THE DIFFRACTION RAINBOW

By Major D. R. English

1. This is a very beautiful natural phenomenon, seen on occasions when the primary bow is a strong one. As will appear later, the occurrence necessitates raindrops of a homogeneous size, and of course bright sunshine at a low angle. The diffraction rainbow is therefore most commonly seen in tropical countries, in a thunderstorm that has already passed over and is moving away from the setting (or rising) sun.

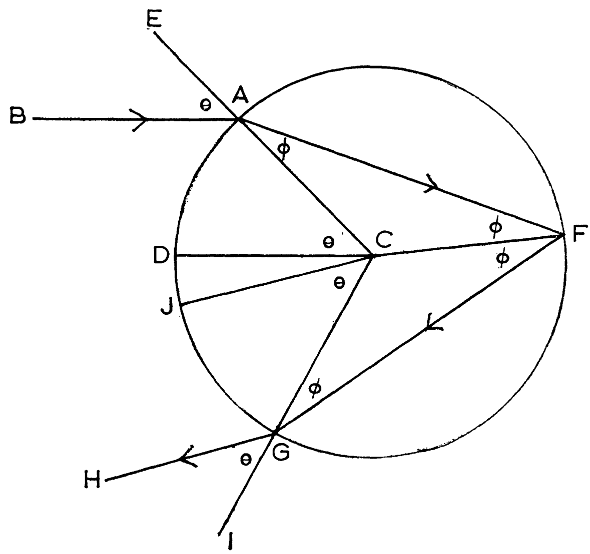

Fig. 1

The writer has not seen the explanation of this effect in any text-book. It is mentioned in R. C. Brown's *Light* (Longmans). The following investigation may therefore be of interest.

2. First it is necessary to remind the reader of the theory of the primary bow. In Fig. 1, the circle represents a raindrop, and $BAFGH$ the direction and position of a portion of an incoming plane wave-front of light from the distant sun, refracted at A,G and internally reflected at F; C is the centre of the drop, and CD, CJ $\parallel$ AB, GH respectively. Let $\theta = \angle BAE =$ angle of incidence at A; then by simple geometry and optics it is clear that

$$\angle CGF = \angle CFG = \angle CFA = \angle CAF = \phi, \text{ say}$$
$$= \text{angle of refraction.}$$

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Then if the index of refraction air/water is μ, we have

$$\sin\theta = \mu \sin\phi.$$

The deviation, D, is given by $D = \pi - \angle DCJ$ and, since $\angle ACD = \angle GCJ = \theta$, we have

$$\begin{aligned} D &= \pi - \angle ACG + 2\theta \\ &= \pi - 4\phi + 2\theta \end{aligned}$$

$$\therefore \quad D = \pi + 2\theta - 4\sin^{-1}\left(\frac{1}{\mu}\sin\theta\right)$$

$$\therefore \quad \frac{dD}{d\theta} = 2 - 4\cos\theta[\mu^2 - \sin^2\theta]^{-1/2}$$

giving a maximum (clearly not a minimum) value for D when:

$$2\cos\theta = (\mu^2 - \sin^2\theta)^{1/2}$$

$$\theta = \cos^{-1}[\tfrac{1}{3}(\mu^2 - 1)]^{1/2} = \theta', \text{ say.}$$

[If $\mu \simeq 1\frac{1}{3}$, this gives $\theta' \simeq 60°$ and $D' \simeq 140°$, as observed in practice, where $D' = D_{\text{max}}$.]

3. Now D is a continuous function of θ, so that there is quite a wide range of values of θ, near θ', which give values of D vanishingly close to D'. This is why an observer, with his back to the sun, looking in the direction $HG = JC$, sees a bright band. The direction JC can be rotated, for all possible positions, about the line from the observer directly away from the sun, i.e. about the direction DC, giving what appears as a circle in the sky. If the sun gave monochromatic light the rainbow would appear as an arc of light, with a sharp outer edge and diffused inner edge. The effect can be observed by seeing a "sodium rainbow" caused by a solitary sodium street lamp on a drizzling night. However, since the sun's light is panchromatic, the diameter of the bow ($2D'$) depends on θ' and therefore on μ, which varies for light of different wave length.

4. As θ passes through the value of θ' it will pass through two values each of which gives the same value for D. This means that light just inside the edge of the rainbow reaches the observer from two points on each drop. As the light is synchronous this means that the observer is receiving synchronous light from a double source, and may expect to see interference effects. These are, in fact, sometimes seen and are known as "spurious bows." They are comparatively faint, and on the *outside* of the main bow. On the other hand the diffraction rainbow can be nearly as bright as the main bow and is immediately *inside* it.

5. In order to consider the cause of the diffraction bow, we may look upon all light between limiting values of θ as being received by the drop and reflected back within a cone of semi-apex angle

$(\pi - D')$. (The limiting values of θ are 0 and some value where $D = \theta + \pi$. However this is of no interest. At this value θ is a large fraction of $\pi/2$, and during the small remaining range for θ up to its maximum possible $\pi/2$, D continues to increase rapidly with resulting lack of intensity of the reflected light).

Within this cone, the reflected light is not homogeneous, but increases in intensity until the surface of the cone is reached, where it attains a maximum value and is then abruptly "switched off." The same effect would be given by a point source of light at the apex of the cone, shining through a circular aperture whose edge is the

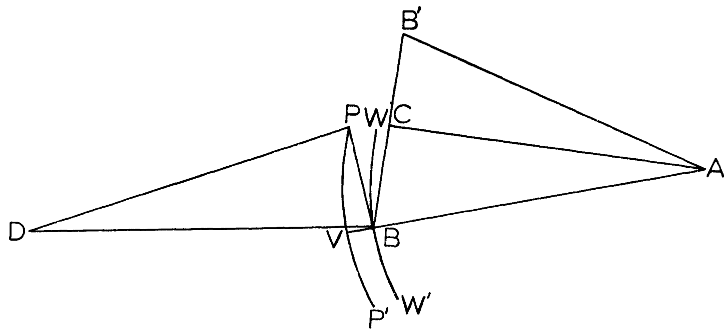

FIG. 2

locus of G, rotated about CD (Fig. 1) where G is at its position when $\theta = \theta'$. Further, near the edge of the aperture the point source can be regarded as of high intensity. This is an ideal condition for diffraction.

6. In Fig. 2, A is the effective point-source, BB' a diameter of the aperture, centre C, WB is a wave front just passing B, and PV is its predecessor. PB is $\perp$ to BA and to BV. The wave fronts have been produced to W' and P' for clarity Then, according to the theory of diffraction, a subsidiary wavefront from P will reinforce one from B at D, inside the cone, if $BD - PD = n\lambda$, where n is an integer and λ is the wave-length of light; and, for diffraction of the first order, $n = 1$ and $BD - PD = \lambda$. An observer at D will see bright light if he looks in the direction DB, which makes a smaller angle with CA (the reverse direction of the sun) than does the direction BA (the direction of the primary bow). Thus the observer will see diffraction *inside* the rainbow.

7. The calculation proceeds as follows:

Let $DB = d$; then $DP = d - \lambda$. In $\triangle ABP$, let $AB = c$; then $AP = c + \lambda$ and

$$BP = [(c+\lambda)^2 - c^2]^{1/2} \simeq (2c\lambda)^{1/2} \quad \text{since } \lambda \ll c.$$

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The angle DBP is given (ignoring λ^2 on the r.h.s. since $\lambda \ll d$) by

$$DB^2 + BP^2 - 2DB\ \ BP \cos \angle DBP = DP^2 = (d - \lambda)^2$$

$$\therefore\quad d^2 + 2c\lambda - 2d(2c\lambda)^{1/2} \cos \angle DBP = d^2 - 2d\lambda,$$

giving

$$\cos \angle DBP = \left(\frac{\lambda}{2c}\right)^{1/2}\left(1 + \frac{c}{d}\right)$$

The difference, to an observer at D, of the direction of the primary bow and that of first order diffraction is clearly $\angle VBD$, $= \psi$, say

Then

$$\psi = \angle VBD = \frac{\pi}{2} - \angle DBP$$

$$\sin \psi = \cos \angle DBP = \left(\frac{\lambda}{2c}\right)^{1/2}\left(1 + \frac{c}{d}\right)$$

We may ignore c/d, since c is of the order of the diameter of a drop, and d is usually of the order of a mile, so that

$$\psi = \sin^{-1}\left(\frac{\lambda}{2c}\right)^{1/2}$$

or

$$\psi = \left(\frac{\lambda}{2c}\right)^{1/2}, \quad \text{since } \lambda \ll c, \text{ as above.}$$

8. Therefore the angle between direction of main bow and that of first order diffraction does not depend on the distance from observer to drop as long as this is large, but does depend on the size of the drop. This explains why the phenomenon is not observed during a "monkey's wedding" in which the observer is in sunshine but still within the rainstorm: such a situation gives an excellent rainbow but no diffraction. Again, no diffraction bow will be seen in light temperate storms in which the drops vary in size considerably.

9. As a matter of interest one may take the average diameter of a drop in a tropical rainstorm as 0·5 cm, and D' as 140°. Then, from Fig. 2 it will be seen that

$$\angle BAC = 40^\circ$$

$$\therefore\quad c = BC \operatorname{cosec} 40^\circ;$$

and from Fig. 1 that BC = perpendicular from G to CD when $\theta = \theta' \simeq 60^\circ$ and $\angle DCJ = 40^\circ$, so that $\angle DCG$ is 100°. BC is therefore $\frac{1}{2}(0{\cdot}5) \sin 100^\circ$ i.e. about 0·24, so that $c = 0{\cdot}375$ cm.

Taking λ for red light as $6{\cdot}44 \times 10^{-5}$ cm, this gives

$$\psi \text{ (red light)} = \left[\frac{6{\cdot}44}{0{\cdot}75} \times 10^{-5}\right]^{1/2}$$

$$= 0{\cdot}0092 \text{ radians}$$

$$= \text{a little over } \tfrac{1}{2}^\circ$$

This agrees with the observed difference.

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10. If the drops are smaller the difference of direction is larger.

Also, λ for blue light is less than λ for red light,

$\therefore$ ψ for blue light is less than ψ for red light,

and so the blue part of the diffraction bow is closer to the blue part of the main bow than the red part of the diffraction bow is to the red part of the main bow; actually taking λ (blue) as about half λ (red), the proportion is $2^{-1/2}$ or about 0·7. But the blue part of the main bow is on the outside (as λ decreases, μ increases, $\therefore$ θ' decreases, and so D' increases). Therefore the colours in the diffraction bow appear in the same order as in the main bow, but further apart. This again agrees with observation.

Rhodes University,
Grahamstown,
South Africa.

D. R. English

ORTHOGONAL CURVILINEAR COORDINATES

By T. A. S. Jackson

Although the gradient of a scalar, and the curl and divergence of a vector can be obtained by various elementary methods in terms of curvilinear components and coordinate vectors, the corresponding results for the components of strain and the vector divergence of stress are usually obtained by specializing the formulae for covariant derivatives. The purpose of this note is to give an elementary approach to these latter results in the spirit of the theory of Cartesian tensors. Apart from standard elementary vector analysis the only prerequisite is a geometrical appreciation of the analysis of deformation. For completeness an elementary analytical derivation of the results about gradient, curl and divergence is given first by a method which, whilst certainly not new, does not appear to be very well known.

Let the surfaces $u_i(x, y, z) = \text{constant}$ $(i = 1, 2, 3)$ intersect orthogonally in a certain region. Then the u_i are a set of orthogonal curvilinear coordinates in this region. The unit coordinate vectors $\mathbf{e}_i$ and the positive scalars h_i are defined by

$$\mathbf{e}_i = h_i \nabla u_i, \qquad h_i = \frac{1}{|\nabla u_i|}, \qquad (i = 1, 2, 3).$$

We may assume that the u_i are numbered in such an order that

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3.$$

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