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ON THE P-NORM OF THE TRUNCATED HILBERT TRANSFORM

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The p-norm of the Hilbert transform is the same as the p-norm of its truncation to any Lebesgue measurable set with strictly positive measure. This fact follows from two symmetry properties, the joint presence of which is essentially unique to the Hilbert transform. Our result applies, in particular, to the finite Hilbert transform taken over $(-1,1)$, and to the one-sided Hilbert transform taken over $(0,\infty)$. A related weaker property holds for integral operators with Hardy kernels.

1. INTRODUCTION

We denote the Hilbert transform of the function u by Hu, that is,

$$Hu(x) = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \frac{u(y)}{y-x}\,dy = \lim_{\epsilon\to 0+} \frac{1}{\pi i} \int_{|y-x|>\epsilon} \frac{u(y)}{y-x}\,dy$$

for $-\infty < x < \infty$. A famous theorem of M. Riesz [7] asserts that, for $1 < p < \infty$, there exists a constant $M_p < \infty$ such that

$$\int_{-\infty}^{\infty} |Hu(x)|^p\,dx \leqslant (M_p)^p \int_{-\infty}^{\infty} |u(x)|^p\,dx \tag{1.1}$$

for every function $u \in L_p(\mathbf{R})$; hence, H is a bounded linear operator on $L_p(\mathbf{R})$ with norm $\leqslant M_p$. During the 1960s, a number of authors made progress towards establishing the value M_p^* of the best constant in (1.1). O'Neil and Weiss [5] gave the upper bound

$$M_p^* \leqslant \frac{2}{\pi}\int_0^{\infty} \frac{\operatorname{arsinh} t}{t^{2-1/p}}\,dt = \frac{q}{\pi^{3/2}}\Gamma(1/2p)\Gamma(1/2q),$$

where q is the conjugate exponent to p, that is, $1/p + 1/q = 1$, and Gohberg and Krupnik [3] gave the lower bound $M_p^* \geqslant \nu(p)$ for $p = 2^n$ $(n = 1, 2, 3, \ldots)$, where

$$\nu(p) = \begin{cases} \tan(\pi/2p), & 1 < p \leqslant 2, \\ \cot(\pi/2p), & 2 \leqslant p < \infty. \end{cases}$$

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The question was settled in 1972 by Pichorides [6], who proved that $M_p^* = \nu(p)$ for $1 < p < \infty$. A recent book by Krupnik [4, Chapter 2] contains a lengthy discussion of some generalisations of this result, involving weighted L_p spaces and quotient norms.

In the present paper, we consider the truncated Hilbert transform, defined by

$$H_E u(x) = \frac{1}{\pi i} \int_E \frac{u(y)}{y-x}\, dy, \qquad x \in E,$$

where E is a measurable subset of $\mathbf{R}$. The most commonly occurring cases are the finite Hilbert transform, for which $E = (-1,1)$, and the one-sided Hilbert transform, for which $E = (0,\infty)$.

It is obvious that, for $1 < p < \infty$, there exists a constant $M_{p,E} < \infty$ such that

$$\int_E |H_E u(x)|^p\, dx \leqslant (M_{p,E})^p \int_E |u(x)|^p\, dx$$

for every $u \in L_p(\mathbf{R})$, and moreover the best constant $M_{p,E}^*$ satisfies $M_{p,E}^* \leqslant M_p^*$. We will show that in fact equality holds, that is,

$$M_{p,E}^* = M_p^* = \nu(p) \qquad \text{for } 1 < p < \infty,$$

provided the Lebesgue measure of E is not zero. The method of proof involves two symmetry properties of the Hilbert transform, and yields as a by-product a related result for integral operators with Hardy kernels.

2. Translations and Dilatations

Denote the norm of a function f in $L_p(\mathbf{R})$ by

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx \right)^{1/p}, \qquad 1 < p < \infty,$$

and the norm of a linear operator $A : L_p(\mathbf{R}) \to L_p(\mathbf{R})$ by

$$\|A\|_p = \sup_{u \in L_p(\mathbf{R}),\, \|u\|_p = 1} \|Au\|_p.$$

Let $\chi_E : L_p(\mathbf{R}) \to L_p(\mathbf{R})$ be the operator of pointwise multiplication by the characteristic function of the measurable set $E \subseteq \mathbf{R}$; in other words

$$\chi_E u(x) = \begin{cases} u(x), & x \in E, \\ 0, & x \in \mathbf{R} \setminus E. \end{cases}$$

Since $f = \chi_E f + (1-\chi_E)f$, the space $L_p(\mathbf{R})$ is the direct sum

$$L_p(\mathbf{R}) = L_p(E) \oplus L_p(\mathbf{R} \setminus E).$$

In this way, we can think of $L_p(E)$ as a closed subspace of $L_p(\mathbf{R})$, and write the truncated Hilbert transform as

$$H_E = \chi_E H \chi_E : L_p(\mathbf{R}) \to L_p(\mathbf{R}). \tag{2.1}$$

The best constants mentioned in the Introduction are simply the p-norms

$$M_p^* = \|H\|_p, \qquad M_{p,E}^* = \|H_E\|_p.$$

Introduce the translation operator

$$\mathcal{T}_a : L_p(\mathbf{R}) \to L_p(\mathbf{R}), \qquad a \in \mathbf{R},$$

and the dilatation operator

$$\mathcal{D}_m : L_p(\mathbf{R}) \to L_p(\mathbf{R}), \qquad m > 0,$$

defined by

$$\mathcal{T}_a f(x) = f(x-a), \qquad \mathcal{D}_m f(x) = m^{-1/p} f(x/m).$$

Both $\mathcal{T}_a$ and $\mathcal{D}_m$ are isomorphisms, indeed,

$$(\mathcal{T}_a)^{-1} = \mathcal{T}_{-a}, \qquad (\mathcal{D}_m)^{-1} = \mathcal{D}_{1/m},$$

and moreover, both operators are isometries, that is,

$$\|\mathcal{T}_a f\|_p = \|f\|_p, \qquad \|\mathcal{D}_m f\|_p = \|f\|_p$$

for every $f \in L_p(\mathbf{R})$.

A bounded linear operator $A : L_p(\mathbf{R}) \to L_p(\mathbf{R})$ is said to *commute with translations* if

$$\mathcal{T}_a A = A \mathcal{T}_a \qquad \text{for all } a \in \mathbf{R},$$

and similarly, A is said to *commute with dilatations* if

$$\mathcal{D}_m A = A \mathcal{D}_m \qquad \text{for all } m > 0.$$

For example, consider an integral operator (possibly singular)

$$Ku(x) = \int_{-\infty}^{\infty} k(x,y) u(y)\, dy, \qquad -\infty < x < \infty,$$

and assume that K is a bounded linear operator on $L_p(\mathbb{R})$. One can easily verify that K commutes with translations if and only if k is a difference kernel, that is, if and only if

$$k(x,y) = k(x-y,0) = k(0,y-x),$$

whereas K commutes with dilatations if and only if k is a Hardy kernel, that is, if and only if

$$k(mx,my) = m^{-1}k(x,y) \qquad \text{for all } m > 0.$$

Obviously, both properties hold for the Hilbert transform, and we will now show that H is essentially the only integral operator for which this is the case.

THEOREM 2.1. *Let $1 < p < \infty$ and suppose $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is a bounded linear operator. If A commutes both with translations and with dilatations, then*

$$A = aI + bH,$$

where a and b are constants, and I is the identity operator.

PROOF: Denote the Fourier transform of u by

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x)\, dx, \qquad \xi \in \mathbb{R},$$

then since A commutes with translations and is bounded on $L_p(\mathbb{R})$, there exists a function $\sigma \in L_\infty(\mathbb{R})$ satisfying

$$\widehat{Au}\;\; (\xi) = \sigma(\xi)\hat{u}(\xi), \qquad \xi \in \mathbb{R};$$

see Bergh and Löfström [**1**, p.132–133] and Stein [**8**, p.28]. The Fourier transform has the property

$$\widehat{\mathcal{D}_m u}\;\; (\xi) = m^{-1-1/p}\hat{u}(m\xi),$$

therefore, since A commutes with dilatations,

$$\sigma(\xi) = \sigma(m\xi) \qquad \text{for } \xi \in \mathbb{R} \text{ and } m > 0,$$

and so

$$\sigma(\xi) = \begin{cases} \sigma(+1), & \text{if } \xi > 0 \\ \sigma(-1), & \text{if } \xi < 0. \end{cases}$$

Put

$$a = \frac{\sigma(+1) + \sigma(-1)}{2}, \qquad b = \frac{\sigma(+1) - \sigma(-1)}{2},$$

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then $\sigma(\xi) = a + b\,\mathrm{sgn}\,(\xi)$, where

$$\mathrm{sgn}\,(\xi) = \begin{cases} +1, & \text{if } \xi > 0 \\ -1, & \text{if } \xi < 0. \end{cases}$$

The result follows at once, because $\widehat{Hu}\ (\xi) = \mathrm{sgn}\,(\xi)\hat{u}(\xi)$. ∎

For any bounded linear operator $A : L_p(\mathbf{R}) \to L_p(\mathbf{R})$, we define by analogy with (2.1) the truncated operator

$$A_E = \chi_E A \chi_E : L_p(\mathbf{R}) \to L_p(\mathbf{R}).$$

Write

$$a + E = \{\, a + x : x \in E \,\}, \qquad mE = \{\, mx : x \in E \,\}$$

then

$$\chi_E \mathcal{T}_a = \mathcal{T}_a \chi_{a+E}, \qquad \chi_E \mathcal{D}_m = \mathcal{D}_m \chi_{mE}$$

for all $a \in \mathbf{R}$ and $m > 0$; consequently, the following holds.

THEOREM 2.2. *Let E be any measurable subset of $\mathbf{R}$.*

(i) *If A commutes with translations, then*

$$\|A_{a+E}\|_p = \|A_E\|_p \qquad \text{for all } a \in \mathbf{R}.$$

(ii) *If A commutes with dilatations, then*

$$\|A_{mE}\|_p = \|A_E\|_p \qquad \text{for all } m > 0.$$

PROOF: Assume that A commutes with translations, then

$$\begin{aligned} \mathcal{T}_a A_{a+E} &= \mathcal{T}_a \chi_{a+E} A \chi_{a+E} = \chi_E \mathcal{T}_a A \chi_{a+E} = \chi_E A \mathcal{T}_a \chi_{a+E} \\ &= \chi_E A \chi_E \mathcal{T}_a, \end{aligned}$$

and so $A_{a+E} = (\mathcal{T}_a)^{-1} A_E \mathcal{T}_a$. Likewise, if A commutes with dilatations, then $A_{mE} = (\mathcal{D}_m)^{-1} A_E \mathcal{D}_m$. Now recall that $\mathcal{T}_a$ and $\mathcal{D}_m$ are isometric isomorphisms. ∎

3. DENSITY

If A commutes with dilatations, then it is tempting to send m to ∞ in part (ii) of Theorem 2.2, and to conclude that $\|A\|_p = \|A_E\|_p$ because $\|A_{mE}\|_p \to \|A\|_p$. We will see that this can be done, provided the set E is 'sufficiently dense' at zero.

Denote the Lebesgue measure of E by $|E|$, and denote the open interval of length 2δ centred at x by

$$J_\delta(x) = (x-\delta, x+\delta), \qquad x \in \mathbb{R},\ \delta > 0.$$

When it exists, the limit

$$d_E(x) = \lim_{\delta\to 0+} \frac{|E \cap J_\delta(x)|}{|J_\delta(x)|}$$

is called the *density* of E at x; obviously, $0 \leqslant d_E(x) \leqslant 1$. If x belongs to the interior of E, then $d_E(x) = 1$, whereas if x lies outside the closure of E, then $d_E(x) = 0$. For an interval $E = (a,b)$, the density at the end points is $d_E(a) = d_E(b) = 1/2$. The Lebesgue Density Theorem [**2**, p.184] asserts that

$$(3.1) \qquad d_E(x) = 1 \qquad \text{for almost every } x \in E,$$

so provided $|E| \neq 0$, the set E has plenty of points with density 1.

The density of E at zero can be characterized in terms of the dilatations of E.

Lemma 3.1. *If J is a bounded interval centred at 0, then*

$$\lim_{m\to\infty} |J \cap mE| = d_E(0)|J|.$$

Proof: Note that $|mE| = m|E|$ and $m(E_1 \cap E_2) = (mE_1) \cap (mE_2)$ for any $m > 0$ and any measurable sets E, E_1 and E_2. Suppose $J = (-M, M)$ and let $m = M/\delta$, then $mJ_\delta(0) = J$ and so

$$d_E(0) = \lim_{\delta\to 0+} \frac{|E \cap J_\delta(0)|}{|J_\delta(0)|} = \lim_{m\to\infty} \frac{|(mE) \cap J|}{|J|},$$

which means $|J \cap mE| \to d_E(0)|J|$ as $m \to \infty$. ∎

Lemma 3.2. *For $1 \leqslant p < \infty$, the following are equivalent:*

(i) $d_E(0) = 1$;

(ii) $\lim_{m\to\infty} \|\chi_{mE} f\|_p = \|f\|_p$ *for all* $f \in L_p(\mathbb{R})$;

(iii) $\lim_{m\to\infty} \|(1-\chi_{mE}) f\|_p = 0$ *for all* $f \in L_p(\mathbb{R})$.

Proof: Parts (ii) and (iii) are equivalent because

$$(\|f\|_p)^p = (\|\chi_{mE} f\|_p)^p + (\|(1-\chi_{mE}) f\|_p)^p.$$

Suppose (ii) holds. Let J be a bounded interval centred at zero, and put $f = \chi_J 1$, then

$$\lim_{m\to\infty} |J \cap mE| = \lim_{m\to\infty} (\|\chi_{mE} f\|_p)^p = (\|f\|_p)^p = |J|,$$

and so (i) is true by Lemma 3.1; thus (i) $\Longleftarrow$ (ii) $\Longleftrightarrow$ (iii).

We complete the proof by showing that (i) $\Longrightarrow$ (iii). Assume $d_E(0) = 1$, and let $f \in L_p(\mathbb{R})$ and $\varepsilon > 0$. There is a continuous function $g : \mathbb{R} \to \mathbb{R}$ having compact support and satisfying $\|f - g\|_p < \varepsilon$. Choose a bounded interval J, centred at zero and containing the support of g, then

$$(1 - \chi_{mE})f = (1 - \chi_{mE})(f - g) + (1 - \chi_{mE})\chi_J g$$

and so

$$\begin{aligned}\|(1 - \chi_{mE})f\|_p &\leqslant \|f - g\|_p + \|(1 - \chi_{mE})\chi_J 1\|_p \|g\|_p \\ &< \varepsilon + |J \setminus mE|^{1/p}\|g\|_p.\end{aligned}$$

Since $d_E(0) = 1$, Lemma 3.1 implies

$$\lim_{m\to\infty} |J \setminus mE| = |J| - \lim_{m\to\infty} |J \cap mE| = 0,$$

therefore $\lim\limits_{m\to\infty} \|(1 - \chi_{mE})f\|_p \leqslant \varepsilon$, from which (iii) follows because $\varepsilon > 0$ is arbitrary. ∎

We are now ready to prove the result mentioned at the beginning of this section.

THEOREM 3.3. *Suppose* $d_E(0) = 1$. *If* A *commutes with dilatations, then*

$$\|A_E\|_p = \|A\|_p.$$

PROOF: Let $\varepsilon > 0$, and choose $u \in L_p(\mathbb{R})$ such that $\|u\|_p = 1$ and

$$\|A\|_p < \|Au\|_p + \varepsilon. \tag{3.2}$$

The splitting

$$Au = A_{mE}u + (1 - \chi_{mE})Au + \chi_{mE}A(1 - \chi_{mE})u$$

implies

$$\|Au\|_p \leqslant \|A_{mE}\|_p + \|(1 - \chi_{mE})Au\|_p + \|A\|_p\|(1 - \chi_{mE})u\|_p,$$

so by sending m to ∞ and applying Theorem 2.2 (ii) and Lemma 3.2, it follows that $\|Au\|_p \leqslant \|A_E\|_p$. Combining this inequality with (3.2), we see $\|A\|_p < \|A_E\|_p + \varepsilon$, and since ε is arbitrary, $\|A\|_p \leqslant \|A_E\|_p$. The reverse inequality $\|A_E\|_p \leqslant \|A\|_p$ is trivial, so the proof is finished. ∎

As explained in Section 2, Theorem 3.3 applies to integral operators with Hardy kernels; for the Hilbert transform, a stronger result holds.

THEOREM 3.4. *If* $|E| \neq 0$, *then* $\|H_E\|_p = \|H\|_p$.

PROOF: By (3.1), there is a point $x \in E$ such that $d_E(x) = 1$, hence $d_{-x+E}(0) = 1$ and so

$$\|H_E\|_p = \|H_{-x+E}\|_p = \|H\|_p,$$

by Theorems 2.1, 2.2 (i) and 3.3. ∎

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