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Norm One Idempotent *cb*-Multipliers with Applications to the Fourier Algebra in the *cb*-Multiplier Norm

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Abstract. For a locally compact group G, let $A(G)$ be its Fourier algebra, let $M_{cb}A(G)$ denote the completely bounded multipliers of $A(G)$, and let $A_{Mcb}(G)$ stand for the closure of $A(G)$ in $M_{cb}A(G)$. We characterize the norm one idempotents in $M_{cb}A(G)$: the indicator function of a set $E \subset G$ is a norm one idempotent in $M_{cb}A(G)$ if and only if E is a coset of an open subgroup of G. As applications, we describe the closed ideals of $A_{Mcb}(G)$ with an approximate identity bounded by 1, and we characterize those G for which $A_{Mcb}(G)$ is 1-amenable in the sense of B. E. Johnson. (We can even slightly relax the norm bounds.)

Introduction

The Fourier algebra $A(G)$ and Fourier–Stieltjes algebra $B(G)$ of a locally compact group G were introduced by P. Eymard [8]. If G is abelian with dual group $\hat{G}$, these algebras are isometrically isomorphic to $L^1(\hat{G})$, the group algebra of $\hat{G}$, and $M(\hat{G})$, the measure algebra of $\hat{G}$, via the Fourier and Fourier–Stieltjes transform, respectively. For abelian G, the idempotent elements in $B(G) \cong M(\hat{G})$ were described by P. J. Cohen [4]: the indicator function χ_E of $E \subset G$ lies in $B(G)$ if and only if E belongs to the *coset ring* $\Omega(G)$ of G, *i.e.*, the ring of sets generated by the cosets of the open subgroups of G. Later, B. Host showed that this characterization of the idempotents in $B(G)$ holds true for general locally compact groups G [16].

In [12], the Cohen–Host idempotent theorem was crucial in characterizing, for amenable G, those closed ideals of $A(G)$ that have a bounded approximate identity, and in [13, 30], the authors made use of it to characterize those G for which $A(G)$ is amenable in the sense of B. E. Johnson [18].

Besides the given norm on $A(G)$, there are other, from certain points of view even more natural, norms on $A(G)$. Recall that a *multiplier* of $A(G)$ is a function ϕ on G with $\phi A(G) \subset A(G)$. It is immediate from the closed graph theorem that each multiplier ϕ of $A(G)$ induces a bounded multiplication operator M_ϕ on $A(G)$; the operator norm on the multipliers turns them into a Banach algebra. Trivially, $A(G)$ embeds contractively into its multipliers, but the multiplier norm on $A(G)$ is equivalent to the given norm if and only if G is amenable [26].

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An even more natural norm on $A(G)$ arises if we take into account that $A(G)$, being the predual of a von Neumann algebra, has a canonical operator space structure. (Our default reference for operator spaces is [7].) This makes it possible to consider the *completely bounded multipliers* (*cb-multipliers* in short) of $A(G)$ as

$$M_{cb}A(G) := \{\phi\colon G \to \mathbb{C}\colon M_\phi\colon A(G) \to A(G) \text{ is completely bounded}\}.$$

For $\phi \in M_{cb}A(G)$, we denote the completely bounded operator norm of M_ϕ by $\|\phi\|_{Mcb}$. It is not difficult to see that $B(G)$ embeds completely contractively into $M_{cb}A(G)$. However, equality holds if and only if G is amenable. In fact, G is amenable if and only if $\|\cdot\|_{Mcb}$ and the given norm on $A(G)$ are equivalent. (For a discussion of these facts with references to the original literature, see [31].)

Let $A_{Mcb}(G)$ denote the closure of $A(G)$ in $M_{cb}A(G)$ (see [10] for some properties of this algebra). For certain non-amenable G, the (completely contractive) Banach algebra $A_{Mcb}(G)$ is better behaved than $A(G)$. For instance, $A(G)$ has a bounded approximate identity if and only if G is amenable ([24]); in particular, if G is $\mathbb{F}_2$, the free group in two generators, then $A(G)$ is not operator amenable. On the other hand, $A_{Mcb}(\mathbb{F}_2)$ has a bounded approximate identity [6] and even is operator amenable [14] in the sense of [28].

Juxtaposing the main results of [13, 14], the question arises immediately whether $A_{Mcb}(\mathbb{F}_2)$ is amenable in the classical sense of [18], and it is this question that has motivated the present note. The proof of the main result of [13], as well as its alternative proof in [30], rests on the Cohen–Host idempotent theorem. Attempting to emulate these proofs with $A_{Mcb}(G)$ in place of $A(G)$ leads to the problem whether certain idempotent functions can lie in $M_{cb}A(G)$. The main problem is that the Cohen–Host theorem is no longer true with $M_{cb}A(G)$ replacing $B(G)$: as M. Leinert showed [23], there are sets $E \subset \mathbb{F}_2$ such that $\chi_E \in M_{cb}A(G) \setminus B(G)$.

The main result of this note is that, even though $M_{cb}(G)$ may have more idempotents than $B(G)$, both algebras do have the same *norm one* idempotents. With this result we can then characterize the closed ideals in $A_{Mcb}(G)$ having an approximate identity bounded by one as well as those G for which $A_{Mcb}(G)$ is 1-amenable. (Due to the useful fact that idempotent Schur multipliers of norm less than $\frac{2}{\sqrt{3}}$ must have norm one, we can even work with slightly relaxed norm bounds.)

1 The Norm One Idempotents of $M_{cb}A(G)$

For a locally compact group G, the functions in $B(G)$ can be described as coefficient functions of unitary representations of G (see [8]). A related characterization, which immediately yields the contractive inclusion $B(G) \subset M_{cb}A(G)$, is the following theorem due to J. Gilbert ([15]; for a more accessible proof, see [21]):

Gilbert's Theorem *Let G be a locally compact group. Then for $\phi\colon G \to \mathbb{C}$ the following are equivalent:*

(i) $\phi \in M_{cb}A(G)$;

(ii) *there are are a Hilbert space $\mathfrak{H}$ and bounded, continuous functions $\xi, \eta\colon G \to \mathfrak{H}$ such that*

$$\phi(xy^{-1}) = \langle \xi(x), \eta(y) \rangle \quad (x, y \in G). \tag{1.1}$$

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Moreover, if $\phi \in M_{cb}A(G)$ *and* ξ *and* η *are Hilbert space valued, bounded, continuous functions on G satisfying* (1.1), *then*

$$\|\phi\|_{Mcb} \le \|\xi\|_\infty \|\eta\|_\infty \tag{1.2}$$

holds, and ξ *and* η *can be chosen such that we have equality in* (1.2).

The following extends [17, Theorem 2.1].

Theorem 1.1 *Let G be a locally compact group. Then for* $E \subset G$ *the following are equivalent:*

(i) $\chi_E \in B(G)$ *with* $\|\chi_E\|_{B(G)} = 1$;
(ii) $\chi_E \in M_{cb}A(G)$ *with* $\|\chi_E\|_{Mcb} = 1$;
(iii) *E is a coset of an open subgroup.*

Proof (i) $\Rightarrow$ (ii) is clear, and (iii) $\Rightarrow$ (i) is the easy part of [17, Theorem 2.1].

(ii) $\Rightarrow$ (iii). Obviously, E is open. If $x \in E$, then $x^{-1}E$ contains e and satisfies $\|\chi_{x^{-1}E}\|_{Mcb} = 1$. Hence, we can suppose without loss of generality that $e \in E$: otherwise, replace E by $x^{-1}E$ for some $x \in E$. We shall show that E is a subgroup of G.

By Gilbert's Theorem, there are a Hilbert space $\mathfrak{H}$ and bounded, continuous functions $\xi, \eta \colon G \to \mathfrak{H}$ with $1 = \|\xi\|_\infty \|\eta\|_\infty$ such that

$$\chi_E(xy^{-1}) = \langle \xi(x), \eta(y) \rangle \quad (x, y \in G). \tag{1.3}$$

Of course, we can suppose that both $\|\xi\|_\infty = \|\eta\|_\infty = 1$. In view of (1.3) and the Cauchy–Schwarz inequality, we obtain

$$xy^{-1} \in E \iff \langle \xi(x), \eta(y) \rangle = 1 \iff \xi(x) = \eta(y) \quad (x, y \in G).$$

As $e \in E$, this means, in particular, that $\xi(e) = \eta(e) =: \xi$, so that

$$E = \{x \in G : \xi(x) = \xi\} = \{y \in G : \eta(y^{-1}) = \xi\}.$$

Hence, if $x, y \in E$, we get $\chi_E(xy) = \langle \xi(x), \eta(y^{-1}) \rangle = \langle \xi, \xi \rangle = 1$, so that $xy \in E$. Consequently, E is a subsemigroup of G.

Let $x \in E$. Applying the preceding argument to $x^{-1}E$ instead of E, we see that x^{-1} is a subsemigroup of G; since $e \in E$, we have, in particular, $x^{-1}x^{-1} \in x^{-1}E$, which means that $x^{-1} \in E$.

All in all, E is a subgroup of G. ∎

Remark. Let $MA(G)$ denote the algebra of all multipliers of $A(G)$. Defining $\|\phi\|_M$ as the operator norm of M_ϕ, we obtain a Banach algebra norm on $MA(G)$; obviously, $M_{cb}A(G)$ embeds contractively into $MA(G)$. Hence, every norm one idempotent in $M_{cb}A(G)$ is a norm one idempotent in $MA(G)$. By [1], $M_{cb}A(\mathbb{F}_2) \subsetneq MA(\mathbb{F}_2)$ holds, and, as M. Bożejko communicated to the second author, there are sets $E \subset \mathbb{F}_2$ such that $\chi_E \subset MA(\mathbb{F}_2) \setminus M_{cb}A(\mathbb{F}_2)$. We do not know if such E can be chosen such that $\|\chi_E\|_M = 1$.

By [2], the elements of $M_{cb}A(G)$ are precisely the so-called *Herz–Schur multipliers* of $A(G)$. For discrete G, the powerful theory of *Schur multipliers* (see [27] for an account) can thus be applied to the study of $M_{cb}A(G)$. By [25] (see also [22]), for any index set $\mathbb{I}$, an idempotent Schur multiplier of $\mathcal{B}(\ell^2(\mathbb{I}))$ with norm greater than 1 must have norm at least $\frac{2}{\sqrt{3}}$. Hence, we obtain the following.

Corollary 1.2 *Let G be a group. Then for $E \subset G$ the following are equivalent:*

(i) $\chi_E \in B(G)$ *with* $\|\chi_E\|_{B(G)} = 1$*;*
(ii) $\chi_E \in M_{cb}A(G)$ *with* $\|\chi_E\|_{Mcb} = 1$*;*
(iii) $\chi_E \in M_{cb}A(G)$ *with* $\|\chi_E\|_{Mcb} < \frac{2}{\sqrt{3}}$*;*
(iv) *E is a coset of a subgroup.*

2 Ideals of $A_{Mcb}(G)$ with Approximate Identities Bounded by $C < 2/\sqrt{3}$

Let G be a locally compact group. In [12], the first author with E. Kaniuth, A. T.-M. Lau, and N. Spronk characterized, for amenable G, those closed ideals of $A(G)$ that have bounded approximate identities in terms of their hulls. Previously, he had obtained a similar characterization of those closed ideals of $A(G)$ that have approximate identities bounded by one without any amenability hypothesis for G [9, Propositon 3.12].

In this section, we use Theorem 1.1 (or rather Corollary 1.2) to prove an analog of [9, Propositon 3.12] for $A_{Mcb}(G)$.

Let H be an open subgroup of G. It is well known that we can isometrically identify $A(H)$ with the closed ideal of $A(G)$ consisting of those functions whose support lies in H; with a little extra effort, one sees that this identification is, in fact, a complete isometry [11, Proposition 4.3]. From there, it is not difficult to prove the analogous statement for $A_{Mcb}(G)$: there is a canonical isometric isomorphism between $A_{Mcb}(H)$ and those functions in $A_{Mcb}(G)$ with support in H.

Given a closed ideal I of $A_{Mcb}(G)$, we define its *hull* to be

$$h(I) := \{x \in G : f(x) = 0 \text{ for all } f \in I\}.$$

If $E \subset G$ is closed, we set

$$I(E) := \{f \in A_{Mcb}(G) : f(x) = 0 \text{ for all } x \in E\},$$

which is a closed ideal of $A_{Mcb}(G)$ such that $h(I(E)) = E$.

Since translation by a group element is an isometric algebra automorphism of $A_{Mcb}(G)$, in view of the preceding discussion we have the following.

Proposition 2.1 *Let G be a locally compact group, let H be an open subgroup of G, and let $x \in G$. Then we have an isometric algebra isomorphism between $A_{Mcb}(H)$ and $I(G \setminus xH)$.*

Our main result in this section is the following.

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Theorem 2.2 *Let G be a locally compact group. Then for a closed ideal I of $A_{Mcb}(G)$ and $C \in [1, \frac{2}{\sqrt{3}})$ the following are equivalent:*

(i) *I has an approximate identity bounded by C;*

(ii) *$I = I(G \setminus xH)$, where $x \in G$ and H is an open subgroup of G such that $A_{Mcb}(H)$ has an approximate identity bounded by C.*

Proof (ii) $\Rightarrow$ (i) is an immediate consequence of Proposition 2.1.

(i) $\Rightarrow$ (ii). Let $(e_\alpha)_\alpha$ be an approximate identity for I bounded by C. By [31, Corollary 6.3(i)], $M_{cb}A(G)$ embeds (completely) isometrically into $M_{cb}A(G_d)$, where G_d stands for the group G equipped with the discrete topology; we may thus view $(e_\alpha)_\alpha$ as a bounded net in $M_{cb}A(G_d)$. It is easy to see that $(e_\alpha)_\alpha$ converges to $\chi_{G \setminus h(I)}$ pointwise on G and thus in $\sigma(\ell^\infty(G), \ell^1(G))$. With the help of [6, Lemma 1.9], we conclude that $\chi_{G \setminus h(I)} \in M_{cb}A(G_d)$ with $\|\chi_{G\setminus h(I)}\|_{Mcb} \leq C$; hence, $\chi_{G \setminus h(I)}$ is an idempotent in $M_{cb}A(G_d)$ of norm strictly less than $\frac{2}{\sqrt{3}}$. By Corollary 1.2, this means that $G \setminus h(I)$ is of the form xH for $x \in G$ and a subgroup H of G and thus $h(I) = G \setminus xH$. Since $h(I)$ is closed, xH, and thus H, must be open. By [14, Proposition 2.2], the Banach algebra $A_{Mcb}(H)$ is Tauberian. By Proposition 2.1, this means that the set $G \setminus xH$ is of synthesis for $A_{Mcb}(G)$, so that $I = I(G \setminus xH)$. Finally, Proposition 2.1 again yields that $A_{Mcb}(H)$ has an approximate identity bounded by C. ■

In [5], locally compact groups G such that $A(G)$ has an approximate identity bounded in $\|\cdot\|_{Mcb}$ were called *weakly amenable*; this is equivalent to $A_{Mcb}(G)$ having an approximate identity [10, Proposition 1]. For instance, $\mathbb{F}_2$ is weakly amenable [6, Corollary 3.9] without being amenable. Both [6, Corollary 3.9] and [14, Theorem 2.7] suggest that for weakly amenable, but not amenable G, the Banach algebra $A_{Mcb}(G)$ is a more promising object of study than $A(G)$. In view of [9, Proposition 3.13] and Theorem 2.2, one is thus tempted to ask whether a suitable version of [12, Theorem 2.3] holds for $A_{Mcb}(G)$ and weakly amenable G: a closed ideal I of $A_{Mcb}(G)$ has a bounded approximate identity if and only $I = I(E)$ for some closed $E \in \Omega(G_d)$.

We conclude this section with an example which shows that the characterization of the closed ideals of $A_{Mcb}(G)$ with a bounded approximate identity for weakly amenable, but not amenable G cannot be as elegant as for amenable G.

Example. Let $E \subset \mathbb{F}_2$ be such that $\chi_E \in M_{cb}A(\mathbb{F}_2)$, but $E \notin \Omega(\mathbb{F}_2)$: such E exists by [23]. Let $I = I(E)$. Then $I = (1 - \chi_E)A_{Mcb}(\mathbb{F}_2)$ is completely complemented in $A_{Mcb}(\mathbb{F}_2)$. Since $A_{Mcb}(\mathbb{F}_2)$ is operator amenable by [14, Theorem 2.7], it follows from [29, Theorem 2.3.7] — with operator space overtones added — that I has a bounded approximate identity even though $h(I) = E \notin \Omega(\mathbb{F}_2)$.

3 Amenability of $A_{Mcb}(G)$

Recall the definition of an amenable Banach algebra. Given a Banach algebra $\mathfrak{A}$, let $\mathfrak{A} \otimes^\gamma \mathfrak{A}$ denote the Banach space tensor product of $\mathfrak{A}$ with itself. The projective Banach space $\mathfrak{A} \otimes^\gamma \mathfrak{A}$ becomes a Banach $\mathfrak{A}$-bimodule via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathfrak{A}).$$

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Let $\Delta\colon \mathfrak{A} \otimes^\gamma \mathfrak{A} \to \mathfrak{A}$ denote the bounded linear map induced by multiplication, *i.e.*, $\Delta(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$.

Definition 3.1 A Banach algebra $\mathfrak{A}$ is called C-*amenable* with $C \geq 1$ if it has an *approximate diagonal* bounded by C, *i.e.*, a net $(d_\alpha)_\alpha$ in $\mathfrak{A} \otimes^\gamma \mathfrak{A}$ bounded by C such that

$$\begin{aligned} a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in \mathfrak{A}) \\ a\Delta d_\alpha \to a \quad (a \in \mathfrak{A}). \end{aligned} \tag{3.1}$$

We say that $\mathfrak{A}$ is *amenable* if there is $C \geq 1$ such that $\mathfrak{A}$ is C-amenable.

Remark. This is not the original definition of an amenable Banach algebra from [18], but equivalent to it [19]. The idea of considering bounds for approximate diagonals seems to originate in [20].

The question as to which locally compact groups G have an amenable Fourier algebra was first studied in depth in [20]. Until then, it was widely believed, probably with an eye on [24], that these G were precisely the amenable ones. In [20], however, Johnson exhibited compact groups G, such as SO(3), for which $A(G)$ is not amenable. Eventually, the authors showed that $A(G)$ is amenable if and only if G is almost abelian, *i.e.*, has an abelian subgroup of finite index ([13, Theorem 2.3]; see also [30]).

A crucial rôle in the proofs in both [13, 30] is played by the *anti-diagonal* of G; it is defined as

$$\Gamma := \{(x, x^{-1}) : x \in G\}.$$

Its indicator function χ_Γ lies $B(G_d \times G_d)$ if and only if G is almost abelian [30, Proposition 3.2]. If G is locally compact such that $A(G)$ is amenable, then χ_Γ lies in $B(G_d \times G_d)$ [30, Lemma 3.1], forcing G to be almost abelian.

For any $f\colon G \to \mathbb{C}$, we define $\check{f}\colon G \to \mathbb{C}$ by letting $\check{f}(x) := f(x^{-1})$. We denote the map assigning $\check{f}$ to f by $^\vee$; it is an isometry on $A(G)$, but completely bounded if and only if G is almost abelian [13, Proposition 1.5]: this fact is crucial for characterizing those G with an amenable Fourier algebra as the almost abelian ones (see both [13, 30]).

Since $^\vee$ need not be completely bounded, it is not obvious that $^\vee$ is an isometry, or even well defined, on $A_{Mcb}(G)$. Nevertheless, both are true.

Lemma 3.2 *Let G be a locally compact group. Then $^\vee$ is an isometry on $M_{cb}A(G)$) leaving $A_{Mcb}(G)$ invariant.*

Proof Since $^\vee$ leaves $A(G)$ invariant, it is clear that it leaves $A_{Mcb}(G)$ invariant once we have established that it is isometric on $M_{cb}A(G)$.

Let $\phi \in M_{cb}A(G)$. By Gilbert's Theorem, there are a Hilbert space $\mathfrak{H}$ and bounded continuous $\xi, \eta\colon G \to \mathfrak{H}$ such that (1.1) holds and $\|\phi\|_{Mcb} = \|\xi\|_\infty \|\eta\|_\infty$. Since

$$\check{\phi}(xy^{-1}) = \phi(yx^{-1}) = \langle \xi(y), \eta(x) \rangle_{\mathfrak{H}} = \overline{\langle \eta(x), \xi(y) \rangle_{\mathfrak{H}}} = \langle \eta(x), \xi(y) \rangle_{\overline{\mathfrak{H}}} \quad (x, y \in G),$$

where $\overline{\mathfrak{H}}$ denotes the complex conjugate Hilbert space of $\mathfrak{H}$, it follows from Gilbert's Theorem that $\check{\phi} \in M_{cb}A(G)$ with $\|\check{\phi}\|_{Mcb} \leq \|\xi\|_\infty \|\eta\|_\infty = \|\phi\|_{Mcb}$. ■

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With Lemma 3.2 at hand, we can prove a $A_{Mcb}(G)$ version of [30, Lemma 3.1].

Proposition 3.3 *Let G be a locally compact group such that $A_{Mcb}(G)$ is C-amenable with $C \geq 1$. Then χ_Γ belongs to $M_{cb}A(G_d \times G_d)$ with $\|\chi_\Gamma\|_{Mcb} \leq C$.*

Proof Let $(d_\alpha)_\alpha$ be an approximate diagonal for $A_{Mcb}(G)$ bounded by C. By Lemma 3.2, the net $((\mathrm{id} \otimes {}^\vee)((d_\alpha))_\alpha$ lies in $A_{Mcb}(G) \otimes^\gamma A_{Mcb}(G)$ and is also bounded by C. Obviously, $((\mathrm{id} \otimes {}^\vee)((d_\alpha))_\alpha$ converges to χ_Γ in the topology of pointwise convergence. Using more or less the same line of reasoning as in the proof of Theorem 2.2, we conclude that $\chi_\Gamma \in M_{cb}A(G_d \times G_d)$ with $\|\chi_\Gamma\|_{Mcb} \leq C$. ■

Remark. Let $A_M(G)$ be the closure of $A(G)$ in $MA(G)$. The question for which G the Banach algebra $A_M(G)$ is amenable seems to be more natural than the corresponding question for $A_{Mcb}(G)$, but is apparently much less tractable (due to the fact that much less is known about $MA(G)$ than about $M_{cb}A(G)$). For instance, we do not know whether or not an analog of Proposition 3.3 holds for $A_M(G)$.

Extending [30, Theorem 3.5], we obtain eventually the following.

Theorem 3.4 *The following are equivalent for a locally compact group G:*

(i) *G is abelian;*
(ii) *$A(G)$ is 1-amenable;*
(iii) *$A_{Mcb}(G)$ is 1-amenable;*
(iv) *$A_{Mcb}(G)$ is C-amenable with $C < \frac{2}{\sqrt{3}}$.*

Proof (i) ⇔ (ii) is [30, Theorem 3.5] and (ii) ⇒ (iii) ⇒ (iv) are trivial.

(iv) ⇒ (i). If $A_{Mcb}(G)$ is C-amenable with $C < \frac{2}{\sqrt{3}}$, then $\chi_\Gamma \in M_{cb}A(G_d \times G_d)$ is an idempotent with $\|\chi_\Gamma\|_{Mcb} \leq C$ by Proposition 3.3. By Corollary 1.2, this means that Γ is a coset a of subgroup of $G \times G$ and thus a subgroup because $(e, e) \in \Gamma$. This is possible only if G is abelian. ■

Remarks. 1. We do not know if the equivalent conditions in Theorem 3.4 are also equivalent to $A_M(G)$ being 1-amenable.

2. In view of [13, Theorem 2.3] and Theorem 3.4, we believe that $A_{Mcb}(G)$ is amenable if and only if G is almost abelian. However, we have no proof in support of this belief. We do not even know whether or not $A_{Mcb}(G)$ is amenable for $G = \mathbb{F}_2$.

3. As a consequence of Theorem 3.4, we have for non-abelian G that

$$\inf\{C : A_{Mcb}(G) \text{ is } C\text{-amenable}\} \geq \frac{2}{\sqrt{3}}$$

(and possibly infinite). This, of course, entails that

$$\inf\{C : A(G) \text{ is } C\text{-amenable}\} \geq \frac{2}{\sqrt{3}},$$

which answers the question raised in the final remark of [30].

We conclude the paper with an observation on amenable closed ideals of $A_{Mcb}(G)$.

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Corollary 3.5 *Let G be a locally compact group, let $C \in [1, \frac{2}{\sqrt{3}})$, and let I be a non-zero, C-amenable, closed ideal of $A_{Mcb}(G)$. Then I is of the form $I(G \setminus xH)$, where $x \in G$ and H is an open, abelian subgroup of G.*

Proof Let I be a non-zero, C-amenable, closed ideal of $A_{Mcb}(G)$. From (3.1), it is immediate that I has an approximate identity bounded by C, and thus is of the form $I(G \setminus xH)$ for some open subgroup H of G. In view of Proposition 2.1 and Theorem 3.4, H has to be abelian. ∎

Remarks. 1. The restriction on C in Corollary 3.5 cannot be dropped: by [23, (13) Bemerkung], there are infinite subsets E of $\mathbb{F}_2$ such that $\chi_E M_{cb} A(G) \cong \ell^\infty(E)$, where $\cong$ stands for a not necessarily isometric isomorphism of Banach algebras. As $A_{Mcb}(G)$ is Tauberian, it is then easy to see that the ideal $I = \chi_E A_{Mcb}(G) = I(G \setminus E)$ is isomorphic to the commutative commutative C^*-algebra $c_0(E)$ and thus an amenable Banach algebra. Clearly, I is not of the form described in Corollary 3.5. (It can be shown that I is 4-amenable and has an approximate identity bounded by 2; see [3].)

2. It is immediate from Corollary 3.5 that $A_{Mcb}(G)$ can have a non-zero, C-amenable, closed ideal if and only if G has an open, abelian subgroup. In particular, for connected G, such ideals exist only if G is abelian.

Addendum After this paper had been submitted we were informed by Ana-Maria Stan that Theorem 1.1 had been obtained independently in

A.-M. Stan, *On idempotents of completely bounded multipliers of the Fourier algebra $A(G)$*. Indiana Univ. Math. J. **58**(2009), no. 2, 523–535.

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