QUADRICS OVER GF(2) AND THEIR RELEVANCE FOR THE CUBIC SURFACE GROUP

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1. Introduction. About half the following pages are concerned with the "cubic surface group" G, of order 51840, and the geometry cognate to a certain representation thereof. The literature of this group, with its subgroup of order 25920, is already voluminous; the addition of these few pages to it will not, it is hoped, be regarded askance as a perverse and misdirected indulgence in archaism. On p. 104 of **(2)** one reads

> It is thus suggested that we should interpret the symbols of lines as being symbols of points in space of 5 dimensions, every such point being dependent upon 6 points; the points so arising from the lines of a tritangent plane of the cubic surface will then be upon a line. We are thus led to a configuration of 27 points, in this space, lying in threes upon 45 lines, there being 5 of these lines passing through each of these 27 points.

Precisely such a figure is to hand; one has merely to base the geometry on the finite field $\mathbf{F}$, of two marks 0, 1 with $1 + 1 = 0$. There are, in a projective space [5] of 5 dimensions over $\mathbf{F}$, two projectively distinct types of non-singular quadric; one type has planes on it or, as one may say, is ruled, and the other not. It is this latter quadric that underlies the figure so forcibly suggested to Baker, and the geometry associated with it is summarized in Table II below. This geometry cannot be appreciated properly unless the geometry in the subordinate spaces of lower dimension is known in some detail; the earlier pages aim therefore to assemble this essential information.

To lay these foundations, and so adequately to describe the geometry that underlies this aspect of the cubic surface group, is not, however, the sole purpose of these pages. Once the geometry in [5] is known it may help to open the way towards the geometry, over $\mathbf{F}$, in spaces of higher dimension, of quadrics and their groups of automorphisms. There is a quadric in [7] consisting of 135 points and having on it, in accordance with Study's principle of triality **(8**, p. 477**)**, two systems each of 135 solids; its group of automorphisms, of order $2^{13}.3^5.5^2.7$, is isomorphic to a group already investigated in a different setting **(7)**; in a [7], yes, but with the complex field, not $\mathbf{F}$, as base field. Just as Table II below is intimately related to Table I in **(7)** so there is a table allied to Miss Hamill's elaborate Table II. The geometry of quadrics over $\mathbf{F}$ is the avenue towards its compilation and any light that it may shed on the structure of the larger group. The table in question was in fact compiled a year or two ago.

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All the geometry in this paper is based on **F** so that, throughout, a line consists of 3 points, a plane of 7 points lying 3 by 3 on 7 lines, and so on.

2. A symmetric senary quadratic and its automorphisms. A projective space $[n]$, of n dimensions, over **F** consists of $2^{n+1} - 1$ points. Consider, among the 63 points of [5], those whose co-ordinates satisfy

$$2.1 \qquad \sum_{i,j=0}^{5} x_i x_j = 0,$$

the summation being over all pairs i, j with $i < j$, and so embracing 15 products. Such points constitute a quadric $\mathscr{L}$. Each product $x_i x_j$ is 0 or 1, and a point is on or off $\mathscr{L}$ according as an even or an odd number of the 15 products is 1. Now if $x_i x_j = 1$ then $x_i = x_j = 1$; hence when a point has

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6$$

of its 6 co-ordinates non-zero the number of non-zero products is, accordingly,

$$0, \quad 1, \quad 3, \quad 6, \; 10, \; 15.$$

Thus $\mathscr{L}$ consists of

$$\binom{6}{1} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 6 = 27$$

points. Denote any such point by m, and any of the $63 - 27 = 36$ points that are off $\mathscr{L}$ by p. This notation will be used throughout to distinguish between points on and points off a quadric.

The partitioning, as $6 + 15 + 6$, of the 27 m is adventitious in that it depends on the system of co-ordinates; but it throws one cardinal feature into strong relief. The first batch of 6 consists of the vertices X_i of the simplex of reference Σ_0; the last batch of 6 consists of the vertices X_i' of another simplex Σ_0'; Σ_0 and Σ_0' are in perspective from the unit point U (**2**, p. 105) and are in the Möbius relation, each being both inscribed and circumscribed to the other. Furthermore, Σ_0 and Σ_0' are both circumscribed to as well as inscribed in $\mathscr{L}$, the tangent prime of $\mathscr{L}$ at any vertex of either being a bounding prime of the other. All this is manifest from the equations

$$x_i = 0, \qquad x_i = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = \sigma,$$

say, of the bounding primes of Σ_0 and Σ_0', and from the co-ordinates of their vertices; all co-ordinates save one of X_i are 0, all save one of X_i' are 1. That Σ_0 and Σ_0' are related to $\mathscr{L}$ in precisely the same way follows because

$$\sum_{i<j} (\sigma + x_i)(\sigma + x_j) \equiv \sum_{i<j} x_i x_j$$

over **F**; the extra terms on the left amount to $(15 + 5)\sigma^2$ and so, the coefficient being even, disappear.

U is but one of the 36 p. Each p is a centre of perspective for two simplexes,

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inscribed and circumscribed to $\mathscr{L}$ and in the Möbius relation. The correspondence with the 36 double-sixes on a cubic surface is unmistakeable.

3. Subject, now, the variables x_i to the linear substitution

$$x = \mathbf{M}\xi$$

where x and ξ are column vectors, each having six components and $\mathbf{M} = (a_{rs})$ is a 6×6 matrix in which each a_{rs} is an element of the field $\mathbf{F}$. The coefficient of ξ_k^2 on the left of 2.1 is $\Sigma_{i<j}\, a_{ik}a_{jk}$. If the substitution leaves $\mathscr{L}$ invariant each square ξ_k^2 must be absent; hence, foreach k, $\Sigma_{i<j}\, a_{ik}a_{jk} = 0$; in other words, each column of $\mathbf{M}$ is the co-ordinate vector of a point on $\mathscr{L}$. The column thus includes 1, 4, or 5 non-zero co-ordinates. But it is not enough, to ensure invariance, that each square ξ_k^2 be absent; every product $\xi_i\xi_j$ has to be present. The coefficient of, say, $\xi_1\xi_2$ is $\Sigma_{i\neq j} a_{i1}\, a_{j2}$, a sum of 30 products, whose vanishing is the condition for the conjugacy of those 2 points of $\mathscr{L}$ whose co-ordinate vectors are the first 2 columns of $\mathbf{M}$. Thus the columns of $\mathbf{M}$ have not only, each of them, to represent points m; it is also necessary that no two of these 6 m be conjugate, that is, that each of their 15 joins has to be a chord c, never a generator g, of $\mathscr{L}$. These conditions, that the columns of $\mathbf{M}$ are co-ordinate vectors of points on $\mathscr{L}$ no two of which are conjugate, are, when supplemented by the proviso that the columns be not linearly dependent, sufficient as well as necessary for the quadratic form to be invariant under the linear substitution.

It might, in building a matrix $\mathbf{M}$ to satisfy these restrictions, be helpful to observe that they imply that each column *is* conjugate to the sum of any two of the others: the conditions

$$\sum_{i\neq j} a_{il}a_{jm} = 1 = \sum_{i\neq j} a_{il}a_{jn}$$

imply

$$\sum_{i\neq j} a_{il}(a_{jm} + a_{jn}) = 1 + 1 = 0.$$

The equation of $\mathscr{L}$ may take other forms than 2.1; see, for one example, § 17 below. Each such form affords, by its group of automorphisms, a representation of G; the matrices that constitute this representation are subject to appropriate restrictions. Some of these forms may well lead to simpler calculations for the order of G, but the symmetric form 2.1 holds the advantage that the restrictions on $\mathbf{M}$ involve all its columns symmetrically.

4. These discussions apply not merely to senary forms but also to forms in other numbers of variables; they afford an opportunity not only of exhibiting, in this context, certain known facts about the cubic surface group but also of investigating other groups. The indispensable prerequisite is a thorough knowledge of the underlying geometry, and this has to be grounded on the geometry in spaces of lower dimensions. It is advisable to preface the account of this by a section which takes note of the idiosyncrasies due to the base field having characteristic 2, and such a section now follows.

5. The null polarity. A quadratic form

$$f(x) \equiv f(x_0, x_1, \ldots, x_n)$$

is *singular* when it is expressible as a quadratic form in fewer variables x_i', linear forms in the x_i. The quadric $f = 0$ is then a *cone* whose *vertex* V is the $[n - r]$ over which all x_i' simultaneously vanish; r is the rank of the matrix of coefficients in the x_i', and every point of V satisfies $f = 0$. The polar form of any point of V vanishes identically; every point of V is conjugate to every point of $[n]$. The converse is also true *except over fields of characteristic* 2; over such fields a point can be conjugate to every point of $[n]$ without f being singular. If, for example, $n = 2$,

$$f \equiv x_1x_2 + x_2x_0 + x_0x_1$$

is non-singular; yet the point $x_1 = x_2 = x_3$ is conjugate to every point of the plane. This phenomenon can always occur for even n, as is now to be explained.

Recall, first, the genesis of the polar form: it is the coefficient of $\lambda\mu$ in

$$f(\lambda x + \mu y) \equiv f(\lambda x_0 + \mu y_0, \lambda x_1 + \mu y_1, \ldots, \lambda x_n + \mu y_n),$$

and is symmetric in x and y. Since, over $\mathbf{F}$, $\lambda\mu$ has zero coefficient in $(\lambda x_i + \mu y_i)^2$, squares of the variables are ignorable when polarizing; all 2^{n+1} forms

$$a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2 + \sum_{i<j} b_{ij}x_ix_j,$$

with different a_i but fixed b_{ij}, have the same polar $\Sigma b_{ij}(x_iy_j + x_jy_i)$. Whenever $y = x$ every term of this sum vanishes; the polar prime of P *always* passes through P whether P be on $f = 0$ or no; the polarity set up by a quadric is, over $\mathbf{F}$, a null polarity.

6. A point common to the polar primes of the $n + 1$ vertices of the simplex of reference is common to those of all the points of $[n]$, and conversely; such points constitute an $[n - q]$ where q is the rank of

$$\mathbf{B} \equiv \begin{bmatrix} \cdot & b_{01} & b_{02} \ldots b_{0n} \\ b_{01} & \cdot & b_{12} \ldots b_{1n} \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ b_{0n} & b_{1n} & b_{2n} \ldots \cdot \end{bmatrix}$$

there being no common point if $\mathbf{B}$ has its full rank $n + 1$. If n is odd the rank may be full; of this two instances, both convenient to have at hand, may be noted. First: let $b_{ij} = 0$ except that $b_{i,i+1} = b_{i+1,i} = 1$ for each even i. This matrix answers to the canonical form

$$x_0x_1 + x_2x_3 + \ldots + x_{n-1}x_n = 0$$

of a ruled quadric (that is, of a quadric having on it linear spaces of the maximum possible dimension $\frac{1}{2}(n - 1)$), and the non-vanishing of $|\mathbf{B}|$ is obvious. Second: let $b_{ij} = 1$ except for $i = j$. Here too $|\mathbf{B}| = 1$ for odd n, and $\mathbf{B}$ answers to the symmetric form $\Sigma_{i<j}\, x_ix_j$, with $\frac{1}{2}n(n + 1)$ terms in the sum.

7. That $|\mathbf{B}|$ always vanishes when n is even follows because any non-zero term in the determinantal expansion is partnered with another; the expansion of, for example

$$\begin{vmatrix} . & h & g \\ h & . & f \\ g & f & . \end{vmatrix}$$

consists of two equal terms fgh. A set of $n + 1$ units in $\mathbf{B}$ whose product supplies a term of $|\mathbf{B}|$ cannot, $n + 1$ being odd, be placed symmetrically about the leading diagonal; thus any non-vanishing terms of $|\mathbf{B}|$ are paired, those of a pair being products of sets of $n + 1$ units with either set the image of the other in the diagonal. The two products, being equal, have, over $\mathbf{F}$ sum zero. Alternatively: that $|\mathbf{B}| = 0$ for n even is a consequence of $\mathbf{B}$ being skew: over $\mathbf{F}$ a matrix that is symmetric is skew also when its diagonal consists of zeros.

Although the rank q of $\mathbf{B}$ is, when n is even, less than $n + 1$, it need not be less than n, for a principal minor can have rank n. Thus, for even n, there may be a single point conjugate to every point of $[n]$. This point need not satisfy $f = 0$. For take the symmetric form

$$f \equiv \sum_{i<j} x_ix_j$$

whose $\mathbf{B}$, when n is even, has $q = n$. The point conjugate to every point of $[n]$ is the unit point

$$x_0 = x_1 = \ldots = x_n = 1,$$

at which $f = \frac{1}{2}n(n + 1)$; this is zero or not according as $\frac{1}{2}n$ is even or odd.* And if $\frac{1}{2}n$ is even we only (in order to obtain a non-singular quadric with a point conjugate to every point of $[n]$ and yet not lying on the quadric) need to take for $\mathbf{B}$ the direct sum of $\left[\begin{smallmatrix} . & 1 \\ 1 & . \end{smallmatrix}\right]$ and the matrix of the above symmetric form in $n - 2$ variables. If, for example, $n = 4$ this leads to the form

$$yz + zx + xy + uv;$$

*When $\frac{1}{2}n$ is even the symmetric form is singular, being the same as $\sum_{0<i<j}(x_0 + x_i)(x_0+x_j)$ wherein x_0^2 occurs an even number, $\frac{1}{2}n(n - 1)$, of times and x_0x_i an odd number, $n - 1$, of times.

the single point conjugate to every point is $x = y = z = 1$, $u = v = 0$; it does not lie on the quadric.

8. Whenever a point x on $f = 0$ is conjugate to every point of $[n]$ its join to any other point y on $f = 0$ lies entirely on the quadric because

$$f(\lambda x + \mu y) = \lambda^2 f(x) + 0 + \mu^2 f(y) = 0 + 0 + 0 = 0;$$

this implies that x belongs to the vertex. But its join to any point z not on $f = 0$ does not meet the quadric save at x because

$$f(\lambda x + \mu z) = \lambda^2 f(x) + 0 + \mu^2 f(z) = \mu^2 f(z).$$

Note too that if P, P' are any pair of conjugate points then the line PP' has to meet $f = 0$, if at neither P nor P' than at $P + P'$, because

$$f(x + x') = f(x) + 0 + f(x') = f(x) + f(x').$$

Hence if $q < n - 1$, so that there are at least 2 points conjugate to every point of $[n]$, f is necessarily singular.

9. Suppose then, the base field being $\mathbf{F}$, that n is even and that a non-singular quadric Q is given. There is a unique point k conjugate to every point of $[n]$; it does not lie on Q and may be called its *kernel*. Any point m on Q, being conjugate both to itself and to k, is conjugate also to the third point p on km; p is off Q because the join of two conjugate points on Q is entirely on Q, whereas k is not. Conversely: the third point of the join of k to any point p off Q is on Q because the join of two conjugate points has to meet Q. Thus, of the $2^{n+1} - 2$ points of $[n]$ other than k half are m, on Q, and half are p, off Q; Q consists of $2^n - 1$ points one on each line t_0 through k. The m and p on any t_0 have the same polar prime—the tangent prime of Q at m. There are in $[n]$ 2^n primes other than these tangent primes, namely those primes not containing k; these, because of the degeneracy of the null polarity, are not polars of points.

10. Quadrics in [2] and [3]. One now sets out to describe the quadric in the spaces of low dimension over $\mathbf{F}$. The description must cover singular, as well as non-singular, quadrics because singular quadrics occur among the sections of non-singular ones in higher space.

There are 7 binary quadratics over $\mathbf{F}$; of these x^2, $x^2 + y^2$, y^2 are perfect squares, $x^2 + xy$, xy, $xy + y^2$ are reducible and $x^2 + xy + y^2$ is irreducible. The perfect squares are singular; equated to zero they give each a repeated point on the 3-point line L whereon x, y are homogeneous co-ordinates. The other forms are non-singular; the reducible ones give the pairs of points of L, but $x^2 + xy + y^2$ has no zero. These three types of binary quadratic correspond to the three ways in which a line that is not a generator can be related to a quadric Q; there are tangents t, with a single point on Q, chords c, with two points on Q, and lines s that are skew to Q.

A singular quadric in $[n]$ has a vertex $[n - r]$ and projects therefrom a non-singular quadric in $[r - 1]$. Hence, taking $n = 2$, singular conics are of three types; repeated lines ($r = 1$) and either line-pairs or single points ($r = 2$). A non-singular conic consists of 3 points A, B, C, one on each line through its kernel k; of the 7 lines in its plane, 3, namely BC, CA, AB, are c, 3, namely kA, kB, kC, are t, while the remaining line s is skew to the conic. And a plane section of a quadric Q in higher space may be by a plane

e, meeting Q in a single line,
f, meeting Q in two lines,
h, meeting Q in a single point,
j, meeting Q in 3 non-collinear points.

It may also happen, if $n \geqslant 5$, that a plane d lies wholly on Q. But no plane can ever be skew to Q. Since a singular quadric in [3] is the projection of a conic from a point, one can now tabulate the following sections of Q; a section may be by a solid

γ, meeting Q in a single plane,
ϕ, meeting Q in two planes,
χ, meeting Q in a single line,
ψ, meeting Q in 3 concurrent non-coplanar lines.

It may also happen, if $n \geqslant 7$, that a solid ω lies wholly on Q. There are also solids κ meeting Q in ruled quadrics, and solids λ, meeting Q in non-ruled quadrics; these sections are not singular, and will be described in a moment. The letters here used to signify points, lines, planes, solids in their various relations to Q will be retained throughout. The nature of the subspaces in a given space is recognizable at a glance. In, say χ, each of the 3 planes through the g is an e while the other 12 planes are all h; the 16 lines skew to g are all s while the 18 which, 6 at each of its 3 points, meet g are all t. The points in an e other than the 3 m on g are a quadrangle of p; these quadrangles, like a triad of desmic tetrahedra, are such that any two of them are in perspective from every vertex of the third. Geometry in solids other than χ is clear too; there will be occasion to speak of ψ in § 18 below.

11. A non-singular ruled quadric, or hyperboloid, $\mathscr{H}$ in [3] has the canonical equation $xy + zt = 0$; there are on $\mathscr{H}$ two complementary reguli with 3 g in each. Thus $\mathscr{H}$ consists of 9 m; at each m the tangent plane f is spanned by lines one in each regulus; the third line t in f through m does not meet $\mathscr{H}$ again and is, like each g, its own polar. The 6 points of [3] off $\mathscr{H}$ lie **(4**, p. 323**)** 3 on each of two lines s, s'; these are polar lines with respect to $\mathscr{H}$. They may be called the Dandelin lines of $\mathscr{H}$, for Dandelin's classical construction sets out from 3 lines in each of two complementary reguli and is viable over any field. When the field is $\mathbf{F}$ the construction utilizes every point of the space.

12. That the quadric $\mathfrak{p}$, whose equation is

$$xy + xz + xt + yz + yt + zt = 0, \tag{12.1}$$

is non-singular was implied in § 6. The vertices of the tetrahedron of reference T all lie on $\mathfrak{p}$, and each face of T meets $\mathfrak{p}$ in the three vertices of T therein and nowhere else; should any co-ordinate be zero no point other than the vertices of T satisfies 12.1. But there is one point, namely the unit point, none of whose co-ordinates is zero, and this does satisfy 12.1. $\mathfrak{p}$ is, indeed, simply a pentagon: a set of 5 points no 4 of which are coplanar. There would perhaps be no call to treat a pentagon as a quadric if spaces of higher dimensions were not to be encountered; but these pentagons are solid sections of quadrics in higher space and their properties call for interpretation in this context. The pentagon has, for instance, tangent planes; these are

$$y + z + t = 0, \qquad z + t + x = 0, \qquad t + x + y = 0, \qquad x + y + z = 0$$

at the vertices of T, and $x + y + z + t = 0$ at the unit point. They are planes h; whereas the 10 faces of the pentagon are all j. Each j has for its pole the kernel of the conic that is its intersection with $\mathfrak{p}$; this kernel is the point common to j and the join c of the two points of $\mathfrak{p}$ outside j. The 35 lines of [3] are distributed as 10 c (edges of $\mathfrak{p}$), 10 s (1 in each j) and 15 t (3 through each vertex of $\mathfrak{p}$ and lying in the tangent plane h there). Each t is its own polar, but c and s are polars of each other.

13. The non-singular quadric in [4]. A non-singular quadric Q in [4] consists of 15 m, one on each of the 15 lines t_0 through its kernel k. The tangent solid at m contains k, and through k there pass, in this solid, 6 lines other than km. But each such line contains a point of Q which, being conjugate to m, is joined to m by a g; thus each of the three planes which lie in the solid and pass through km contains a g through m, and these g form the section of Q by the solid, a solid which may therefore be designated by ψ. Q is thus the exemplar, over **F**, of the figure (**1**, frontispiece) of 15 points lying 3 by 3 on 15 lines in [4]. The 3 g through an m span ψ, and one idiosyncrasy of the figure consequent upon the base field being **F** is that the 15 ψ are concurrent (at k).

Let m_1 be a given m. There are $15 - 1 - 6 = 8$ m not conjugate to m_1 and whose joins to m_1 are therefore c; hence Q has $\frac{1}{2}(15 \times 8) = 60$ c. The plane ck contains not only m_1, m_2, say on c but also m_3 on the join of k to that point p on c other than m_1, m_2 themselves. Hence there are 20 planes j_0 through k, each meeting Q in 3 m that form a triangle whose sides are all c; such a plane includes a line s skew to Q. As there are in all 35 planes through k they are accounted for by the 20 j_0 and by the 15 e which join k to the 15 g.

The polar of a line μ not passing through k is a plane through k and lies in the polar solids of all points in the plane μk. If μ is g the polar is gk. But if μ is s the polar will be j_0', skew to s because no two points on s are conju-

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gate. Thus the 20 s are paired; the joins j_0, j_0' of a pair s, s' to k are polars of s', s respectively. Every point on s is conjugate to every point on s'; the 9 transversals of s, s' are all t and the solid κ spanned by s and s' meets Q in a hyperboloid with s, s' for its Dandelin lines. These 10 solids $\kappa = [s\, s']$, containing 6 g that fall into complementary reguli of 3 lines each, are the exemplars of what Baker (**1**, p. 115) calls the singular solids. When lines are skew in Baker's frontispiece the corresponding lines on Q are skew; hence the 15 g form 6 quintuples of mutually skew lines, each g belonging to 2 quintuples and being determined as their common line. Each κ answers to a partition of the 6 quintuples into complementary triads and it is, as in the classical case, a consequence of this that the plane common to two solids κ_1, κ_2 is a plane f of a pair of intersecting g. The third solid through f is, of course, the ψ which joins it to k.

Of the solids in [4] 16 do not contain k; those 6 of these that are not κ are λ; each contains 5 m and the triads of g concurrent thereat together account for all 15 g. In the analogous classical figure the corresponding pentagons certainly occur: each of the 15 points corresponds, via the 3 lines concurrent there, to a syntheme (that is, a partitioning into three pairs) of the 6 quintuples; when 5 synthemes constitute a synthematic total (that is, they together account, by the 3 pairs in each, for all 15 pairs of quintuples) the 5 corresponding points form one of the pentagons in question. Since there are 6 synthematic totals there are 6 pentagons; but the vertices of a pentagon need not lie in a solid. That they do so here is another consequence of the base field having characteristic 2.

14. An incidence table for Q is now given; a zero suffix indicates that a space passes through k, whereas its counterpart without the suffix does not. The upper half, above the diagonal, shows the number of subspaces, indicated by a letter on the extreme right, that lie in a given space indicated by a letter at the top. The bottom half, below the diagonal, shows the number of spaces, again indicated by a letter on the right, that contain a given subspace. Either half is deducible from the other: for example, since there are 18 of the 60 c in each of the 10 κ the number of κ through a given c is $18 \times 10/60 = 3$.

15. The whole figure can be described succinctly by using a supernumerary set of 6 homogeneous co-ordinates whose sum is zero. Then the 15 points having two zero co-ordinates constitute Q, and collinearities occur for points such as

$$(0, 0, 1, 1, 1, 1); \qquad (1, 1, 0, 0, 1, 1); \qquad (1, 1, 1, 1, 0, 0).$$

Thus in this representation it is the g, not the m, that correspond to synthemes; the m, not the g, that correspond to pairs. Q is the section of the non-singular quadric $\Sigma_{i<j}\, x_i x_j = 0$ in [5] by the [4]

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0.$$

TABLE I

1	15	15	15	15	45	60	20	15	15	20	60	45	15	10	6	
k	p	m	g	t_0	t	c	s	h	e	j_0	j	f	ψ	κ	λ	
			.	1	.	.	.	.	1	1	.	.	1	.	.	k
			.	1	2	1	3	6	3	3	4	2	7	6	10	p
			3	1	1	2	.	1	3	3	3	5	7	9	5	m
.	.	3						.	1	.	.	2	3	6	.	g
15	1	1						.	3	3	.	.	7	.	.	t_0
.	6	3						3	3	.	3	1	9	9	15	t
.	4	8						.	.	3	3	4	12	18	10	c
.	4	.						4	.	1	1	.	4	2	10	s
.	6	1	.	.	1	.	3						1	.	5	h
15	3	3	1	3	1	.	.						3	.	.	e
20	4	4	.	4	.	1	1						4	.	.	j_0
.	16	12	.	.	4	3	3						4	6	10	j
.	6	15	6	.	1	3	.						3	9	.	f
15	7	7	3	7	3	3	3	1	3	3	1	1				ψ
.	4	6	4	.	2	3	1	.	.	.	1	2				κ
.	4	2	.	.	2	1	3	2	.	.	1	.				λ

Examples of equations for subspaces of this [4] are as follows.

$$\lambda : x_1 = 0.$$
$$\kappa : x_1 + x_2 + x_3 = 0 = x_4 + x_5 + x_0.$$
$$\psi : x_1 + x_2 = 0.$$
$$h : x_1 = x_2 = 0.$$
$$f : x_1 + x_2 = x_3 = x_4.$$
$$j_0 : x_1 = x_2 = x_3.$$
$$j : x_1 + x_2 = x_3 = 0.$$
$$e : x_1 + x_2 = x_3 + x_4 = x_5 + x_0.$$
$$s : x_1 = x_2 = x_3 = 0.$$
$$c : x_1 = 0,\ x_2 = x_3 = x_4.$$
$$t : x_1 = x_2 = x_3 + x_4 = x_5 + x_0.$$
$$t_0 : x_2 = x_3 = x_4 = x_5.$$

The 6 homogeneous co-ordinates related by $\Sigma x_i \equiv 0$ were used, though not over a finite field, by Castelnuovo (**3**, p. 885) in his study of linear complexes in [4]. Such a complex sets up, as does Q, a null polarity which, as with Q, is degenerate (which it cannot but be in a space whose dimension is even); the kernel of Q is analogous to what Castelnuovo calls the centre of the complex. Castelnuovo's statement (**3**, p. 862) that it is from its centre, and its centre only, that the lines of a linear complex in [4] are projected into those of one in [3], is valid in the finite figure. For, since the lines of a screw through a point of [3] are coplanar, any eligible centre of projection has to lie in the solid that contains the lines of the complex at any point of [4]; but the g at a point of Q span ψ, and the only point common to all ψ is k. Thus projection from k onto a [3] which must, as not passing through k, be κ or λ, gives 15 points lying 3 by 3 on 15 lines, the 3 lines through any point being coplanar. These lines constitute a screw (**4**, p. 325). If the projection is onto κ it is at once seen that the 15 lines consist of triads forming complementary reguli and of the tangents to the quadric, one at each of its 9 points; that these constitute a screw is known. (The map of complementary reguli on the Klein quadric in [5] is by conics in a pair of polar planes: when the base field is **F** these planes intersect at the kernel of both conics, and so lie in a [4].) This accords with the known isomorphism (**9**, p. 17), when the base field has characteristic 2, of the orthogonal group on 5 variables with the symplectic group on 4.

16. The non-ruled quadric in [5]. It is now, with the facts assembled in the preceding sections, a straightforward matter to give a conspectus in [5] of the geometry of $\mathscr{L}$, the quadric introduced in § 2. $\mathscr{L}$ consists of 27 m; the tangent prime M at m meets $\mathscr{L}$ in a cone that projects a $\mathfrak{p}$ from a point outside its [3]; instances are the bounding primes of Σ_0 and Σ_0' (cf. the footnote to § 7). The polar prime P of any of the 36 p off $\mathscr{L}$ meets $\mathscr{L}$ in a non-singular quadric having p for kernel; the 15 lines through p in P are all t, and are the only t through p; of the 16 lines through p outside P 6 are (§ 2) c; the remaining 10 must be s. Since the numbers of p on t, c, s are 2, 1, 3 the total number in [5]

$$\begin{aligned} &\text{of } t \text{ is } 36 \times 15/2 = 270,\\ &\text{of } c \text{ is } 36 \times 6 \quad\;\; = 216,\\ &\text{of } s \text{ is } 36 \times 10/3 = 120.\end{aligned}$$

Any other line is a g, and as there are, through any m, 5 g projecting the vertices of a $\mathfrak{p}$ the total number of g on $\mathscr{L}$ is $27 \times 5/3 = 45$.

Each line has a polar solid. The polar of c is λ: for example, each bounding [3] of either Σ_0 or Σ_0' is the polar of an edge of the other. The polar of s is κ: for example, $x_4 = x_5 = x_0$ is the polar of $x_1 = x_2 = x_3 = x_4 + x_5 + x_0 = 0$. As for the polars of t and g, let m_0 be any point of $\mathscr{L}$. All t and all g through m_0 lie in the tangent prime M_0; hence their polar solids all lie in M_0 and contain m_0. They thus project, from m_0, the polar planes j, h with respect to $\mathfrak{p}$ of points p, m in the [3] wherein $\mathfrak{p}$ lies; thus t has a polar solid ψ and g a polar solid χ.

It only remains, before compiling Table II, to discuss the planes in [5] and their relations to $\mathscr{L}$. A plane through g in the polar χ of g meets $\mathscr{L}$ in g only, and so is an e; e is its own polar, every point therein being conjugate to every other. There are 135 e, 3 through each g. Also there are 270 f, spanned by pairs of g. Since f is spanned by a pair of g its polar is the intersection of a pair of χ and so is an h—the only type of plane other than e to lie in χ. The

10 f and 10 h through m project the 10 c and 10 s of a $\mathfrak{p}$. Finally there are j, for instance the plane faces of Σ_0. Since j contains an s its polar lies in a κ and so is also a j—the only type of plane other than f to lie in κ. Now while each κ contains 6 j the κ through a given j is, with the s in the polar j, unique. Hence there are 720 j, consisting of 360 polar pairs. The kernel of the section of $\mathscr{L}$ by j, being conjugate to every point of j, lies in the polar plane; thus the j of a polar pair have in common the kernel of both their sections of $\mathscr{L}$. There are 10 of the 360 polar pairs through each of the 36 p, and they are (§ 13) the pairs of polar j_0 in P.

17. Table II can now be compiled. In order that it be endowed with central symmetry one refers the lower half, registering the numbers of spaces through any given subspace, to the column of symbols on the left and the row of symbols below; the upper half, registering the numbers of subspaces in any given space, is indexed, as was Table I, by means of the row above and the column on the right. This enables one to arrange the symbols on the left and right so that those in either column indicate the polar spaces of those in the other read in *exactly the reverse order*; the same applies to the rows at the top and bottom.

A notable feature of the geometry is the distribution of the 120 s in trios s, s', s'' such that the polar κ of each member of the trio is the solid spanned by the other two. If the vertices of the simplex of reference were chosen 2 on each of s, s', s'' the equation of $\mathscr{L}$ would be, since the quadratic giving the intersections with any of s, s', s'' is irreducible,

$$y_0^2 + y_0y_1 + y_1^2 + y_2^2 + y_2y_3 + y_3^2 + y_4^2 + y_4y_5 + y_5^2 = 0.$$

G is thus representable as the group of automorphisms of this quadratic form, and its order is instantly calculable now it is known that there are 40 trios. For there are on each member of a trio 3 choices for the pair of reference points which, having been chosen, may yet be transposed; also the members of the trio may undergo permutation. Hence G has order

$$40 . (3.2)^3 . 3! = 51840.$$

18. Table II is intimately related to Table I of **(7)**; it is natural that this should happen because the two tables depict closely allied representations of the same group—the cubic surface group of order 51840. Both representations are in projective space of 5 dimensions, but here the base field **F** is finite whereas in **(7)** it is the field of complex numbers. The main feature of contrast, among a multitude of similarities, is, however, that no analogues of the 27 m appear in **(7)**, where the whole figure is based on 36 points p_0; this forestalls any possibility of there appearing in **(7)** the analogues of any spaces whose p are inadequate to span them; that is, of g, whereon no p occurs, of c, whereon there is but a single p, of f, whereon the p are collinear.

TABLE II

	36	27	45	270	216	120	270	135	720	270	120	216	270	45	27	36	
	p	m	g	t	c	s	h	e	j	f	κ	λ	ψ	χ	M	P	
p			.	2	1	3	6	4	4	2	6	10	8	12	20	16	p
m			3	1	2	.	1	3	3	1+4	9	5	1+6	3	1+10	15	m
g	.	5					.	1	.	2	6	.	3	1	5	15	g
t	15	10					3	6	3	1	9	15	4+12	18	10+60	60	t
c	6	16					.	.	3	4	18	10	12	.	40	60	c
s	10	.					4	.	1	.	2	10	4	16	40	20	s
h	45	10	.	3	.	9					.	5	1	12	10+40	15	h
e	15	15	3	3	.	.					6	10	8	.	80	80	j
j	80	80	.	8	10	6					.	.	3	3	15	15	e
f	15	10+40	12	1	5	.					9	.	3	.	10	45	f
κ	20	40	16	4	10	2	.	1	.	4					.	10	κ
λ	60	40	.	12	10	18	4	3	.	.					16	6	λ
ψ	60	10+60	18	4+12	15	9	1	3	6	3					10	15	ψ
χ	15	5	1	3	.	6	2	.	1	.					5	.	χ
M	15	1+10	3	1+6	5	9	1+4	3	3	1	.	2	1	3			M
P	16	20	12	8	10	6	2	4	4	6	3	1	2	.			P
	p	m	g	t	c	s	h	j	e	f	κ	λ	ψ	χ	M	P	

Certain entries are shown as sums of two numbers. This indicates, in the upper right-hand half of the table, that when a space meets $\mathscr{L}$ in a singular quadric subspaces of the same dimension may be related in different ways to the vertex of the section; and dually for the lower left-hand half of the table. For example: of the 16 t in ψ, 4 pass through the concurrence of the 3 g therein, whereas the other 12 do not; and of the 50 h in an M, 10 pass through the contact m, whereas the other 40 do not.

Miss Hamill's nomenclature for the subspaces of the figure in **(7)** is easily translated into that used here, but she builds solids and primes of more varied kinds and in larger numbers, and one must explain why this is. The following short dictionary

$$\begin{array}{cccccc} p_0 & e_1 & \gamma_1 & \gamma_1 \times p_0 & c_2 & \gamma_1 \times \gamma_1 \\ p & t & s & j & h & \kappa \end{array}$$

shows how spaces whose symbols are below are denoted, in **(7)**, by the symbols above: e_1 is defined as a line spanned by two conjugate p_0 whereas t is spanned by two conjugate p, while $\gamma_1 \times \gamma_1$ is the translation of the fact that κ is spanned by its Dandelin lines. But consider, for a moment, a solid ψ. It contains g_1, g_2, g_3, concurrent at m_0, and 8 j, namely those planes in ψ which do not contain m_0, consisting of 4 pairs, a pair together containing all points on g_1, g_2, g_3 other than m_0. The intersection of such a pair of j is an s, and the 4 s are joined to m_0 by the same h—the only plane in ψ through m_0 that does not contain any of g_1, g_2, g_3. Now the kernel of the conic in any of the 8 j is on t, the line in ψ through m_0 that is not in any of the planes g_2g_3, g_3g_1, g_1g_2; thus, m_0 itself being conjugate to each point in ψ, each point of t is conjugate to each point of h. Hence ψ may be spanned by t and any of the s in h, and it is because there are 4 choices for this s that there occur 1080 solids $\gamma_1 \times e_1$, in **(7)**—4 times the number 270 of ψ. Moreover ψ can be spanned by h and either of the p on t, and it is because there are 2 choices for this p that there occur 540 solids $c_2 \times p_0$ in **(7)**. Likewise for primes. There are in P 6 λ, which explains why there are 6 times as many primes $\gamma_3 \times p_0$ in **(7)** as there are P; and there are 20 s falling into 10 pairs, each pair spanning a κ as its Dandelin lines, which explains why there are 10 times as many primes $\gamma_1 \times \gamma_1 \times p_0$ as there are P.

19. The cubic surface group. The order of the group of senary linear transformations that leave $\mathscr{L}$ invariant can be calculated by appealing to the properties (§ 3) that the columns of the corresponding matrices $\mathbf{M}$ must have. One has to choose a hexad of m whose 15 joins are all c. For m_1 there are 27 choices; then 16, the number of c through m_1, for m_2; then 10, the number of j through c, for m_3; m_4 has to lie in a λ through j, and there are 3 such λ, but each contains two further m, vertices of a $\mathfrak{p}$ in λ, that are not conjugate to any of m_1, m_2, m_3. Thus there are 6 choices for m_4. Denote, for the moment, the vertex of $\mathfrak{p}$ that is not chosen by m'; it is not then eligible to furnish another column of $\mathbf{M}$, since there must be no linear dependence between the 6 columns. On each of

$$m_2m_3, \qquad m_3m_1, \qquad m_1m_2, \qquad m_1m_4, \qquad m_2m_4, \qquad m_3m_4$$

there is a p, and this has to be (see the remark towards the end of § 3) conjugate to each of the m yet to be chosen. The six p are coplanar, constituting (§ 12) with m' the tangent plane to $\mathfrak{p}$ at m'; their plane h has for its polar an

f wherein the 2 g meet at m'. The polar c of λ lies in f, and the m thereon are not eligible; the others, one on each g, are, and complete the tally. Hence the number of **M** is

$$27.16.10.6.2 = 51840.$$

It is true that the classical properties of the 27 lines on a cubic surface also afford a means of obtaining this number, and have long done so; the choice for m_4 after m_1, m_2, m_3 are fixed is analogous to the choice, in Schläfli's notation, of say, a_4 or c_{56} towards filling half a double-six once a_1, a_2, a_3 are fixed. But for corresponding "orthogonal" groups in other numbers of variables there is no such precedent.

20. The 6-rowed permutation matrices π constitute, 2.1 being symmetric in all 6 co-ordinates, a subgroup $\mathscr{S}_6$ of G. Each π has certain latent column vectors; the nature of the space σ of whose points these are co-ordinate vectors depends on the cyclic decomposition of the permutation, and so on a partition of 6. For example: the partition $\mathfrak{Z}^2$ corresponds to the cyclic decomposition $(xxx)(xxx)$ and so to a σ with equations of the type

$$x_0 = x_1 = x_2, \qquad x_3 = x_4 = x_5;$$

this is a line s. For a partition $\lambda_1\lambda_2 \ldots$, σ has dimension $5 - \Sigma(\lambda_i - 1)$. No two among the 11 partitions yield the same type of space for σ, so that the π provide representatives of 11 conjugate sets in G.

21. If π is given, one can find the number of matrices conjugate thereto in G by calculating the order of the normalizer. One writes down the most general matrix, with all its elements in **F**, that commutes with π, and then imposes all the conditions on its columns that are necessary for it to belong to G. If N is the number of such matrices the number of operations in G that are conjugate to π is $51840/N$. As an example take

$$\pi = \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix}$$

answering to $\mathfrak{Z}^2$. Any matrix that commutes with π has the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & c_1 & c_2 & c_3 \\ a_3 & a_1 & a_2 & c_3 & c_1 & c_2 \\ a_2 & a_3 & a_1 & c_2 & c_3 & c_1 \\ b_1 & b_2 & b_3 & d_1 & d_2 & d_3 \\ b_3 & b_1 & b_2 & d_3 & d_1 & d_2 \\ b_2 & b_3 & b_1 & d_2 & d_3 & d_1 \end{bmatrix}. \tag{21.1}$$

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If $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$, this is singular. Thus

either $$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1,$$

when it is necessary, for 21.1 to belong to G, that

$$a_2a_3 + a_3a_1 + a_1a_2 + b_2b_3 + b_3b_1 + b_1b_2 = 1,$$

or one of $a_1 + a_2 + a_3$ and $b_1 + b_2 + b_3$ is 0 and the other 1, when

$$a_2a_3 + a_3a_1 + a_1a_2 = b_2b_3 + b_3b_1 + b_1b_2 = 1.$$

It is not possible to satisfy simultaneously the two conditions

$$yz + zx + xy = x + y + z = 0;$$

they represent a conic and the line in its plane that is skew to it. This enables one to tabulate the whole set of solutions of the above conditions. The outcome is, since c, d are subject to precisely the same restrictions as a, b, that the first 3, as the last 3, columns of 21.1 must be one of the blocks

$$\begin{array}{ccc} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{array} \qquad \begin{array}{ccc} . & 1 & 1 \\ 1 & . & 1 \\ 1 & 1 & . \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ . & 1 & 1 \\ 1 & . & 1 \\ 1 & 1 & . \end{array}$$

or else be derived therefrom by cyclic permutation of its 3 columns. The order of the normalizer of π is therefore, since the same block must not occur twice, $4.3.3^2 = 108$, and π belongs to a conjugate set of 480 matrices in G.

The above 12 columns are co-ordinate vectors, of points outside $x_0 + x_1 + x_2 = x_3 + x_4 + x_5 = 0$, no two of which are conjugate.

22. There is another way of finding how many operations compose a conjugate set in G once the space σ constituted by the invariant points is known. One first writes down the most general matrix, with all its elements in $\mathbf{F}$, having all the points of σ among its latent column vectors: this is secured by stipulating the invariance of a set of points that span σ. One next imposes the restrictions on the columns of this matrix to ensure that it belongs to G. One has then to exclude any matrices that leave invariant not merely the points of σ itself but also those of some space of higher dimension that contains σ; this last stage is liable to be the more complicated the lower the dimension of σ. As an example, let σ be a plane h. This does not happen for the permutation matrices, but it illustrates the points at issue equally well and provides another conjugate set. Take, for h,

$$x_0 = x_1 = x_2 + x_3 + x_4 + x_5 = 0.$$

Every point therein is invariant if 3 linearly independent points are; hence one requires a matrix having

$$\begin{matrix} . & . & . \\ . & . & . \\ 1 & . & . \\ 1 & 1 & . \\ . & 1 & 1 \\ . & . & 1 \end{matrix}$$

among its latent column vectors, and so having its last 4 columns of the form

$$\begin{matrix} a & a & a & a \\ b & b & b & b \\ c+1 & c & c & c \\ d & d+1 & d & d \\ e & e & e+1 & e \\ f & f & f & f+1 \end{matrix}$$

Since each, as the co-ordinate vector of a point, must be conjugate to the sum of any two others, $c = d = e = f$; and since each must be the co-ordinate vector of an m,

22.1 $$(a+1)(b+1) = c+1.$$

One now has to prefix 2 columns, each conjugate to the sum of any 2 of the last 4; thus there ensues

$$\begin{bmatrix} A_1 & A_2 & a & a & a & a \\ B_1 & B_2 & b & b & b & b \\ \alpha & \beta & c+1 & c & c & c \\ \alpha & \beta & c & c+1 & c & c \\ \alpha & \beta & c & c & c+1 & c \\ \alpha & \beta & c & c & c & c+1 \end{bmatrix},$$

The conditions on the first two columns are

$$A_1B_2 + A_2B_1 = 1, \qquad A_1B_1 = A_2B_2 = 0.$$

These allow $A_1 = B_2 = 1$, $A_2 = B_1 = 0$; but this is inadmissible since it permits every point of the space $x_0 + x_1 = x_2 + x_3 + x_4 + x_5 = 0$ to be invariant. Hence

$$A_1 = B_2 = 0, \qquad A_2 = B_1 = 1.$$

Next, the first and third columns would be conjugate unless $\alpha = a$, the second and third unless $\beta = b$; moreover α, β must be unequal, for otherwise the ψ $x_0 + x_1 = x_2 + x_3 + x_4 + x_5 = 0$ would have all its points invariant. Since a, b are thus unequal, 22.1 gives $c = 1$, and the matrix becomes

$$\begin{bmatrix} . & 1 & a & a & a & a \\ 1 & . & a+1 & a+1 & a+1 & a+1 \\ a & a+1 & . & 1 & 1 & 1 \\ a & a+1 & 1 & . & 1 & 1 \\ a & a+1 & 1 & 1 & . & 1 \\ a & a+1 & 1 & 1 & 1 & . \end{bmatrix}.$$

Hence there are, corresponding to $a = 0$ and $a = 1$, two matrices in G leaving invariant the points of a given h and no point outside this plane. That they are conjugate in G follows because each is obtainable from the other by simultaneous transpositions of the first two rows and of the first two columns, and so by transformation by a permutation matrix. Each is inverse to the other, and both have for their square an involution for which every point in the ψ $x_0 + x_1 = x_2 + x_3 + x_4 + x_5 = 0$ is invariant. They are representative of a conjugate set of 540 operations of period 4 whose invariant points consist of 6 p and an m, a fact that must be mirrored in other representations of G.

23. The following list shows, for each partition of 6, the type of equations for the space σ of invariant points of the corresponding permutation matrices; it also says what kind of space σ is, gives the Roman numeral that labels the corresponding conjugate set in **(6)** and **(5)**, and the number of operations of G in this set. Several facts that are evident from this list have to agree with known results; the period of an operation can be seen from the partition, as can the nature of the different powers. For instance: since 6 has 3^2 for its square and 2^3 for its cube, the operations labelled XXIII must have their squares labelled IX and their cubes labelled XVII. Moreover, the numbers of m and p that are invariant for a given operation, being those of points in its invariant space, are known and must be the same as numbers already encountered in other representations of G.

TABLE III

1^6		[5]	I	1
$1^4 2$	$x_1 = x_2$	P	XVI	36
$1^3 3$	$x_1 = x_2 = x_3$	κ	VI	240
$1^2 4$	$x_1 = x_2 = x_3 = x_4$	f	XVIII	1620
$1^2 2^2$	$x_1 = x_2,\ x_3 = x_4$	ψ	II	270
123	$x_1 = x_2,\ x_3 = x_4 = x_5$	j	XXI	1440
15	$x_1 = x_2 = x_3 = x_4 = x_5$	c	XV	5184
6	$x_0 = x_1 = x_2 = x_3 = x_4 = x_5$	p	XXIII	4320
24	$x_0 = x_1,\ x_2 = x_3 = x_4 = x_5$	t	V	3240
2^3	$x_0 = x_1,\ x_2 = x_3,\ x_4 = x_5$	e	XVII	540
3^2	$x_0 = x_1 = x_2,\ x_3 = x_4 = x_5$	s	IX	480

24. Not every type of subspace in [5] has appeared in the role of σ. Some cannot. Were λ invariant so would its polar c be, and on c lies a single p which would be invariant too. Thus the hypothesis that every point of λ is invariant, implying the invariance of a p outside λ, implies the invariance of every point of the join P of λ and p. Nor, as is easily seen by taking a face of Σ_0, can every point of an M be invariant. It has, however, been seen that σ might be h; this gives the set XIX. Furthermore, every point of

$$\chi : x_0 + x_1 = x_2 + x_3 = x_4 + x_5$$

is invariant for the involution

$$\begin{bmatrix} 1 & . & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & . & 1 & 1 \\ 1 & 1 & . & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & . \\ 1 & 1 & 1 & 1 & . & 1 \end{bmatrix}$$

and for no other operation of G save I. This involution is one of 45 in the set III.

25. The 36 involutions J associated one with each p are conspicuous as being the only operations in G whose invariant points fill a [4]. Suppose that p_1, p_2 are such that $J_1J_2 = J_2J_1$. If m is outside both P_1 and P_2, and if m_1 and m_2, the remaining points on mp_1 and mp_2 respectively, are those points into which m is transformed by J_1 and J_2, commutation demands that the intersection m_{12} of m_1p_2 and m_2p_1 is on $\mathscr{L}$. The quadrangle $mm_1m_2m_{12}$ has p_1, p_2 for diagonal points; since **F** has characteristic 2 the third diagonal point is the remaining point of p_1p_2. Now any plane that includes 4 m has to be an f and include 5; hence p_1p_2 is a t. Conversely, let p_1p_2 be a t, ψ its polar solid; as p_1, p_2 are conjugate P_1 and P_2 both contain ψ. Every point in ψ is invariant for both J_1 and J_2; the 16 points of P_1 outside ψ, collinear in pairs with p_2, are transposed in pairs by J_2; the 16 points of P_2 outside ψ are transposed in pairs by J_1; hence the successive application of J_1, J_2 has the same effect, in either order, on any point in either P_1 or P_2. Any point of [5] that is outside both P_1 and P_2 is on lines that meet them both, and is the only point on any such line, it follows that it, too, is changed into the same point by successive application of J_1, J_2, in either order. Thus $J_1J_2 = J_2J_1$.

This discussion has shown that J_1, J_2 do or do not commute according as p_1p_2 is a t or an s. Thus non-commuting J occur (**6**, p. 86) in associated sets of 3. The product of any number of mutually commutative J is itself an involution; each factor is associated with a p and every pair among these p has its join a t. In particular: the 4 points in an e not on the g therein have this property; the product of the 4 corresponding J is the involution whose invariant points compose the polar χ of g. As there are 3 e through g in χ there are **(6**, p. 88**)** three ways of expressing this same involution as a product of 4 commutative J.

The conjugate sets in Table III are identified by representatives that impose certain permutations on the vertices of Σ_0. The mutually disjoint cycles that compose the permutation correspond, when appropriately expressed as products of transpositions, to those r-chains used by Frame that have $r \leqslant 6$. This is why the same 11 partition labels have appeared among the 18 at the bottom of p. 91 of **(6)**.

26. G is a subgroup, of index 28, of the "group of the bitangents," and this larger group happens too to have a representation in 6 variables over **F**, being isomorphic to the symplectic group of matrices μ for which $\mu'\mathbf{S}\mu = \mathbf{S}$, when **S** is a non-singular skew matrix. A referee has proposed that this occurrence as a subgroup of the symplectic group be shown in the present context, and so this § 26 is appended.

Any projectivity that leaves $\mathscr{L}$ invariant turns conjugate points into conjugate points and non-conjugate points into non-conjugate points; hence, with $\mathscr{L}$ in the symmetric form of § 2, the bilinear form $\sum_{i \neq j} y_i z_j$ must be unaltered under each of the 51840 projectivities. But this bilinear form is $y'\mathbf{B}z$, z being the column vector whose 6 components are the z_j, y' the row vector with components y_i, and **B** the non-singular 6-rowed matrix mentioned at the end of § 6. But **B**, symmetric and with a zero diagonal, is skew over **F**, so that all 51840 **M** belong to the symplectic group Γ on **B**.

The condition that **M** belong to Γ is less stringent than the condition that **M** belong to G; for the symplectic property is not that a quadric but that a polarity is left invariant and, as was noted in § 5, the same polarity arises from 64 quadrics. The wider latitude of the symplectic condition permits these quadrics to undergo permutation. Yet a ruled quadric can be permuted only with other ruled quadrics, and likewise for quadrics, such as $\mathscr{L}$, that are not ruled. One presumes, knowing the index of G in Γ, that 28 of the 64 quadrics are not ruled while the remaining 36 are ruled.

All 64 quadrics, inducing as they do a non-degenerate polarity, are non-singular; in order that one be ruled it is enough for it to contain a single plane. If A is a point on the quadric, any plane thereon through A has to lie in the null prime of A. When $\mathscr{L}$ has the form 2.1 and **B** corresponds thereto the null prime of (1, 1, 1, 1, 1, 1) is

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

and the conditions for

26.1 $$\sum_{i=0}^{5} a_i x_i^2 + \sum_{i<j} x_i x_j = 0$$

to contain the plane $x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = 0$ are

$$a_0 + a_1 = a_2 + a_3 = a_4 + a_5 = 1.$$

Thus 3 squares must be absent from 26.1, 3 present. Again: the null prime of X_0 (see § 2) is

$$x_0 = x_0 + x_1 + x_2 + x_3 + x_4 + x_5,$$

and the conditions for 26.1 to contain the plane $x_1 + x_2 = x_3 + x_4 = x_5 = 0$ are

$$a_0 = 0, \qquad a_1 + a_2 = a_3 + a_4 = 1.$$

The mark a_5 can be either 0 or 1; taking $a_5 = 0$ shows 26.1 to be ruled when 2 squares are present, 4 absent. So $20 + 15 = 35$ ruled quadrics are accounted for.

The ruling would, of course, be patent if 26.1 could be thrown into Plücker's form. For example: if all 6 squares are present 26.1 is

$$(x_0 + x_1 + x_4)(x_0 + x_1 + x_5) + (x_2 + x_3 + x_0)(x_2 + x_3 + x_1) \\ + (x_4 + x_5 + x_2)(x_4 + x_5 + x_3) = 0.$$

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