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ISOMORPHISMS BETWEEN LINEAR GROUPS OVER DIVISION RINGS

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ABSTRACT In the present paper we completely describe the isomorphisms between projective elementary groups PSL_n and PSL_m ($n \geq 2, m \geq 2$) over division rings It was found that such groups can be isomorphic only if $n = m$, the division rings are isomorphic or anti-isomorphic, except for the following groups

$$\mathrm{PSL}(2, F_7) \text{ and } \mathrm{PSL}(3, F_2), \quad \mathrm{PSL}(2, F_4) \text{ and } \mathrm{PSL}(2, F_5)$$

The idea is based on a deepening of the classical Hua's approach This problem has been solved independently by H Ren, Z Wan and X Wu using a different way

In the present paper we completely determine the isomorphisms between projective elementary groups $\mathrm{PSL}(n, R)$ and $\mathrm{PSL}(m, S)$ ($n \geq 2, m \geq 2$) over division rings. It was found that such groups can be isomorphic only if $n = m$; the division rings R and S are isomorphic or anti-isomorphic, except for the following groups:

$$\mathrm{PSL}(2, F_7) \text{ and } \mathrm{PSL}(3, F_2), \quad \mathrm{PSL}(2, F_4) \text{ and } \mathrm{PSL}(2, F_5).$$

The first step in the description of the isomorphisms between projective elementary groups was taken by Schreier and van der Waerden [1]. They determined the isomorphisms between $\mathrm{PSL}(n, R)$ and $\mathrm{PSL}(m, S)$ ($n, m \geq 3$) over fields R and S. Their paper was the beginning of the construction of the theory of homomorphisms of algebraic and classical groups over rings. The most complete survey of the modern state of this theory may be found in [2]. Dieudonné [6], Rickart, Hua [5] described the isomorphisms between $\mathrm{PSL}(n, R)$ and $\mathrm{PSL}(m, S)$ ($n, m \geq 3$) over division rings R and S, developing the involution method idea which goes back to Mackey. The case when one of the m, n is equal to 2, needs additional considerations. The most difficult cases of small m and n have been studied by Hua and Wan [4]. For $m = n = 2$ Hua [5] determined the isomorphisms between $\mathrm{PSL}(n, R)$ and $\mathrm{PSL}(m, S)$ over fields R and S.

Almost all of the results of the description of the isomorphisms between projective elementary groups over division rings that have become classical may be found in Dieudonné's book [3]. There (3, Chapter IV) it is pointed out that the problem of the description of automorphisms of $\mathrm{SL}(2, R)$ is unsolved in the case when (-1) does not belong to the commutator subgroup of R^* and R is a division ring of zero characteristic. It should be pointed out that Ren [7] has solved Dieudonné's problem under the condition that there exist elements in R, the squares of which are equal to 2 and 3, correspondingly.

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The problem of the isomorphisms between $\mathrm{PSL}(n,R)$ and $\mathrm{PSL}(m,S)$ over division rings R and S in the case of $m \geq 3$, $n \geq 3$ follows from O'Meara-Sosnowski theorem, the new proof of which is proposed by the author in [8]; for the rest of the cases the description follows from Theorems 1 and 2. In this case Hahn's [13] deep work should be noted. Theorems 1 and 2 are a partial solution of the problem of the construction of the isomorphism theory for two-dimensional linear groups which are rich in transvections. This problem has been formulated by Merzlyakov in [9]. They are also tightly connected with the description of linear group isomorphisms over integral domains (see [14]).

The complete solution of the description problem for the isomorphisms between projective groups over division rings was reported in [11]. Independently, using quite different methods, H. Ren, Z. Wan and X. Wu have solved the description problem for automorphisms between linear groups over division rings in [12]. More precisely, for a division ring R they have determined automorphisms of $\mathrm{PSL}(n,R)$ and reported a of complete solution of the problem for isomorphisms between $\mathrm{PSL}(n,R)$ and $\mathrm{PSL}(m,S)$ over division rings. Their basic idea is an imbedding of the division ring into a ring of matrices such that images of transvections would be triangulable, first over some extension and then over the basic division ring.

The author's idea is based on a deepening of the classical approach of Hua. In the two-dimensional case the author succeeded in calculating the first and the second commutator subgroups of the centralizer of an arbitrary set of scalar matrices. It was found that almost always they can be represented as linear groups. This allowed passing to isomorphisms between linear groups over fields and then proving that almost always at least one transvection under isomorphism maps into a unipotent element.

An isomorphism between $\mathrm{PSL}(n,R)$ and $\mathrm{PSL}(m,S)$ is called *standard* if $n = m$ and, up to conjugation, it is the result of an isomorphism or an anti-isomorphism between R and S.

Let R^* be the group of units of R; R_n be the ring of all $n \times n$ matrices over R; $\mathrm{GL}(n,R) = R_n^*$; $\mathrm{SL}(n,R) = gr\big(t_{ij}(x) = E + xe_{ij} \mid e_{ij}-$ standard matrix unit$\big) = \ker\det$, where $\det\colon \mathrm{GL}(n,R) \longrightarrow R^*/[R^*,R^*]$ is the Dieudonné determinant.

An element $\sigma \in \mathrm{GL}(n,R)$ is called *unipotent* if there exists a number k such that $(\sigma - E)^k = 0$. If $k = 2$ and $\dim(\sigma - E) = 1$ then σ is called a *transvection*. Let G be a group and ζG its centre. $G/\zeta G$ is called a *projective group* and is denoted by PG or $\bar{G}$. If g is an arbitrary element of G, then projective image of g is denoted by $\bar{g}$. Let $\bar{A} = \{a\zeta G \mid a \in A\}$ and $C_G(A) = \{g \in G \mid ga = ag \text{ for all } a \in A\}$.

The commutator subgroup of a group G is denoted by $[G,G]$ or G', $G'' = [G',G']$; $[a,b] = aba^{-1}b^{-1}$, $b^a = aba^{-1}$. The equation $a = usv$ means that the element a is represented by Bruhat decomposition , where u is the higher unimodular, v is the higher triangular and s is monomial matrices. F_q is a finite field with q elements.

It is easy to see that for the elements a, b, c of a group G, it follows from commutativity of $\bar{a}$ with $\bar{b}$ and $\bar{c}$ that a commutes with $[b,c]$. Hence, $C'_{\bar{G}}(\bar{A}) \subset \overline{C_G(A)}$.

The elements of order two are called *involutions*.

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LEMMA 1. *Let G be a group with finite centre. If $\bar{G}$ includes an infinite number of pairwise commutative involutions then G also includes an infinite number of pairwise commutative involutions.*

PROOF. Let $\bar{a}$ be any involution from $\bar{G}$. Then there exists an infinite set of involutions $\bar{b}$ in the group $\bar{G}$ such that $[a,b] = \alpha$, where α is some fixed element from the centre of the group G. It is easy to see that the equalities $[a,b_1] = \alpha = [a,b_2]$ imply $ab_1a^{-1} = b_1\alpha$ and $ab_2^{-1}a^{-1} = b_2^{-1}\alpha^{-1}$. So the element a commutes with $b_1b_2^{-1}$. It is obvious that there is an infinite number of elements of the form $b_1b_2^{-1}$. Applying similar arguments to the elements of this set, we have that there exists an infinite number of pairwise commutative elements in G, the images of which in $\bar{G}$ are involutions.

Since the centre of the group G is finite, there exists an infinite number of pairwise commutative elements c in G such that $c^2 = \beta$, where β is some fixed element from the centre of the group G. It follows from the equalities $c_1^2 = \beta = c_2^2$ that $c_1c_2^{-1}$ is an involution. Since there exists an infinite number of elements of the form $c_1c_2^{-1}$, these elements form an infinite set of pairwise commutative involutions in the group G.

LEMMA 2. *Let R be a field. Then for arbitrary $\alpha, \delta \in R$, $\alpha \neq 0$, $C_{\mathrm{GL}(2,R)}\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}$ is an abelian group.*

PROOF. Let $\mathrm{GL}(n,R) = \mathrm{GL}(n,V)$, where $n = \dim V \geq 2$. Let a be an element from R_n such that V as $R[a]$-module is cyclic. Then the diagram

$$R[a] \longrightarrow \mathrm{Hom}_{R[a]}(V,V) = C_{R_n}(a)$$

implies that $C_{R_n}(a)$ is a commutative ring and, therefore, its group of units is abelian.

We'll need the Hua formula:

$$([a,b] - [a,b-1])b = 1 - [a,b-1]. \tag{1}$$

This formula implies, in particular, that an element of R which commutes with each element from $[R^*,R^*]$ belongs to the centre of R. Moreover, if R^* is nilpotent, then R is a field. From (1) we have that if $\big[[R^*,R^*],[R^*,R^*]\big]$ belongs to the centre of R, then R is a field. Indeed, let $a \in [R^*,R^*]$, $a \neq 1$, and $b \in R^*$. Then

$$[a,b^a] = [a,b]^a = \gamma_1[a,b]$$

and

$$[a-1,b^a] = [a-1,b]^a = \gamma_2[a-1,b],$$

where γ_1 and γ_2 are elements from the centre of R. From these equalities we have that $(\gamma_1^{-1} - \gamma_2^{-1})a^b = 1 - \gamma_2^{-1}$.

If $\gamma_1 \neq \gamma_2$ then a^b belongs to the centre of R and, therefore, $[a,b]$ commutes with a. It is easy to show that this statement remains valid under the condition that $\gamma_1 = \gamma_2 = 1$. Applying formula (1) again, a belongs to the centre of R. So, we have proved that R is a field.

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As a corollary, we obtain that all the elements of R which commute with all the elements of $[R^*, R^*]'$ form a subfield F Clearly for each inner automorphism ϕ of R^* we have $\phi(F) = F$ Using (1) for a non-zero element $a \in F$ and for arbitrary $b \in R^*$, $b \neq 1$, we obtain that $ab = ba$ This means that F is the centre of R

Another important corollary from the formula (1) is that the commutativity of $C'_{\mathrm{PSL}(2,R)}\overline{t_{12}(1)}$ implies that R is a field Indeed, let a_1, a_2, x_1, x_2, y_1, y_2 be arbitrary elements of $[R^*, R^*]$ Then $[a_1, a_2]$ commutes with elements

$$\big[[x_1, x_2], [y_1, y_2]\big],$$

$\big[x_1 x_2 E, t_{12}[y_1, y_2]\big] = t_{12}\Big(\big(\big[[x_1, x_2], [y_1, y_2]\big] - 1\big)[y_1, y_2]\Big)$, and consequently with $[y_1, y_2]$ Since $[a_1, a_2]$ commutes with $[y_1 E, t_{12}(y_2)] = t_{12}\big(([y_1, y_2] - 1)y_2\big)$, $[a_1, a_2]$ commutes with an arbitrary element y_2 and, therefore, belongs to the centre of R Hence, $[R^*, R^*]'$ belongs to the centre of R and, thus, R is a field

LEMMA 3 *Let R and S be division rings, and Λ $\mathrm{PSL}(n, R) \longrightarrow \mathrm{PSL}(2, S)$ be an isomorphism which maps, at least, one transvection into a transvection*

Then $n = 2$ and Λ is a standard isomorphism

PROOF It is known that a nilpotent group is unitriangulable if its lower central series is generated by unipotent elements of $\mathrm{PSL}(2, S)$ Since $\Lambda\,\mathrm{PSL}'(n, R) = 1$ for $n \geq 3$, it follows that $n = 2$ Up to an inner automorphism we can assume that $\Lambda\overline{t_{12}(1)} = \overline{t_{12}(1)}$

Let $\Lambda\bar{a} = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$ and $a = usv$ be the Bruhat decomposition of matrix a A suitable selection of basis gives $u = E$ Let $vt_{12}(1)v^{-1} = t_{12}(m)$ where m is an element of R Then

$$\Lambda\overline{t_{12}(m)}^{s} = \Lambda s v\overline{t_{12}(1)}v^{-1}s^{-1} = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\,\overline{t_{12}(1)}\,\overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}^{-1} = \overline{t_{21}(-1)}$$

Since matrices $t_{12}(1)$ and $t_{21}(-1)$ do not commute with one another it follows that $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Lambda\overline{t_{21}(m)} = \overline{t_{21}(-1)}$

From the formula

$$t_{12}(x)t_{21}(-x^{-1})t_{12}(x) = t_{21}(-x^{-1})t_{12}(x)t_{21}(-x^{-1}), \tag{2}$$

for $x = 1$ we obtain $m = -1$ Let $\Lambda\overline{t_{12}(x)} = \overline{\alpha(x)t_{12}\big(\delta(x)\big)}$, where $\alpha(x)$ and $\delta(x)$ are elements of S which depend on x It is easy to show that for arbitrary elements x and y from R, $\overline{\alpha(x+y)} = \overline{\alpha(x)\alpha(y)}$ and $\overline{\alpha(0)} = \overline{\alpha(1)} = \bar{1}$ From the formula

$$t_{12}(x)^{\mathrm{diag}(x, x^{-1})} = t_{12}(x^2) \tag{3}$$

we get that $\overline{\alpha(x^2)} = \bar{1}$ and (2) implies $\delta(x^2)\delta(x^{-2}) = -1$ for all nonzero elements $x \in R$ As a corollary we obtain $\alpha(x)\delta(x^2) = \delta(x^2)\alpha(x)$, $\overline{\alpha(2x)} = \overline{\alpha\big((x+1)^2 - x^2 - 1\big)} = 1$ and $\Lambda\overline{\mathrm{diag}(x^2, x^{-2})} = \overline{\mathrm{diag}(\delta(x^2), \delta(x^{-2})}$

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When $\operatorname{char} R \neq 2$, applying the formula

$$t_{12}(4x)^{\operatorname{diag}(1/2,2)} = t_{12}(x),$$

we find that $\overline{\alpha(x)} = \overline{\alpha(4x)} = \bar{1}$ for all $x \in R$. In the case when $\operatorname{char} R = 2$, the last statement follows from the formula

$$[t_{12}(x), \operatorname{diag}(x^2; x^{-2})]^{\operatorname{diag}((1+x)^{-2},(1+x)^2)} = t_{12}(x).$$

Thus, we have proved that $\Lambda\overline{t_{12}(x)} = \overline{t_{12}(\delta x)}$ for all $x \in R$. It is easy to show that the mapping $\delta: x \longrightarrow \delta(x)$ which is additive (it is because there may be only one transvection in a coset) and $\delta(1) = 1$, $\delta(x^{-1}) = \delta(x)^{-1}$ for all nonzero elements x from R. By the Hua theorem, δ is an isomorphism or an anti-isomorphism between the division rings R and S. Hence, the isomorphism Λ is standard.

LEMMA 4. *Let R and S be division rings of equal characteristic $p > 0$. Let $\Lambda: \mathrm{PSL}(n, R) \longrightarrow \mathrm{PSL}(2, S)$ be an arbitrary isomorphism. Then $n = 2$ and Λ is a standard isomorphism.*

PROOF. It is easy to see that for elements a, b of an arbitrary linear group over a division ring of characteristic $p > 0$, the equalities $\bar{a}^p = \bar{E}$ and $\bar{a}[\bar{a}, \bar{b}] = [\bar{a}, \bar{b}]\bar{a}$ imply $a[a, b] = [a, b]a$ and

$$1 = [a^p, b] = [a, b]^{a^{p-1}}[a^{p-1}, b] = [a, b][a^{p-1}, b] = [a, b]^p.$$

Let δ_{ij} be the Kronecker symbol. Applying formula $[t_{ij}(x), t_{sk}(y)] = t_{ik}(\delta_{js}xy)$ for $n \geq 3$ and (3) for $n = 2$, provided that $|R| > 3$ or $|S| > 3$, we get that Λ maps transvections into projective images of elements of order p, and, therefore, into transvections. It can be verified directly that when $n = 2$, $R = F_3$, $S = F_3$ or $n = 2$, $R = F_2$, $S = F_2$, the element $\Lambda\overline{t_{12}(1)}$ is a transvection.

An application of Lemma 3 completes the proof of Lemma 4.

LEMMA 5. *Let R be a division ring and S be a field. Let $\Lambda: \mathrm{PSL}(n, R) \longrightarrow \mathrm{PSL}(m, S)$ be an arbitrary isomorphism. Then R is a field and the isomorphism Λ is standard except for the groups* $\mathrm{PSL}(3, F_2)$ *and* $\mathrm{PSL}(2, F_7)$, $\mathrm{PSL}(2, F_4)$ *and* $\mathrm{PSL}(2, F_5)$.

PROOF. Let R be a noncommutative division ring. Let us prove that the isomorphism Λ is impossible. Consider the solvable subgroup in $\mathrm{PSL}(n, R)$ generated by elements $t_{12}(y)$, where y is an arbitrary element of R, and by matrices $\operatorname{diag}(x; x)$ and $\operatorname{diag}(x; x^{-1})$, where x is a fixed noncentral element from the commutator subgroup of R^*. According to Lie-Colchin-Maltsev theorem this subgroup contains a subgroup N of finite index such that $\Lambda[\bar{N}, \bar{N}]$ consists of unipotent elements and the identity. Since R is infinite, there exist elements $y \neq 0$, y_1, y_2 in R such that matrices

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x & y_1 \\ 0 & x \end{pmatrix}, \quad \begin{pmatrix} x & y_2 \\ 0 & x^{-1} \end{pmatrix}$$

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belong to N. It is easy to show that if

$$\left[\begin{pmatrix} x & y_1 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right] = E, \text{ then } \left[\begin{pmatrix} x & y_2 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right] \neq E.$$

Thus, we have proved that the isomorphism Λ maps a transvection into a unipotent element. From Lemma 8 [8] it follows for $m \geq 3$ and $n \geq 3$ and from Lemma 3 for other cases that Λ maps transvections into transvections. Hence, R is isomorphic or anti-isomorphic to S; that is impossible. Hence, R is a field.

If R is an infinite field, then from similar considerations as in the case when R is a division ring we have that Λ is a standard isomorphism. Let R and S be finite fields. If $m \geq 3$ and $n \geq 3$ then it is proved in [8] that Λ is a standard isomorphism. Then, without loss of generality, we may consider that $m = 2$ and $\Lambda\overline{t_1(1)} = \overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$, where $\alpha, \delta \in S$. According to Lemma 2,

$$C_{\mathrm{PSL}(2,S)}\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$$

is commutative. Then $C'_{\mathrm{PSL}(n,R)}\overline{t_{1n}(1)}$ is also commutative. Then $n = 3$ and $R = F_2$ or $n = 2$. If $n = 3$ and $R = F_2$, then $S = F_7$ and the mapping $\Lambda\colon \mathrm{PSL}(3, F_2) \longrightarrow \mathrm{PSL}(2, F_7)$:

$$\Lambda\overline{t_{12}(1)} = \overline{\begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}}; \quad \Lambda\overline{t_{23}(1)} = \overline{\begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}}; \quad \Lambda\overline{t_{31}(1)} = \overline{\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}}$$

induces a nonstandard isomorphism. Let $n = 2$. It is easy to show that $|\mathrm{PSL}(2, R)| = (|R|^2 - 1)|R|/2$, if $\operatorname{char} R \neq 2$ and $|\mathrm{PSL}(2, R)| = (|R|^2 - 1)|R|$, if $\operatorname{char} R = 2$.

If $\operatorname{char} R \neq 2$ and $\operatorname{char} S \neq 2$ then $|R| = |S|$ and $\operatorname{char} R = \operatorname{char} S = p > 0$. Then, without loss of generality, we may assume that $|R| = 2^t$ and $|S| = p^s$, where $p = \operatorname{char} S > 2$. Then

$$2(|R|^2 - 1)|R| = (|S|^2 - 1)|S|. \tag{4}$$

Since $|S|$ does not divide $|R|$ and $|R| + 1 = (|R| - 1) = 2$, $|S| \leq |R| + 1$. On the other hand, the equality (4) is possible only if $|R| \leq |S| - 1$. Hence, $|S| = |R| + 1 = 5$. In this case there exists a nonstandard isomorphism $\Lambda\colon \mathrm{PSL}(2, R) \longrightarrow \mathrm{PSL}(2, S)$, where $|R| = 5$ and $S = \{0, 1, \alpha, \alpha^2, \alpha^3 = 1\}$, which may be determined on generators of $\mathrm{PSL}(2, R)$ in the following way:

$$\Lambda\overline{t_{12}(1)} = \overline{\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}}; \quad \Lambda\overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \overline{\begin{pmatrix} 0 & \alpha \\ \alpha^2 & 1 \end{pmatrix}}.$$

LEMMA 6. *Let R be a division ring and $A = \{aE\} \subset \mathrm{SL}(2, R)$. Let $R_1 = \{r \mid ar = ra$ for all $a \in A\}$ and $\bar{H} = C'_{\mathrm{PSL}(2,R)}(\bar{A})$.*

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Then the following results hold:

$\lvert R_1\rvert$	$\bar{H}$	$[\bar{H},\bar{H}]$	$\lvert\bar{H}\rvert$	$\lvert[\bar{H},\bar{H}]\rvert$
>3		$\overline{\mathrm{SL}(2,R_1)}$		≥ 60
$3;\quad -1\notin[R^*,R^*]$	$\mathrm{PSL}'(2,F_3)$	$\bar{E}$	4	4
	$\mathrm{PSL}(2,F_3)$	$\mathrm{PSL}'(2,F_3)$	12	4
$3;\quad -1\in[R^*,R^*]$	$\mathrm{PSL}(2,F_3)$	$\mathrm{PSL}'(2,F_3)$	12	4
	$\mathrm{PGL}(2,F_3)$	$\mathrm{PSL}(2,F_3)$	24	12
2	$\mathrm{PSL}'(2,F_2)$	$\bar{E}$	3	1
	$\mathrm{PSL}(2,F_2)$	$\mathrm{PSL}'(2,F_2)$	6	3

PROOF. The inclusions $\overline{\mathrm{SL}(2,R_1)}\subset C_{\mathrm{PSL}(2,R)}(\bar{A})$ and

$$\overline{\mathrm{SL}'(2,R_1)}\subset\bar{H}\subset\overline{\mathrm{GL}(2,R_1)}$$

are obvious. It should be noted that $\mathrm{GL}(2,R_1)$ normalizes $C_{\mathrm{PSL}(2,R)}(\bar{A})$ and, consequently, $\bar{H}$ as well. If $x^2\neq 1$, then the following formula holds:

$$\Big[t_{12}\big(x(1-x^2)^{-1}\big),\mathrm{diag}(x;x^{-1})\Big]=t_{12}(x). \tag{5}$$

Therefore, in the case when $\lvert R_1\rvert>3$ the following inclusion is valid:

$$\overline{\mathrm{SL}(2,R_1)}=\overline{\mathrm{SL}'(2,R_1)}\subset\bar{H}.$$

Since $\mathrm{GL}'(2,R_1)=SL(2,R_1)$, $[\bar{H},\bar{H}]=\overline{\mathrm{SL}(2,R_1)}$. Let $R_1=F_3$. Then $\zeta R=R_1=F_3$.

Let $-1\notin[R^*,R^*]$. If H contains a transvection then $\bar{H}=\overline{\mathrm{SL}(2,F_3)}=\mathrm{PSL}(2,F_3)$ and

$$[\bar{H},\bar{H}]=\mathrm{PSL}'(2,F_3)=\left\{\bar{E},\overline{\begin{pmatrix}0&1\\-1&0\end{pmatrix}},\overline{\begin{pmatrix}-1&1\\1&1\end{pmatrix}},\overline{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}\right\}.$$

Therefore, we'll consider the case when $\bar{H}$ does not contain any transvection.

$\bar{H}$ contains $\overline{\mathrm{SL}'(2,F_3)}=\mathrm{PSL}'(2,F_3)$. Obviously,

$$\lvert\,\mathrm{PGL}(2,F_3)/\,\mathrm{PSL}'(2,F_3)\rvert=6 \text{ and } \mathrm{PGL}(2,F_3)/\,\mathrm{PSL}'(2,F_3)$$

are generated by $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&1\end{pmatrix}$. It is clear that these elements and their products do not belong to $\bar{H}$. Therefore, $\bar{H}=\mathrm{PSL}'(2,F_3)$ and $[\bar{H},\bar{H}]=\bar{E}$.

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Let $-1 \in [R^*, R^*]$ Then $\mathrm{PGL}(2, F_3) = \overline{\mathrm{GL}(2, F_3)} \subset C_{\mathrm{PSL}(2\ R)}(\bar{A})$ and

$$\mathrm{PSL}(2, F_3) = \mathrm{PGL}'(2, F_3) \subset \bar{H} \subset \mathrm{PGL}(2, F_3)$$

Two cases are possible $\bar{H} = \mathrm{PSL}(2, F_3)$ or $\bar{H} = \mathrm{PGL}(2, F_3)$ In the first case $[\bar{H}, \bar{H}] = \mathrm{PSL}'(2, F_3)$, and in the second $[\bar{H}, \bar{H}] = \mathrm{PSL}(2, F_3)$

Finally, let $R_1 = F_2$ Since $\zeta R = R_1 = F_2$ then

$$\mathrm{PSL}'(2, F_2) = \overline{\mathrm{SL}'(2, F_2)} \subset \bar{H} \subset \overline{\mathrm{GL}(2, F_2)} = \mathrm{PSL}(2, F_2)$$

Hence, $\bar{H} = \mathrm{PSL}'(2, F_2)$ and $[\bar{H}, \bar{H}] = \bar{E}$ or $\bar{H} = \mathrm{PSL}(2, F_2)$ and $[\bar{H}, \bar{H}] = \mathrm{PSL}'(2, F_2)$

It is obvious that $\mathrm{char}\, R_1 = \mathrm{char}\, R$ Therefore, if Λ is an arbitrary isomorphism between $\mathrm{PSL}(2, R)$ and $\mathrm{PSL}(2, S)$ such that $\Lambda\overline{xE} = \overline{yE}$, where x, y are some elements from R and S and $R = \{r \in R \mid xr = rx\}$, $S_1 = \{s \in S \mid sy = ys\}$, then either the isomorphism Λ induces an isomorphism between $\mathrm{PSL}(2, R_1)$ and $\mathrm{PSL}(2, S_1)$, or $\mathrm{char}\, R = \mathrm{char}\, R_1 = \mathrm{char}\, S_1 = \mathrm{char}\, S = p > 0$, where $p = 2$ or $p = 3$

LEMMA 7 *Let R and S be noncommutative division rings Then there is no isomorphism* Λ $\mathrm{PSL}(n, R) \longrightarrow \mathrm{PSL}(2, S)$, *for which the following equality holds*

$$\Lambda t_{1n}(1) = a\bar{E},$$

where a is some non-central element of S

PROOF Let us suppose that such an isomorphism exists Let $S = \{s \in S \mid sa = as\}$ Since S includes the non-central element a, it includes more than three elements, and $\mathrm{SL}(2, S_1) = \mathrm{SL}'(2, S_1) = \mathrm{GL}'(2, S_1)$

It is easy to see that for arbitrary elements x from $[R^*, R^*]$ and $r \in R$, the element $[x\bar{E}, \overline{t_{1n}(r)}] = \overline{t_{1n}(xrx^{-1} - r)}$ belongs to $C'_{\mathrm{PSL}(n\ R)}\overline{t_{1n}(1)}$

Let $\bar{N}$ be a group consisting of all elements $\overline{t_{1n}(1)}$ which belong to the group $C'_{\mathrm{PSL}(n\ R)}\overline{t_{1n}(1)}$ Then $\Lambda\bar{N}$ is a commutative normal subgroup of $C_{\mathrm{PSL}(2\ S)}a\bar{E}$ Obviously

$$\overline{\mathrm{SL}(2, S_1)} \subset C'_{\mathrm{PSL}(2\ S)}a\bar{E} \subset \overline{\mathrm{GL}(2, S_1)}$$

So $\Lambda\bar{N}$ belongs to the centre of the group $C'_{\mathrm{PSL}(2\ S)}a\bar{E}$ It follows that the group $\bar{N}$ belongs to the centre of $C'_{\mathrm{PSL}(n\ R)}\overline{t_{1n}(1)}$ Hence all the elements commute with the elements of the group $(R^*)''$ and as a consequence, belong to the centre of R This, however, contradicts the non-commutativity of the division ring R Really, if $xr_0x^{-1} - r_0 = \alpha \neq 0$ for some $r_0 \in R$, then for an arbitrary element $r \in R$, the following equalities hold $xrx^{-1} = r + \beta$, $xr_0rx^{-1} = r_0r+\gamma$, where α, β, γ are elements from the centre of R Since $r_0\beta+\alpha r+\alpha\beta = \gamma$, then R is generated by the elements of the centre and by r_0 which proves commutativity of R

THEOREM 1 *Let R and S be division rings and*

$$\Lambda\ \mathrm{PSL}(2, R) \longrightarrow \mathrm{PSL}(2, S)$$

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be an arbitrary isomorphism. Then Λ *is standard, except for the groups* $\mathrm{PSL}(2, F_4)$ *and* $\mathrm{PSL}(2, F_5)$.

PROOF. Taking into account Lemmas 4 and 5, we may assume that division rings R and S are noncommutative and do not have the same characteristic $p > 0$. Without loss of generality, we may assume that

$$\Lambda\overline{t_{12}(1)} = \overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}} \text{ and } \Lambda\overline{sv} = \overline{t_{12}(1)}.$$

Let $vt_{12}(1)v^{-1} = t_{12}(x)$. Then $\Lambda\overline{st_{12}(x)s^{-1}} = \overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$.

If $\operatorname{char} R = 2$, then $\operatorname{char} S \neq 2$, $\delta = 0$, and α is an element from the centre of S. If $\overline{\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}}$ commutes with $\overline{\begin{pmatrix} 1 & \alpha-1 \\ 1 & -1 \end{pmatrix}}$, then $\alpha = 1/2$ and $s = E$. Since $[\overline{t_{12}(1)}, \bar{v}]$ is a transvection and

$$\left(\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, t_{12}(1)\right) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{-1},$$

then, without loss of generality, we may assume that the element $\overline{\begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}}$ does not commute with $\overline{\begin{pmatrix} 1 & \alpha-1 \\ 1 & -1 \end{pmatrix}}$ and $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

If $\operatorname{char} R \neq 2$ and $\operatorname{char} S \neq 2$, then, by Lemma 7, $\delta \neq 0$. Hence the elements $\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ and $\overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$ are non-commutative and $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

If $\alpha\delta \neq \delta\alpha$ and $\Lambda\overline{u_1 s_1 v_1} = \alpha\bar{E}$, then, since the element $[t_{12}(1), u_1 s_1 v_1]$ is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & * \end{pmatrix}$ and

$$\left(\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}, \alpha E\right) = t_{12}\alpha(\alpha^{-1}\delta - \delta\alpha^{-1})\alpha^{-1},$$

we may assume that $\alpha\delta = \delta\alpha$, considering the isomorphism Λ^{-1} instead of the isomorphism Λ.

Commutativity of $\overline{\alpha E}$ with the elements $\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ and $\overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$ implies $\Lambda\beta\bar{E} = \alpha\bar{E}$, where β is some element of division ring R which commutes with α.

Let $R_1 = \{r \in R \mid r\beta = \beta r\}$ and $S_1 = \{s \in S \mid s\alpha = \alpha s\}$. If α is a non-central element of R, then, by Lemma 6, $\Lambda\,\mathrm{PSL}(2, R_1) = \mathrm{PSL}(2, S_1)$ and the element α belongs to the centre S_1.

Thus, without loss of generality, we may assume that in all the cases $\Lambda\overline{t_{12}(1)} = \overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ and $\Lambda\overline{t_{21}(x)} = \overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$, where α is an element of the centre of S.

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It is easy to see that the group $C'_{\mathrm{PSL}(2,R)}\overline{t_{12}(1)}$ is non-commutative. Hence $\bar{H} = C_{\mathrm{PSL}(2,S)}\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ is also non-commutative.

Let $\bar{h}$ be an arbitrary element of H and $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$. Let T denote a division ring generated by the elements h_1, h_2, h_3, h_4 for all the elements $\bar{h} \in \bar{H}$. Obviously, all the elements of the division ring T commute with α and δ. By Lemma 2, T is non-commutative.

Let γ be an arbitrary element from the group $[T^*, T^*]$. It follows from commutativity of $\gamma\bar{E}$ with the elements $\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ and $\overline{\begin{pmatrix} 1 & 1+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$ that $\Lambda x(\gamma)\bar{E} = \gamma\bar{E}$, where $x(\gamma)$ is an element of R which depends on γ and commutes with x.

Let $R_2 = \{r \in R \mid rx(\gamma) = x(\gamma)r \text{ for all } \gamma \in [T^*, T^*]\}$ and $S_2 = \{s \in S \mid s\gamma = \gamma s$ for all $\gamma \in [T^*, T^*]\}$. Since $\operatorname{char} R_2 \neq \operatorname{char} S_2$, it follows from Lemma 6 that $|R_2| > 3$, $|S_2| > 3$, and the isomorphism Λ induces an isomorphism of groups $\mathrm{PSL}(2, R_2)$ and $\mathrm{PSL}(2, S_2)$. The intersection of the division rings $T \cap S_2$ belongs to the centre of T; by Lemma 2, the groups

$$C'_{\mathrm{PSL}(2,S_2)}\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}} \text{ and } C'_{\mathrm{PSL}(2,R_2)}\overline{t_{12}(1)}$$

are commutative and R_2 is a field.

According to Lemma 5 either Λ maps transvections into transvections or Λ induces a non-standard isomorphism between $\mathrm{PSL}(2, F_4)$ and $\mathrm{PSL}(2, F_5)$. In the first case, by Lemma 3 Λ is standard. In the second case, since the centres of R and S belong to R_2 and S_2, by Lemma 1 the groups $\mathrm{SL}(2, R)$ and $\mathrm{SL}(2, S)$ include infinite sets of pairwise commutative involutions. That is impossible in a linear group over a division ring with characteristic not equal to 2.

We have proved that the isomorphism Λ is standard.

THEOREM 2. *Let R and S be division rings and $\Lambda\colon \mathrm{PSL}(n, R) \longrightarrow \mathrm{PSL}(2, S)$ be an arbitrary isomorphism. If $n > 2$, then the isomorphism Λ is impossible, except for the groups* $\mathrm{PSL}(3, F_2)$ *and* $\mathrm{PSL}(2, F_7)$.

PROOF. Let $n > 2$. By the Lemmas 4 and 5 we may assume that the division rings R and S are non-commutative and do not have the same characteristic $p > 0$. Without loss of generality, we may assume that $\Lambda\overline{t_{1n}(1)} = \overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}$ and $\Lambda\overline{sv} = \overline{t_{12}(1)}$.

Let $vt_{1n}(1)v^{-1} = t_{1n}(x)$. Then $\Lambda\overline{st_{1n}(x)s^{-1}} = \overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$.

First consider the case: $\operatorname{char} S \neq 2$ or $\delta \neq 1$. Then the group

$$C'_{\mathrm{PSL}(2,S)}\left\{\overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}}, \overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}\right\}$$

consists of the projective images of diagonal matrices with equal diagonal elements.

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According to Lemma 7 we may assume that the group

$$C'_{\mathrm{PSL}(n,R)}\{\overline{t_{1n}(1)}, s\overline{t_{1n}(x)}s^{-1}\}$$

includes no transvection. Hence $n = 3$ and $st_{13}(x)s^{-1} = t_{31}(x)$.

Thus, up to the conjugacy we may assume that

$$\Lambda\overline{t_{23}(1)} = \overline{\begin{pmatrix} 0 & \alpha \\ 1 & \delta \end{pmatrix}} \text{ and } \Lambda\overline{t_{32}(x)} = \overline{\begin{pmatrix} 1 & \alpha+\delta-1 \\ 1 & \delta-1 \end{pmatrix}}$$

Then $\Lambda\overline{\mathrm{diag}(t,1,1)} = \gamma(t)\bar{E}$ for all the elements $t \in (R^*)''$.

Let $S_1 = \{s \in S \mid s\gamma(t) = \gamma(t)s$ for all $t \in (R^*)''\}$. The group $C''_{\mathrm{PSL}(3,R)}\{\overline{\mathrm{diag}(t,1,1)}$ for all $t \in (R^*)''\} = \overline{\mathrm{diag}\big(1, \mathrm{SL}(2,R)\big)}$ is isomorphic to $\mathrm{SL}(2,R)$. $\mathrm{SL}(2,R)$ is an infinite group, by Lemma 6, $C''_{\mathrm{PSL}(2,S)}\{\gamma(t)\bar{E}\} = \overline{\mathrm{SL}(2,S_1)}$. Thus, the isomorphism Λ induces an isomorphism of the groups $\mathrm{PSL}(2,R)$ and $\mathrm{PSL}(2,S_1)$ which is, by Theorem 1, standard. Hence, the isomorphism Λ preserves transvections and thus is impossible.

Consider the last case. Since

$$\Lambda\overline{t_{1n}(-1)} = \overline{\begin{pmatrix} -\delta\alpha^{-1} & 1 \\ \alpha^{-1} & 0 \end{pmatrix}} = \overline{\begin{pmatrix} 1 & \alpha \\ 0 & \alpha \end{pmatrix}}^{-1}\overline{\begin{pmatrix} 0 & \alpha^{-1} \\ 1 & -\delta\alpha^{-1} \end{pmatrix}}\overline{\begin{pmatrix} 1 & \delta \\ 0 & \alpha \end{pmatrix}},$$

it is enough to consider the case $\mathrm{char}\, S = 2$, $\delta = 1$, $\alpha = 1$. Then $\mathrm{char}\, R = 3$ and $st_{1n}(x)s^{-1} = t_{1n}(-1)$. Hence, $x = -1$ and $vt_{1n}(1)v^{-1} = t_{1n}(-1)$.

Let z be an arbitrary non-zero element of the centre of R and $\Lambda\overline{t_{1n}(z)} = \bar{\sigma}$. Then $svt_{1n}(z)v^{-1}s^{-1} = t_{1n}(-z)$. Hence

$$\overline{t_{12}(1)}\bar{\sigma}\overline{t_{12}(-1)} = \bar{\sigma}^{-1}, \quad \bar{\sigma}^3 = \bar{E}, \quad \bar{\sigma}\overline{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} = \overline{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}\bar{\sigma}$$

It follows from these equalities that z is equal to 1 or -1. Hence, the centre of the division ring R coincides with F_3.

Since there is an infinite number of commutative involutions in the group $\mathrm{PSL}(2,S)$, their number in the group $\mathrm{PSL}(n,R)$ is also infinite. Because the centre of R is finite, by Lemma 1, the group $\mathrm{SL}(n,R)$ includes an infinite number of pairwise commutative involutions, that is impossible.

As a result, we have proved that the groups $\mathrm{PSL}(n,R)$ $(n > 2)$ and $\mathrm{PSL}(2,S)$ over division rings R and S are non-isomorphic, except for the groups $\mathrm{PSL}(3,F_2)$ and $\mathrm{PSL}(2,F_7)$.

REFERENCES

1. O. Schreirer and B. L. van der Waerden, *Die Automorphismen der projektiven Gruppen*, Abh. Mat. Sem. Hamburg **6**(1928), 303–322.

2. A. J. Hahn, D. G. James and B. Weisfeiler, *Homomorphisms of algebraic and classical groups: a survey*, Can. Math. Soc. Conf. Proc. **4**(1984), 249–296.

3. J. Dieudonne, *La geometrie des groupes classiques*, Berlin, Hottingen, Heidelberg, troisieme edition, 1971.

4. L. K. Hua and C. H. Wan, *On the automorphisms and isomorphisms of linear groups*, J. Chinese Mat. Soc. (1953), 1–32.

5. ______, *On the automorphisms of the symplectic groups over any field*, Ann. of Math. **49**(1948), 739–759.

https://doi.org/10.4153/CJM-1993-055-3 Published online by Cambridge University Press

6. J Dieudonne, *On the automorphisms of the classical groups*, Mem Amer Math Soc **2**(1951), 1–95
7. H Ren, *Automorphisms of* $\mathrm{SL}_2(k)$ *over a class of skew fields*, (Chinese), Acta Math Sin **424**(1981), 566–577
8. V M Petechuk, *Isomorphisms of groups that are rich in projective transvections*, Mat Zametki (2) **39** (1986), 186-195
9. Yu I Merzlyakov, *A survey of the latest results on classical group automorphisms* In The classical group automorphisms, Moscow, "Mir" Publishers, 1976, 250–259
10. ———, *The isomorphisms of classical groups over integral rings*, Moscow, "Mir" Publishers, 1980, 1–272
11. V M Petechuk, *The isomorphisms of linear groups over division rings*, Lviv, Abstracts of the XIX-th All-Union Algebraic Conference (2) (1987), 220–221
12. H Ren, Z Wan and X Wu, *Automorphisms and isomorphisms of linear groups over skew fields*, Proceedings of Symposia in Pure Mathematics **47**(1987), 473–476
13. A I Hahn, *Category equivalences and linear groups over rings*, J Algebra **77**(1982), 505–543
14. ———, *Isomorphism theory for orthogonal groups over arbitrary integral domains*, J Algebra (1) **51** (1978), 233–287

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