An extension of de Longchamps' chain of theorems.

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1. Points, n in number, A, B, C, D, E, ..., are taken at random in a plane, and through each is drawn a line in a random direction. The only condition imposed is that no two of these lines may be parallel.

(i). Two points A, B, define a circle $S\,(AB)$ which passes through A, B and the intersection of the random lines through A and B. Its centre is denoted by (AB). Each pair of the points gives such a circle and centre.

(ii). Three points A, B, C, define three circles $S\,(BC)$, $S\,(CA)$, $S\,(AB)$, and three centres (BC), (CA), (AB). The three circles are found to meet in a point $P\,(ABC)$; the three centres lie on a circle $S\,(ABC)$, whose centre is denoted by (ABC).

(iii). Four points A, B, C, D, define four circles such as $S\,(ABC)$, and four centres such as (ABC). The four circles are found to meet in a point $P\,(ABCD)$; the four centres are found to lie on a circle $S\,(ABCD)$ whose centre is denoted by $(ABCD)$.

(iv). Five points in the same way define five circles which meet in a point and five centres which lie on a circle. These chains of theorems may be continued indefinitely.

Moreover a subsidiary figure of points a, b, c, d, e, ..., associated with A, B, C, D, E, ... respectively and lying on a circle, may be constructed such that the three centres in (ii) and the four centres in (iii) . . . . form figures similar to abc and to $abcd$ . . . . This also continues to hold good in the later stages.

It will be noticed that the rules under which the circles were drawn ensure that one intersection of each pair of circles is known at every stage. At stage (iii) for example the circles $S\,(ABC)$ and $S\,(ABD)$ are known to pass through the point (AB). The second intersections all coincide in the point $P\,(ABCD)$. After stage (i) (in which the random lines make their one and only appearance) the procedure is perfectly regular, the number of points and circles increasing by one at each stage. In one very special case this may be avoided, viz. when the random lines are concurrent. If O is their point of concurrence, the points (AB) are determined as the intersections of lines which bisect OA, OB, OC, ... perpendicularly and the chains of theorems stated above follow as theorems upon the points of intersections of n random lines in a plane.

2. *Historical.* This special case was discovered by de Longchamps[1] in 1877. It was rediscovered independently by Pesci[2] in 1891 and again in part by Morley[3] (who overlooked the fact that the circles have a common point) in 1900. Morley proved his part of the theorems of the special case by an ingenious algebraic method which is capable of wide extension, and Grace proved the same by a method of striking originality. Later White[4], Lob[5] and Grace[6] applied geometry of many dimensions to this and to Clifford's well known chain, Grace in this way filling the hiatus in Morley's results. To the present writer it appeared almost certain that these very advanced methods would have simple algebraical equivalents in plane geometry[7]. They did in fact suggest the equations used here, which are simpler than Morley's and prove the chains of theorems in the general extended form at the same time as those of de Longchamps. But de Longchamps was able to prove the theorems in the special case without using any but the most elementary methods; they are presented in a rigorous form by Coolidge[8], and they apply to some extent here. The algebraic formulæ in this paper are, I think, better fitted to show how far the special case is in harmony with the general one and how far each is connected with Clifford's earlier chain.

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[1] De Longchamps, *Nouville correspondence mathématique*, 3 (1877), 306 and 340. Published in Brussels; the periodical is now continued under the title *Mathesis.*

[2] Pesci, *Periodico di Mathematica*, **5** (1891), 120.

[3] Morley, *Trans. American Math. Soc.*, **1** (1900), 97.

[4] F. P. White, *Proc. Camb. Phil. Soc.*, **22** (1925), 684.

[5] H. Lob, *Proc. Camb. Phil. Soc.*, **29** (1933), 45.

[6] J. H. Grace, *Proc Camb. Phil. Soc.*, **24** (1928), 10; also *Proc. London Math. Soc.* (2), **33** (1908), 193.

[7] H. W. Richmond, *Proc. Camb. Phil. Soc.*, **29** (1933), 165. See also the obituary notice of Frank Morley and the note which follows it in *Journal London Math. Soc.*, **14** (1939), 73 and 78.

[8] J. L. Coolidge, *A treatise on the circle and the sphere* (Oxford, 1916), 92. The date given for de Longchamps' discovery is wrongly stated as 1887.

3. Besides the figure described in §1, now called the Z-figure, we construct a subsidiary z-figure. We take a circle of unit radius with centre w, and from a point p on its circumference we draw chords pa, pb, pc, ... parallel to the random lines drawn through A, B, C, ... in §1; these determine points a, b, c, ... on the unit circle. A rotation of all the random lines through the same angle does not affect the Z-figure: the relative distances of a, b, c, ... on the unit circle are unaltered by this, or by a change in the position of p.

A letter which represents any point may represent also the value of Z or z, the complex variable, pertaining to that point. In the subsidiary figure z is a variable point on the unit circle, so that z; a, b, c, ... all have modulus 1, the centre of the circle being the origin. In the Z-figure the origin is not specified.

Consider the most general integral symmetric function of a, b, c, ... containing no power above the first, viz.,

$$F(a, b, c, \ldots. k, l) = r_0 + r_1 \Sigma_1 + r_2 \Sigma_2 + r_3 \Sigma_3 + \ldots. + r_n \Sigma_n, \quad (3\cdot1)$$

where $\Sigma_1, \Sigma_2, \Sigma_3, \ldots. \Sigma_n$ denote the sums of products of $a, b, c, \ldots.$ taken $1, 2, \ldots. n$ at a time. When we put one of the letters equal to zero (as we shall always do) r_n disappears; but n free constants r are left, and values can be found for these so that F takes the respective values A, B, C, ... when a, b, c, ... is put equal to zero. This is always possible if a, b, c, ... are unequal, *i.e.* if no two of the random lines in § 1 are parallel. Thus

$$F(0, b, c, \ldots. k, l) = A;\ F(a, 0, c, \ldots. k, l) = B;\ F(a, b, 0, \ldots. k, l) = C;$$
$$\ldots.\ F(a, b, c, \ldots. 0, l) = K;\ F(a, b, c, \ldots. k, 0) = L. \quad (3\cdot2)$$

Replace a and b in F by z and 0, or by 0 and z;—F being a symmetric function the order is immaterial;—and consider the relation

$$Z = F(z, 0, c, \ldots. k, l). \quad (3\cdot3)$$

Here Z is a linear function of z and corresponding values of Z and z give similar configurations. As z traces the unit circle Z traces a circle passing through A and B as z passes through b and a. Hence to $z = 0$, the centre of the unit circle corresponds (AB) the centre of Z's circle. (See the note below.) Hence

$$\text{At } (AB), \qquad Z = F(0, 0, c, d, \ldots. k, l). \quad (3\cdot4)$$

Repeat the reasoning, replacing a, b, c in F by z, 0, 0. Z traces a circle passing through (BC), (CA), (AB) as z passes through a, b, c. The centres correspond; hence

$$\text{At } (ABC), \qquad Z = F(0, 0, 0, d, \ldots. k, l). \quad (3\cdot5)$$

Similarly At $(ABCD)$, $Z = F(0, 0, 0, 0, e, \ldots .)$. (3·6)

Moreover, since corresponding values of Z and z give similar figures, (BC), (CA), (AB) form a triangle similar to abc, and (BCD), (CDA), (DAB), (ABC) form a cyclic quadrangle similar to $abcd$. The centres of the circles in the Z-figure are known definitely.

NOTE. The two figures are not directly similar; each is directly similar to a reflection of the other. This removes any ambiguity as to the position of (AB).

4. To state the formulæ in detail in a simple case, take four points A, B, C, D, associated (by means of the random lines drawn through them) with four points a, b, c, d of the subsidiary figure. We have

At A, $Z = r_0 + r_1(b + c + d) + r_2(bc + bd + cd) + r_3 bcd$. (4·1)

A circle $Z = r_0 + r_1(z + c + d) + r_2(zc + zd + cd) + r_3 zcd$. (4·2)

At (AB), $Z = r_0 + r_1(c + d) + r_2 cd$. (4·3)

A circle $Z = r_0 + r_1(z + d) + r_2 zd$. (4·4)

At (ABC), $Z = r_0 + r_1 d$. (4·5)

A circle $Z = r_0 + r_1 z$. (4·6)

At $(ABCD)$, $Z = r_0$. (4·7)

Here (4·1) is one of the four equations which express the Z-coordinates of A, B, C, D in terms of a, b, c, d. (4·2) is the circle which passes through A and B when z has the values b and a and has the point (AB) for its centre. (4·4) is the circle which passes through (BC), (CA), (AB) when z is a, b, c, etc.

Given similar equations for any number of points it is obvious that the equations for a smaller number of points can be deduced. But it is possible also to deduce equations for an increased number. To deduce the equations for five points from those given above we replace r_0, r_1, r_2, r_3 by r_0', r_1', r_2', r_3', where

$$r_0' = r_0 + r_1 e;\; r_1' = r_1 + r_2 e;\; r_2' = r_2 + r_3 e;\; r_3' = r_3 + r_4 e. \quad (4·8)$$

5. The concurrence of the circles in the points $P(ABC)$ and $P(ABCD)$ .... of § 1 has not yet been proved. To do this we must introduce the complex numbers conjugate to those used hitherto. For the conjugate of Z we shall use V; for those of $r_0, r_1, r_2, \ldots .$ we shall use $s_0, s_1, s_2, \ldots .$ The conjugates of $z, a, b, \ldots .,$ which have modulus 1, are $1/z$, $1/a$, $1/b$, ... respectively. We consider first the circle of (4·4) and the three others in which d is replaced by a, b or c. By eliminating z from (4·4) and its conjugate

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$$V = s_0 + s_1(1/z + 1/d) + s_2/(zd), \tag{5·1}$$

we find the equation of the circle to be

$$(r_0 + r_1 d - Z)(s_0 + s_1/d - V) = (r_1 + r_2 d)(s_1 + s_2/d), \tag{5·2}$$

or

$$[(r_0 - Z)(s_0 - V) - r_2 s_2] + [r_1(s_0 - V) - r_2 s_1]d + [s_1(r_0 - Z) - s_2 r_1]/d = 0. \tag{5·3}$$

Now if the terms in the last two square brackets vanish the first square bracket also vanishes, and the whole expression vanishes whatever number takes the place of d; the four circles of (4·4) therefore pass through the point

$$Z = r_0 - s_2 r_1/s_1; \qquad V = s_0 - r_2 s_1/r_1. \tag{5·4}$$

Whatever the number of points $A, B, C, \ldots$ this proves the theorem for the final stage, when for the first time all the n points are involved. It follows that if the theorems are known to be true for $n - 1$ points they are true for n points, and therefore are true for any number of points.

The three types of theorems proved by de Longchamps for a set of random lines in a plane have thus been proved to hold in the wider form enunciated in § 1, based upon n random points and a random line through each.

6. Some modification of the foregoing equations is to be expected in the simple case discovered by de Longchamps, yet we have had so far no indication of what this can be; indeed it is difficult to see how the formulæ of §§ 1-4 can be modified in the direction of greater simplicity. The suggestion made in § 1 will be verified, and gives the clue.

Given n random lines, we take any origin O. We take the points inverse to O in each line for the points $A, B, C, \ldots$ of § 1, and the lines $AO, BO, CO, \ldots$ for the lines drawn through the points. A being as before the value of Z pertaining to the point A, the conjugate complex number V is aA; the equation of the given line, the perpendicular bisector of OA is

$$ZV = (Z - A)(V - aA)$$

or

$$V + aZ = aA. \tag{6·1}$$

At the point of intersection of two of the given lines, which we note is the point (AB) of 1 (i), we have a result which is seen to agree with (3·4), viz.

$$Z = (aA - bB)/(a - b). \tag{6·2}$$

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For de Longchamps' special case § 1 should be:—

"Lines n in number, A, B, C, ... are taken at random in a plane, no two being parallel. (i) Two lines A and B determine by their intersection a point (AB). (ii) Three lines A, B, C determine a triangle which has a circumcircle $S(ABC)$ and a circumcentre (ABC). In (iii) and (iv) the second word should be 'lines.' Further changes of a word here and there need not be catalogued." The algebra which follows is valid without alteration.

The modifications of results in de Longchamps' special case depend upon the fact that A and aA are conjugate complex numbers. In § 4, the formulæ when four lines A, B, C, D are taken, the value of A is shown in (4·1). The conjugate number

$$V = s_0 + s_1(1/b + 1/c + 1/d) + s_2(1/bc + \dots) + s_3/bcd \tag{6·3}$$

equated to aA shows that

$$s_3 + s_2(b + c + d) + s_1(bc + bd + cd) + s_0\,bcd$$
$$= abcd\,[r_0 + r_1(b + c + d) + r_2(bc + bd + cd) + r_3\,bcd].$$

Therefore $\quad s_0/r_3 = s_1/r_2 = s_2/r_1 = s_3/r_0 = abcd. \qquad (6·4)$

In the same way when five lines A, B, C, D, E are under consideration we have

$$s_0/r_4 = s_1/r_3 = s_2/r_2 = s_3/r_1 = s_4/r_0 = abcde, \tag{6·5}$$

and so for higher numbers of lines. It is therefore possible to avoid s_0, s_1, s_2, altogether, but there are other consequences of a different kind. In (4·4), since $|r_1|$ and $|r_2|$ are now equal, $r_1 z + r_2$ vanishes for a value of z whose modulus is 1, *i.e.* a permissible value of z. When z receives this value d disappears, and we arrive at the common point of all the circles more easily than in § 5.

7. *Miquel's Theorem*[1]. With four lines A, B, C, D, the four circumcircles of the triangles they form have been proved in 5·4 to meet in a point $P(ABCD)$, viz.

$$Z = r_0 - s_2 r_1/s_1 = r_0 - r_1^2/r_2. \tag{7·1}$$

[1] Miquel, *Liouville's Journal*, 10 (1845), 349. See also Clifford, *Messenger of Math.*, 5 (1870), 124; or *Collected Papers*, 38.

When we deal with five lines A, B, C, D, E, each four provide such a point and the theorem asserts that they lie on a circle. By the rule given in (4·8), at $P(ABCD)$

$$Z = (r_0 + r_1 e) - (r_1 + r_2 e)^2/(r_2 + r_3 e) \tag{7·2}$$

or, $$Z - r_0 + r_1^2/r_2 = r_1(r_1 + r_2 e)/r_2 - (r_1 + r_2 e)^2/(r_2 + r_3 e)$$

that is, $$r_2 Z - r_0 r_2 + r_1^2 = \left(\frac{r_1 + r_2 e}{r_2 + r_3 e}\right) \times [r_1(r_2 + r_3 e) - r_2(r_1 + r_2 e)]$$

$$= \frac{r_1 + r_2 e}{s_1 + s_2/e} \times (r_1 r_3 - r_2^2)\, abcde.$$

The four other points have a, b, c, d in place of e. But the numerator and denominator of the fraction are conjugate complex numbers, so that

$$|r_2 Z - r_0 r_2 + r_1^2| = |r_1 r_3 - r_2^2|,$$

which proves that the five points are on a circle. It may be added that they do not lie on a circle in the general case discussed in the early part of this paper.

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