ON THE AUTOMORPHISMS OF INFINITE CHEVALLEY GROUPS

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In (**8**, § 3.2) Steinberg proved the following result.

THEOREM. *Let K be a finite field, G' a simple Chevalley group ("normal type") over K. Then every automorphism of G' is the composite of inner, graph, field, and diagonal automorphisms.*

For the meaning of these notions, see (**8**). Our aim in this note is to indicate how the Theorem may be extended to arbitrary *infinite* fields K, provided we replace G' by the group denoted G in (**5**) and $\hat{G}$ in (**8**). This amounts to proving the Theorem for automorphisms of G' which are induced by automorphisms of G; when K is finite, Steinberg's results show that all automorphisms of G' arise in this way. As Steinberg points out, the sole use made of the finiteness of K in his argument is in the proof of the following statement: Let U be the subgroup of G' corresponding to the set of positive roots, and let σ be any automorphism of G'; then U^σ is conjugate to U in G'. (For finite K, this amounts to an application of one of the Sylow theorems.) *Henceforth, assume that K is infinite.* We shall prove the following lemma, which, together with Steinberg's arguments (modified slightly), yields the above Theorem for G.

LEMMA. *Let U be the subgroup of G corresponding to the set of positive roots, and let σ be any automorphism of G. Then U^σ is conjugate to U in G, and hence (by an easy application of the Bruhat decomposition) in G'.*

Before proving the Lemma we need to recall the relationship between Chevalley's earlier notion of algebraic linear group (**4**) and that of Weil; see (**1**, § 2.4) for details. Let $\Omega \supset K$ be a universal domain. If $H \subset \mathrm{GL}(n, K)$ is an algebraic linear group in the sense of (**4**), then its Zariski closure H^* in $\mathrm{GL}(n, \Omega)$ is an algebraic linear group defined over K, with K-rational points $(H^*)_K = H^* \cap \mathrm{GL}(n, K)$ equal to H. H is connected (solvable, etc.) if and only if H^* has the same property, and the dimension of H coincides with $\dim H^*$ (cf. **4**, *vol.* II, p. 113, *proposition* 5). In particular, let G (with derived group G') be the group over K defined by Chevalley in (**5**), say $G \subset \mathrm{GL}(n, K)$. According to a theorem of Ono (**7**, Theorem 2), G is an algebraic linear group

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*I am indebted to G. B. Seligman and R. Steinberg for their critical comments on an earlier version of the paper. A proof of Steinberg's theorem for G' (K any perfect field) appears in (**9**). An analogous theorem for a much wider class of groups is to appear eventually in (**3**); Borel informs me that he and Tits proved the main result of the present note some time ago (unpublished).

in the sense of (**4**), and $G^* = (G^*)'$ is the simple Chevalley group over Ω of the same type. Moreover, the group U above and the standard diagonal group H are closed in G (**7**, Propositions 1–5); if $B = H \cdot U$, then it follows readily, using the structure theory of (**6**), that $B^* = H^* \cdot U^*$ is a Borel subgroup of G^* with maximal torus H^* and maximal connected unipotent group U^*, all defined over K (in fact, K-split).

Proof of the Lemma. Suppose that S is a solvable subgroup of G including B; then its closure in G is also solvable. Thus, if B is not maximal solvable, it lies in a larger closed solvable subgroup S, and $S^* \supset B^*$ with S^* solvable. Since Borel groups are maximal solvable (**6**, 9–05) we conclude that $S^* = B^*$, $S = (S^*)_K = (B^*)_K = B$. This shows that B, and hence also B^σ, is maximal solvable; in particular, B^σ is *closed* in G. For the remainder of the proof we distinguish two cases.

(a) *K has prime characteristic p.* In this case, U is precisely the set of p-elements in B (elements of order a power of p) since U^* is unipotent and H^* is a torus. Thus, U^σ is precisely the set of p-elements in B^σ. However, the closure of U^σ, which lies in the closed group B^σ, evidently consists of p-elements, and therefore U^σ is already closed.

Consider the standard decomposition $U = \pi X_r$, r ranging over the m positive roots, where $m = \dim U = \dim U^*$ (**8**, § 2). Let w_r and w in $N(H)$ represent the symmetry with respect to r and the element of the Weyl group which interchanges positive and negative roots, respectively. If r is a *simple* root, then $X_r = U \cap w_r w U w^{-1} w_r^{-1}$. This implies that $(X_r)^\sigma$ is the intersection of two closed groups, hence is closed. Since each root is conjugate under the Weyl group to a simple root, we conclude that all $(X_r)^\sigma$ are closed. Evidently, $\dim (X_r)^\sigma \geqq 1$; this implies that $\dim U^\sigma \geqq m$, the product being semidirect.

Now the identity component of $(U^\sigma)^*$ is a connected unipotent group of dimension at least m. Since U^* is a maximal connected unipotent group, and all such groups are conjugate, this forces $\dim U^\sigma = m$. Moreover, the identity component of $(U^\sigma)^*$ is the unipotent radical of a Borel group of G^* (namely, its normalizer in G^*), whence we see that $(U^\sigma)^*$ is already connected.

We next consider the dimension of H^σ. Let $l = \dim H = \dim H^*$ be the rank of G^*. Since H^* is a maximal torus and H is dense in H^*, H must contain a regular semisimple element x of G^*; then $H^* = Z_{G^*}(x)$, whence $H = Z_G(x)$ and $H^\sigma = Z_G(x^\sigma)$. In particular, H^σ is closed. Now choose a large enough power q of p so that $u^q = 1$ for all unipotent matrices u in $\mathrm{GL}(n, \Omega)$. In G^* the criterion for an element of H^* to be regular is that each simple root have value there different from 1; this implies at once that x^q is again regular. Write $x^\sigma = s \cdot u$ as a product of commuting semisimple and unipotent elements in G^* (s and u need not be in G). Then $(x^q)^\sigma = s^q$ and $H = Z_G(x^q)$, $H^\sigma = Z_G(y)$, where $y = (x^q)^\sigma = s^q$ is a semisimple element of H^σ. Now the identity component Z of $Z_{G^*}(y)$ is connected and defined over K (**2**, § 2.15d), and of maximal rank in G^*; thus, Z includes a maximal torus T' of G^* defined over

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K (**2**, § 2.14a). By (**2**, § 2.14c), the group of K-rational points $T = (T')_K$ is dense in T', i.e., $T^* = T'$. However, T lies in $Z_G(y) = H^\sigma$; thus, $\dim H^\sigma \geqq l$.

Since the product $B^\sigma = H^\sigma \cdot U^\sigma$ is semidirect, we conclude that $(B^\sigma)^*$ has dimension at least $l \cdot m = \dim B^*$; it follows easily that $(B^\sigma)^*$ is a Borel group, defined over K. According to (**2**, § 4.13), B^* is conjugate to $(B^\sigma)^*$ by an element of $(G^*)_K = G$. It is clear that U^* is taken to the unipotent radical $(U^\sigma)^*$ of $(B^\sigma)^*$; thus finally, U is conjugate to U^σ in G, as required.

This completes the proof in case (a).

(b) *K has characteristic* 0. As before, let $l = \dim H = \dim H^*$. We will use some standard facts about Cartan subgroups valid for arbitrary fields of characteristic 0 (**4**, *vol.* III, *chapitre* VI; **1**, § 20). In the first place, H^* is a Cartan subgroup of G^* (**6**, 12–09, Theorem 2); therefore, H is a Cartan subgroup of G (**4**, *vol.* III, p. 224, *Proposition* 22). Since the definition of Cartan subgroup is purely group-theoretic, H^σ is also a Cartan subgroup. Now it follows that H^σ is closed, connected, of dimension l; and that $(H^\sigma)^*$ is a maximal torus of G^*. In particular, H^σ consists of semisimple elements.

Since H^σ is connected, it lies in the identity component of B^σ. From the standard structural properties of G, notably the generation of G' by copies of $\mathrm{PSL}(2, K)$ (cf. **5**, p. 47, Lemma 1), we find at once that B has no normal subgroup of finite index including H except itself (thus, B^σ has a similar property and must already be connected), and that U is precisely the derived group of B (thus, U^σ is the derived group of B^σ). Now B^σ is closed, connected, and solvable; therefore, it lies in a Borel group, and its derived group U^σ lies in the unipotent radical of that Borel group. Since $(H^\sigma)^*$ is a torus, it is clear that U^σ contains all the unipotent elements of B^σ: recall that in characteristic 0, G contains the semisimple and unipotent parts of its elements (**4**, *vol.* II, p. 184, *théorème* 18). The closure of U^σ consists of unipotents (**1**, § 6.3), therefore, U^σ must be closed.

As in part (a), we can now argue that each $(X_r)^\sigma$ is closed and then, that $\dim U^\sigma = m$. This makes $(U^\sigma)^*$ a maximal connected unipotent group (every unipotent group is connected in characteristic 0), defined over K. From (**2**, § 8.2) it follows that this group is K-conjugate to U^*, and finally U^σ is conjugate to U in G.

Remark. The automorphism σ was merely assumed to be an automorphism of G *as an abstract group*. Steinberg's remark (**8**, p. 614) that the Lemma is proved for algebraically closed K in (**6**) is therefore misleading, since the "automorphisms" discussed there are always required to be *birational*. However, it is quite easy to give a direct proof of the Lemma when K is algebraically closed, bypassing our complicated arguments above.

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