique Sylow $p$-subgroup of $G$. Moreover, the Hall $p'$-subgroups of $G$ are abelian (note that $G$ is soluble). Assume $A_pB < G$. Then $A_pB$ satisfies the hypotheses of the theorem. By the choice of $G$, we have that $(A_pB)^{\mathfrak{U}} = B^{\mathfrak{U}}$. Note that $G' \leq A_pB$. Hence, $A_pB$ is a normal subgroup of $G$. This implies that $B^{\mathfrak{U}}$ is normal in $G$, a contradiction. Hence, $G = A_pB$ and $A_p$ and $B$ satisfy the hypotheses of the theorem. If $A \neq A_p$, the choice of $(G, A, B)$ implies that $G^{\mathfrak{U}} = B^{\mathfrak{U}}$, a contradiction. Consequently, we have that $A = A_p$.

Write $T = AB_{p'}$. By Theorem C, $T$ is supersoluble. Moreover, since $G = F(G)B_{p'}$, it follows that every minimal normal subgroup $N$ of $G$ contained in $T$ is a minimal normal subgroup of $T$. Thus, $|N| = p$. Consequently, $N$ is $\mathfrak{U}$-central in $G$. By [6, V, 3.2], $N$ is contained in every supersoluble normalizer of $G$. Let $E$ be one of them. Then $G = G^{\mathfrak{U}}E$ and $G^{\mathfrak{U}} \cap E = 1$. However, $N \leq G^{\mathfrak{U}} \cap E = 1$. This final contradiction proves the theorem. □

**Proof of Corollary C.** Since $\mathfrak{U} \subseteq w\mathfrak{U}$, we have $G^{w\mathfrak{U}} \leq G^{\mathfrak{U}} \leq G'$. Then $G/G^{w\mathfrak{U}}$ is a metanilpotent w-supersoluble group. Applying [8, Theorem 2.11], we have that $G/G^{w\mathfrak{U}}$ is supersoluble. Hence, $G^{\mathfrak{U}} \leq G^{w\mathfrak{U}}$, and therefore $G^{\mathfrak{U}} = G^{w\mathfrak{U}}$, and the same is true for $A$ and $B$. Therefore, by Theorem D, $G^{w\mathfrak{U}} = A^{w\mathfrak{U}}B^{w\mathfrak{U}}$, as desired. □

## 4. An analogue of Monakhov's result

The following two results are the key to prove Corollary D.

**Lemma 4.1.** [**4, Theorem A**] *Let the group $G = HK$ be the product of the subgroups $H$ and $K$. Assume that $H$ permutes with every maximal subgroup of $K$ and $K$ permutes with*

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*every maximal subgroup of $H$. If $H$ is supersoluble, $K$ is nilpotent and $K$ is $\delta$-permutable in $H$, where $\delta$ is a complete set of Sylow subgroups of $H$, then $G$ is supersoluble.*

**Proposition 4.2.** *Let $G = AB$ be a weak normal product of $A$ and $B$, with $A$ and $B$ supersoluble and $A$ normal in $G$. Then $B^{'}$ is a subnormal subgroup of $G$.*

**Proof.** Assume the result is not true, and let $G$ be a counterexample of minimal order with $|A|$ as small as possible. Let $p$ be the largest prime dividing the order of $A$. Then $A$ has a normal Sylow $p$-subgroup $A_p$, which is also a normal subgroup of $G$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq A_p$. It is clear that $A_pB$ satisfies the hypotheses of the theorem. Assume that $A_pB$ is a proper subgroup of $G$. By the minimality of $G$, $B^{'}$ is a subnormal subgroup of $A_pB$. Hence, $B' \leq F(A_pB)$. By Lemma 2.1(a), $G/N$ is a weak normal product of $A/N$ and $BN/N$. By the minimality of $G$, we have that $B'N$ is a subnormal subgroup of $G$. Since $N \leq F(A_pB)$, it follows that $B'N \leq F(A_{p'}B)$. Hence, $B'N$ is a subnormal nilpotent subgroup of $G$. Consequently, $B'N \leq F(G)$. Thus, $B^{'}$ is a subnormal subgroup of $G$, a contradiction. Hence, we may assume that $G = A_pB$. The minimality of $|A|$ implies that $A = A_p$. Applying now the above Lemma, we conclude that $G$ is supersoluble and therefore $G'$ is nilpotent. Hence, $B^{'}$ is subnormal in $G$. This final contradiction proves the proposition. □

**Proof of Corollary D.** Arguing as in [7, Theorem 1], we obtain $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$. Moreover, by [7, Lemma 1(3)], we have that $G' = A'B'[A,B] = (A')^G(B')^G[A,B]$. Since $A$ is a normal subgroup of $G$, then $A^{'}$ is a subnormal subgroup of $G$. Also the application of Proposition 4.2 yields $B^{'}$ is subnormal in $G$ and both $A^{'}$ and $B^{'}$ are nilpotent. Hence, $(A')^G(B')^G$ is a normal nilpotent subgroup of $G$. By [6, II, Lemma II.2.12], $(G')^{\mathfrak{N}} = ((A')^G(B')^G)^{\mathfrak{N}}[A,B]^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}$, as desired. □

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