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RESEARCH ARTICLE

Extreme cases for Frostman's theorem

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Abstract
We construct measures whose Fourier coefficients exhibit extreme behaviour among those whose support have Hausdorff h-measure zero.

1. Introduction

We work on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and write $|I|$ for the length of an interval I. We shall only look a closed subsets E of $\mathbb{T}$. All measures will be bounded positive Borel measures, and we write $\mathcal{P}(E)$ for the collection of probability measures with support contained in E. We recall the definition of the Hausdorff h-measure.

Definition 1.1. If $h : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function with $h(0) = 0$, we say that a closed set E has Hausdorff-h measure

$$h(E) = \lim_{\delta\to 0+} \inf\left\{\sum_{j=1}^{n} h(|I_j|) \ : \ \bigcup_{j=1}^{n} I_j \supseteq E,\ I_j \text{ an interval}, |I_j| \leq \delta\right\}.$$

If $1 \geq \alpha > 0$, we write $h_\alpha(t) = t^\alpha$.

It is not difficult to see that, given E, there exists a number d (with $1 \geq d \geq 0$) called the Hausdorff dimension $\dim_H(E)$ of E such that $h_\alpha(E) = \infty$ for $d > \alpha$ and $h_\alpha(E) = 0$ for $\alpha > d$. It is often hard to compute the Hausdorff dimension of a given set, and an important tool is provided by Frostman's theorem [2], which asserts that

$$\dim_H(E) = \sup\left\{\alpha \ : \ \exists\mu \in \mathcal{P}(E) \text{ with } \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} < \infty\right\}.$$

Using Parseval's theorem (for details, see [6] Section 3.5), we can express the result in Fourier analytic form as

$$\dim_H(E) = \sup\left\{\alpha \ : \ \exists\mu \in \mathcal{P}(E) \text{ with } \sum_{r\neq 0} \frac{|\hat{\mu}(r)|^2}{|r|^{1-\alpha}} < \infty\right\}.$$

Some further, more precise, information along these lines is obtained as Theorem V of Chapter III of the book of Kahane and Salem [3]. (The authors acknowledge inspiration from a talk by Beurling [1]. See also [8].)

Theorem 1.2 (Kahane and Salem). *Suppose that* $1 \geq \alpha > 0$. *Then, if* μ *is a probability measure with*

$$\sum_{r \neq 0} \frac{|\hat{\mu}(r)|^2}{|r|^{1-\alpha}} < \infty,$$

the support of μ *has strictly positive* h_α*-measure.*

In the first part of this paper, we show that Theorem 1.2 is sharp in the following sense.

Theorem 1.3. *Suppose that* $1 > \alpha > 0$, $\gamma_r > 0$ *and* $\gamma_r \to 0$ *as* $r \to \infty$. *Then we can find a probability measure* μ *with support of zero* h_α*-measure, but*

$$\sum_{r \neq 0} \gamma_{|r|} \frac{|\hat{\mu}(r)|^2}{|r|^{1-\alpha}} < \infty.$$

We actually go a little further. The arguments of [3] go through essentially unchanged to give the following version of Theorem 1.2.

Theorem 1.4. *Suppose that* $h : [0, 1] \to [0, \infty)$ *is an increasing concave function with* $h(0) = 0$. *Suppose, in addition, that there exists an* α *with* $1 \geq \alpha > 0$ *such that* $h(t)t^{-\alpha}$ *increases with t.*

Then, if μ *is a probability measure with*

$$\sum_{r \neq 0} \frac{|\hat{\mu}(r)|^2}{|r| h(|r^{-1}|)} < \infty,$$

the support of μ *has strictly positive h-measure.*

We prove the following.

Theorem 1.5. *Suppose that* $h : [0, 1] \to [0, \infty)$ *is a continuously differentiable increasing function with* $h(0) = 0$ *and* $t^{-1}h(t)$ *increasing to* ∞ *as* $t \to 0+$. *Suppose further that there exists an* α *with* $1 > \alpha > 0$ *such that* $h(t)t^{-\alpha}$ *increases with t. Then, if* $\gamma_r > 0$ *and* $\gamma_r \to 0$ *as* $r \to \infty$, *we can find a probability measure* μ *with support of Hausdorff h-measure zero with*

$$\sum_{r \neq 0} \gamma_r \frac{|\hat{\mu}(r)|^2}{|r| h(|r|^{-1})} < \infty.$$

Notice that, for example, if $h(t) = t^\beta \big(\log(1/t)\big)^\gamma$ for t small with $1 > \beta > 0$ and γ real, or with $\beta = 1$ and $0 > \gamma$, then h satisfies the conditions of Theorem 1.5. Readers will lose little if they take $h(t) = t^\alpha$ throughout. On the other hand, the additional work involved to obtain the more general case is not great and the proof of Lemma 2.3 appears as something more than a numerical coincidence.

By choosing $\tilde{h}$ in such a way that

$$\tilde{h}(t)/h(t) \to 0, \text{ but } \gamma_n h(1/n)/\tilde{h}(1/n) \to 0$$

and then applying Theorem 1.5 to $\tilde{h}$, we may replace the condition $t^{-1}h(t)$ increasing to ∞ as $t \to 0+$ in Theorem 1.5 by the condition $t^{-1}h(t)$ nondecreasing as as $t \to 0+$. This allows us to take $h(t) = t$ and recover a result of Salem (see [9] and Theorem VI of Chapter 3 in [3]).

The individual Fourier coefficients of the particular measure μ that we construct to prove Theorem 1.5 are not all small. In particular, we have $\limsup_{|n| \to \infty} |\hat{\mu}(n)| = 1$ (see the remark at the end of Sub-section 2.3). However, this is not an inherent feature, and by modifying our construction, we can obtain a new proof of a result obtained in [5] concerning individual coefficients.

Theorem 1.6. *Suppose that* $h : [0, 1] \to [0, \infty)$ *is an increasing function with* $h(0) = 0$ *and* $t^{-1}h(t)$ *increasing to* ∞ *as* $t \to 0+$. *Suppose further that there exists an* α *with* $1 > \alpha > 0$ *such that* $h(t)t^{-\alpha}$

increases with t. Let $L : [1, \infty) \to [1, \infty)$ be a continuous increasing function such that $x^{-1}L(x)$ decreases with x and there exists a γ with $\gamma > \alpha$, a K with $L(x^{3/\gamma}) \leq KL(x)$ for all $x \in [1, \infty)$ and, further,

$$\int_1^\infty \frac{1}{xL(x)\log x}\,dx = \infty.$$

Then there exists a probability measure μ with support of Hausdorff h-measure zero with

$$|\hat{\mu}(r)|^2 \leq \frac{1}{h(1/r)L(r)}$$

for all $r \neq 1$.

Notice that if $L(x) = \log\log x$ for x large and $h(t) = h_\alpha$, this gives a probability measure μ with support of Hausdorff-α measure zero, but

$$|\hat{\mu}(r)|^2 \leq \frac{|r|^{-\alpha}}{\log\log|r|}$$

for $|r|$ sufficiently large that the formula makes sense.

The reader will see from the proofs that we could prove a portmanteau theorem to the effect that if the conditions of Theorems 1.5 and 1.6 apply, then we can find a measure satisfying the conclusions of both. However, I think that although the two proofs have many points in common, the ideas become clearer if we look at them separately. (The reader will also see, without surprise, that if the conditions of Theorems 1.5 and 1.6 apply, we can find a measure μ satisfying the conclusion of Theorems 1.6 such that there exists a sequence $\gamma_r > 0$ with $\gamma_r \to 0$ as $r \to \infty$, but $\sum_{r\neq 0} \gamma_r \frac{|\hat{\mu}(r)|^2}{|r|h(|r|^{-1})} = \infty$.)

The condition that $t^{-\alpha}h(t)$ increases for some α with $\alpha > 0$ (or something very close to that condition) is essential for our proofs (and seems to be needed for proving Theorem 1.4). Section 3, where we see that the corresponding results for Hausdorff logarithmic measure take a different form, shows that the restriction is not artificial. The following example from [4] confirms this.

Theorem 1.7. *We can find a decreasing positive convex sequence c_n with $c_n \to 0$ as $n \to \infty$ such that $\sum_{n=1}^\infty c_n^2$ diverges, but if μ is a nonzero measure with $|\hat{\mu}(n)| \leq c_n$, then* $\operatorname{supp}\mu = \mathbb{T}$.

In my opinion, the central idea of the paper is contained in Lemma 2.3, which states that there exists a well-behaved function g_M whose behaviour echos the behaviour that we desire from the measure μ in Theorem 1.5 and is such that

$$\hat{g}_M(r) = 0 \text{ for all } 1 \leq |r| \leq M-1,$$

where we can choose M as large as we want. We now choose an extremely rapidly increasing sequence of $M(j)$ and seek to show (see Lemma 2.8) that $\prod_{j=1}^{n+1} g_{M(j)}$ preserves the desirable properties of $\prod_{j=1}^{n} g_{M(j)}$. Standard limiting arguments complete the proof of Theorem 1.5. However, our proof of Lemma 2.8 depends on an estimate described in Definition 2.4 and Lemma 2.5, so we need to obtain this first.

A search for information on problems related to those discussed in this paper could start with [6].

2. Proof of Theorem 1.5

2.1. The building blocks

Several of our formulae take a slightly simpler form if we work with $k(x) = 1/h(1/x)$ and make use of the following remark.

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Lemma 2.1. *Suppose that* $h : [0,1] \to [0,\infty)$ *is a continuously differentiable increasing function with* $h(0) = 0$ *and* $k(x) = \dfrac{1}{h(1/x)}$.

(A) If $1 \geq \alpha > 0$, *the following statements are equivalent:*

(i) $h(t)t^{-\alpha}$ *increases with* t.
(ii) $k(x)x^{-\alpha}$ *increases with* x.
(iii) $xk'(x) \geq \alpha k(x)$ *for all* x.

(B) The following statements are equivalent:

(i) $h(t)t^{-1}$ *increases to* ∞ *as* $t \to 0+$.
(ii) $k(x)x^{-1}$ *increases to* ∞ *as* $x \to \infty$.

Proof. Immediate. □

(In the typical case $h(t) = t^\alpha$, then $k(x) = x^\alpha$, but we are interested in what happens when t is small and when x is large.) Notice that the final formula of Theorem 1.5 can be rewritten as

$$\sum_{r\neq 0} \gamma_r \frac{k(|r|)}{|r|} |\hat{\mu}(r)|^2 < \infty,$$

and we shall use that form in proving the theorem.

Many of the constants we introduce will depend on α, but we shall usually suppress the reference to α and write $C = C(\alpha)$. We shall use the convention that constants with suffices like C_j have only local importance.

We fix some positive function $u \in C^\infty(\mathbb{R})$ with $\int_{\mathbb{T}} u(t)\, dt = 1$ and $\operatorname{supp} u \subseteq [-1/4, 1/4]$. The following lemma is standard.

Lemma 2.2. *If* $R \geq 1$ *and we define* $u_R : \mathbb{T} \to \mathbb{R}$ *by*

$$u_R(t) = R^{-1} u(Rt) \text{ for } |t| \leq 1/2,$$

then u_R *is a positive* $C^\infty(\mathbb{T})$ *function with the following properties:*

(i) $|\hat{u}_R(j)| \leq \hat{u}_R(0) = 1$ *for all* j.
(ii) *There are constants* β_q *with* $|\hat{u}_R(r)| \leq \beta_q R^q |r|^{-q}$ *for all* $r \neq 0$.
(iii) $\operatorname{supp} \mu \subseteq [-R^{-1}/4, R^{-1}/4]$.

Proof. Use change of variables and integration by parts. □

Here is our basic building block.

Lemma 2.3. *Let* $1 > \alpha > 0$. *Then if the conditions of Theorem 1.5 hold, there exists an* $A = A(\alpha)$ *for which the following is true. There exists an* M_0 *(depending on* h*) such that, if* $M \geq M_0$, *there exists a positive function* $g \in C^\infty(\mathbb{T})$ *with the following properties:*

(i) $\hat{g}(0) = 1$.
(ii) $\hat{g}(r) = 0$ *if* r *is not divisible by* M.
(iii) $\displaystyle\sum_{r\neq 0} \frac{k(|r|)}{|r|} |\hat{g}(r)|^2 \leq A$.
(iv) *There is a finite collection of intervals* $\mathcal{I}$ *such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} g, \text{ but } \sum_{I\in\mathcal{I}} h(|I|) \leq 1.$$

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Proof. Chose M_0 such that $k(r) \le r$ for all $r \ge M_0$. If $M \ge M_0$, take $R = k^{-1}(M)$ and

$$g(t) = u_R * \frac{\delta_0 + \delta_{1/M} + \delta_{2/M} + \ldots + \delta_{(M-1)/M}}{M}$$

with u_R as in Lemma 2.2, $*$ denoting convolution and δ_t the Dirac delta measure at t. We observe that

$$\hat{g}(r) = \begin{cases} \hat{u}_R(jM) & \text{if } r = jM \text{ for some } j \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Conditions (i) and (ii) are immediate. Condition (iv) follows on choosing $\mathcal{I}$ to be the collection of intervals

$$[jM^{-1} - R^{-1}/2, jM^{-1} + R^{-1}/2] \text{ with } 0 \le j \le M-1.$$

We now look at condition (iii). We have

$$\begin{aligned} \sum_{r\neq 0} \frac{k(|r|)}{|r|}|\hat{g}(r)|^2 &= \sum_{j\neq 0} \frac{k(|jM|)}{|jM|}|\hat{g}(jM)|^2 \\ &= \sum_{1\le |j|<R/M} \frac{k(|jM|)}{|jM|}|\hat{u}_R(jM)|^2 + \sum_{|j|\ge R/M} \frac{k(|jM|)}{|jM|}|\hat{u}_R(jM)|^2. \end{aligned}$$

We bound the two sums separately. Since $s^{-1}k(s)$ decreases as s increases, we have

$$\frac{1}{M}\int_{(j-1)M}^{jM-1} \frac{k(s)}{s}\, ds \ge \frac{k(jM)}{jM}$$

for $j \ge 2$. Thus, since $|\hat{u}_R(r)| \le 1$ and $k(M) \le M$,

$$\begin{aligned} \sum_{1\le |j|<R/M} \frac{k(|jM|)}{|jM|}|\hat{u}_R(jM)|^2 &\le 2\sum_{1\le j<R/M} \frac{k(jM)}{jM} \\ &\le 2 + \frac{2}{M}\int_M^R \frac{k(s)}{s}\, ds \le 2 + \frac{2}{M}\int_M^R \alpha^{-1}k'(s)\, ds \end{aligned}$$

from Lemma 2.1 (iii). But

$$\int_M^R k'(s)\, ds = k(R) - k(M) \le k(R) = M,$$

so

$$\sum_{1\le |j|<R/M} \frac{k(|jM|)}{|jM|}|\hat{u}_R(jM)|^2 \le 2 + 2\alpha^{-1}.$$

Since $s^{-1}k(s)$ decreases and $k(R) = M$, Lemma 2.2 (ii) yields

$$\begin{aligned} \sum_{|j|\ge R/M} \frac{k(|jM|)}{|jM|}|\hat{u}_R(jM)|^2 &\le \sum_{|j|\ge R/M} \frac{k(R)}{R}|\hat{u}_R(jM)|^2 \\ &\le 2\sum_{j\ge R/M} \frac{M}{R}\beta_1^2 \frac{R^2}{(jM)^2} = 2\beta_1^2 M^{-1}R \sum_{j\ge R/M} \frac{1}{j^2} = A_2 \end{aligned}$$

for an appropriate constant A_2. Condition (iv) follows. ◻

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Observe that $|\exp(iMt) - 1| \leq M/(2R) = M/\big(2k^{-1}(M)\big)$ must be small when M is large and $t \in \operatorname{supp} g$.

2.2. An intermediate step

Before we complete the proof, we need to extract a little more information about the function g of Lemma 2.3. We make the following ad hoc definition.

Definition 2.4. If $f \in C(\mathbb{T})$, we write

$$\tilde{f}(r) = |\hat{f}(r)| + \sum_{s \neq 0} \frac{|\hat{f}(r-s)|}{s^2}.$$

Note that $|\tilde{f}(r)| \leq 3\|f\|_\infty$. The object of this section is to prove the following estimate.

Lemma 2.5. *Let $\alpha > 0$. Then there exists a constant K such that the function g of Lemma 2.3 satisfies the condition*

$$\sum_{r \neq 0} \frac{k(|r|)}{|r|} |\tilde{g}(r)|^2 \leq K.$$

We use a preliminary lemma.

Lemma 2.6. *If $b_j \in \mathbb{C}$ and $n, m \geq 1$, then*

$$\sum_{|j| \leq n} \left(|b_j| + \sum_{1 \leq |s| \leq m} \frac{|b_{j-s}|}{s^2} \right)^2 \leq 2^4 \sum_{|j| \leq n+m} |b_j|^2.$$

Proof. Observe that if $0 \leq \theta \leq 1$,

$$(|a| + \theta|b| + \theta|c|)^2 \leq |a|^2 + 3\theta(|a|^2 + |b|^2 + |c|^2).$$

Thus induction on m gives

$$\sum_{|j| \leq n} \left(|b_j| + \sum_{1 \leq |s| \leq m} \frac{|b_{j-s}|}{s^2} \right)^2 \leq \prod_{s=1}^{m} \left(1 + \frac{9}{s^2}\right) \sum_{|j| \leq n+m} |b_j|^2.$$

Since

$$B = \prod_{s=1}^{\infty} \left(1 + \frac{9}{s^2}\right) = \frac{3\pi^2}{2} \leq \frac{2^4}{9},$$

we obtain the required result. □

Proof of Lemma 2.5. We start by bounding $\sum_{2^n \leq r < 2^{n+1}} \tilde{g}(r)^2$. Set

$$\tilde{g}_1(r) = |\hat{g}(r)| + \sum_{1 \leq s \leq 2^{n-1}} \frac{|\hat{g}(r-s)|}{s^2} \text{ and } \tilde{g}_2(r) = \sum_{2^{n-1} < s} \frac{|\hat{g}(r-s)|}{s^2}.$$

By Lemma 2.6,

$$\sum_{2^n \leq r < 2^{n+1}} \tilde{g}_1(r)^2 \leq B \sum_{2^{n-1} \leq r < 2^{n+1} + 2^{n-1}} |\hat{g}(r)|^2.$$

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We know that $|\hat{g}(r-s)| \leq 1$, so

$$\tilde{g}_2(r) \leq \sum_{s>2^{n-1}} \frac{1}{s^2} \leq 2^{-n+2}$$

and thus

$$\sum_{2^n \leq r < 2^{n+1}} \tilde{g}_2(r)^2 \leq 2^{-n+4}.$$

Since $\tilde{g} = \tilde{g}_1 + \tilde{g}_2$, it follows that

$$\begin{aligned}
\sum_{2^n \leq r < 2^{n+1}} \tilde{g}(r)^2 &= \sum_{2^n \leq r < 2^{n+1}} (\tilde{g}_1(r) + \tilde{g}_2(r))^2 \\
&\leq 4 \sum_{2^n \leq r < 2^{n+1}} (\tilde{g}_1(r)^2 + \tilde{g}_2(r)^2) \\
&\leq 2^6 \sum_{2^{n-1} \leq r < 2^{n+1}+2^{n-1}} |\hat{g}(r)|^2 + 2^{-n+6}.
\end{aligned}$$

By considering sums over $-2^{n+1} < r \leq -2^n$ in the same manner, we obtain

$$\sum_{2^n \leq |r| < 2^{n+1}} \tilde{g}(r)^2 \leq 2^7 \sum_{2^{n-1} \leq |r| < 2^{n+2}} |\hat{g}(r)|^2 + 2^{-n+7}.$$

Next we observe that there are constants K_1 and K_2 such that

$$K_1 \frac{k(2^n)}{2^n} \geq \frac{k(|r|)}{|r|} \geq K_2 \frac{k(2^n)}{2^n}$$

for all $2^{n-1} \leq |r| < 2^{n+1}$; so, taking $B_1 = 2^7 K_1/K_2$, we have

$$\sum_{2^n \leq |r| < 2^{n+1}} \frac{k(|r|)}{|r|} \tilde{g}(r)^2 \leq B_1 \sum_{2^{n-1} \leq |r| < 2^{n+2}} \frac{k(|r|)}{|r|} |\hat{g}(r)|^2 + 2^{-n+7}.$$

Thus

$$\begin{aligned}
\sum_{r \neq 0} \frac{k(|r|)}{|r|} \tilde{g}(r)^2 &\leq B_2 + \sum_{|r| \geq 4} \frac{k(|r|)}{|r|} \tilde{g}(r)^2 \\
&= B_2 + \sum_{n=2}^{\infty} \sum_{2^n \leq |r| < 2^{n+1}} \frac{k(|r|)}{|r|} \tilde{g}(r)^2 \\
&\leq B_2 + B_1 \sum_{n=2}^{\infty} \sum_{2^{n-1} \leq |r| < 2^{n+2}} \frac{k(|r|)}{|r|} |\hat{g}(r)|^2 \\
&\leq B_2 + B_1 \sum_{r \neq 0} \frac{k(|r|)}{|r|} |\hat{g}(r)|^2 \leq K
\end{aligned}$$

for appropriate choices of constants B_2 and K. □

We can now draw an obvious conclusion.

Lemma 2.7. *Let $\alpha > 0$. Then if the conditions of Theorem 1.5 hold and $1 > \delta > 0$, there exists a K for which the following is true. There exists a $M_0(\delta)$ (depending on h) such that, if $M \geq M_0(\delta)$, there exists a positive function $g \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{g}(0) = 1$.
(ii) $\hat{g}(r) = 0$ *if r is not divisible by M.*
(iii) $\displaystyle\sum_{r\neq 0} \frac{k(|r|)}{|r|}|\tilde{g}(r)|^2 \leq K\delta^{-1}$.
(iv) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} g, \ \textit{but} \ \sum_{I\in\mathcal{I}} h(|I|) \leq \delta.$$

Proof. Replace h by $\tilde{h} = \delta^{-1}h$ in Lemma 2.5. □

2.3. Completion of the proof of Theorem 1.5

The rest of the proof of Theorem 1.5 is essentially contained in the next lemma.

Lemma 2.8. *Suppose that $\alpha > 0$, $\gamma_r > 0$ and $\gamma_r \to 0$ as $r \to \infty$. Then if $f \in C^\infty(\mathbb{T})$ is a positive function with $\hat{f}(0) = 1$ and $\epsilon, \eta > 0$ and $Q \geq 1$ are given, we can find a positive function $F \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{F}(0) = 1$.
(ii) $|\hat{F}(r) - \hat{f}(r)| \leq \epsilon$ *for all* $|r| \leq Q$.
(iii) $\displaystyle\sum_{r\neq 0} \gamma_r \frac{k(|r|)}{r}|\hat{F}(r)|^2 \leq \epsilon + \sum_{r\neq 0} \gamma_r \frac{k(|r|)}{r}|\hat{f}(r)|^2$.
(iv) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} F, \ \textit{but} \ \bigcup_{I\in\mathcal{I}} h(|I|) \leq \eta.$$

(v) $\operatorname{supp} F \subseteq \operatorname{supp} f$.

Proof. Observe that it suffices to find a $\tilde{F}$ satisfying conditions (ii), (iii), (iv) and (v) with ϵ replaced by a sufficiently small $\tilde{\epsilon}$ and then taking $F = \left(\hat{F}(0)\right)^{-1}\tilde{F}$. We therefore ignore condition (i).

We observe that, since f is infinitely differentiable, we can find a $C > 1$ with $|\hat{f}(r)| \leq Cr^{-2}$ for $r \neq 0$. Let K be as in Lemma 2.7, and choose N sufficiently large that $C^{-2}K^{-1}\delta\epsilon/2 > \gamma_r$ for all $r \geq N$. Now let M be an integer with $M > Q$, to be fixed later, take g to be the corresponding function satisfying the conditions of Lemma 2.7 and set $F(t) = g(t)f(t)$. Conditions (iv) and (v) are immediate.

Since $f \in C^\infty$, we have $\hat{f} \in l^1$. Further, $\hat{g}(0) = 1$ and $\hat{g}(r - j) = 0$ for $1 \leq |r - j| < M$, so

$$|\hat{F}(r) - \hat{f}(r)| = \left|\sum_{j\neq r} \hat{f}(j)\hat{g}(r-j)\right| \leq \sum_{|r-j|\geq M} |\hat{f}(j)| \leq \sum_{|j|\geq M-Q} |\hat{f}(j)|$$

and condition (ii) will hold provided only that M is sufficiently large.

Since N is fixed, we may also choose M sufficiently large

$$\sum_{1\leq|r|\leq N} \gamma_r \frac{k(|r|)}{|r|}|\hat{F}(r)|^2 \leq \frac{\epsilon}{2} + \sum_{1\leq|r|\leq N} \gamma_r \frac{k(|r|)}{|r|}|\hat{f}(r)|^2$$

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so, automatically,

$$\sum_{1\le |r|\le N}\gamma_r\frac{k(|r|)}{|r|}|\hat F(r)|^2 \le \frac{\epsilon}{2}+\sum_{r\ne 0}\gamma_r\frac{k(|r|)}{|r|}|\hat f(r)|^2.$$

Now, $|\hat f|\le Cr^{-2}$ for $r\ne 0$ and $|\hat F(r)|\le \tilde g(r)$, so, using Lemma 2.7, we have

$$\begin{aligned}\sum_{|r|>N}\gamma_r\frac{k(|r|)}{|r|}|\hat F(r)|^2 &\le \frac{\epsilon}{2C^2K\delta^{-1}}\sum_{|r|>N}\frac{k(|r|)}{|r|}|\hat F(r)|^2\\ &\le \frac{\epsilon}{2C^2K\delta^{-1}}\sum_{|r|>N}\frac{k(|r|)}{|r|}\tilde g(r)^2\\ &\le \frac{\epsilon}{2K\delta^{-1}}\sum_{r\ne 0}\frac{k(|r|)}{|r|}\tilde g(r)^2 \le \frac{\epsilon}{2}.\end{aligned}$$

Combining this result with the final formula of the previous paragraph, we see that condition (iii) holds and the proof is complete. □

The rest of the argument is standard.

Proof of Theorem 1.5. Take $f_0 = 1$. By Lemma 2.8, we can find a sequence of positive function $f_n\in C^\infty(\mathbb{T})$ with the following properties:

(i)$_n$ $\hat f_n(0)=1$.
(ii)$_n$ $|\hat f_n(r)-\hat f_{n-1}(r)|\le 2^{-n}$ for all $|r|\le n$.
(iii)$_n$ $\sum_{r\ne 0}\gamma_r\frac{k(|r|)}{|r|}|\hat f_n(r)|^2\le 1-2^{-n}$.
(iv)$_n$ There is a finite collection of intervals $\mathcal{I}_n$ such that

$$\bigcup_{I\in\mathcal{I}_n} I\supseteq \operatorname{supp} f_n,\ \text{ but } \bigcup_{I\in\mathcal{I}_n} h(|I|)\le 2^{-n}.$$

(v)$_n$ $\operatorname{supp} f_{n-1}\subseteq \operatorname{supp} f_n$.

Standard theorems now tell us that the measures $f_n m$ (where m is Lebesgue measure) converge weakly to a probability measure μ with $\operatorname{supp}\mu\subseteq\operatorname{supp} f_n$ and that μ has the properties we require. □

Observe that the final sentence of Subsection 2.2 implies that there is a sequence $M(j)\to\infty$ such that $\exp(iM(j)t)\to 1$ uniformly on $\operatorname{supp}\mu$, so $\hat\mu\big(M(j)\big)\to 1$ as $j\to\infty$.

3. A result for logarithmic Hausdorff measure

Theorem V of Chapter III of [3] also states the following result, where we write $h_L(t)=\big(\log(2/t)\big)^{-1}$ for $t\in(0,1]$, $h(0)=0$.

Theorem 3.1 (Kahane and Salem). *Then if μ is a probability measure with*

$$\sum_{r\ne 0}\frac{|\hat\mu(r)|^2}{|r|}<\infty,$$

the support of μ has strictly positive h_L-measure.

Again, we can show that Theorem 3.1 is sharp in the following sense.

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Theorem 3.2. *Suppose $\gamma_r > 0$ and $\gamma_r \to 0$ as $r \to \infty$. Then we can find a probability measure μ with support zero h_L-measure, but*

$$\sum_{r\neq 0} \gamma_{|r|} \frac{|\hat{\mu}(r)|^2}{|r|} < \infty.$$

The proof depends on the following close (but simpler) analogue of Lemma 2.3.

Lemma 3.3. *There exists a constant C for which the following is true. Given any $M \geq 1$, there exists exists a positive function $g \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{g}(0) = 1$.
(ii) $\hat{g}(r) = 0$ *for* $1 \leq |r| \leq M - 1$.
(iii) $\displaystyle\sum_{r\neq 0} \frac{|\hat{g}(r)|^2}{|r|} \leq C$.
(iv) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} g, \ \textit{but} \ \sum_{I\in\mathcal{I}} h_L(|I|) \leq 1.$$

Proof. We set

$$g(t) = u_{\exp(M)} * \frac{\delta_0 + \delta_{1/M} + \delta_{2/M} + \ldots + \delta_{(M-1)/M}}{M},$$

with u_R as in Lemma 2.2, and follow the proof of Lemma 2.3.

Conditions (i) and (ii) are immediate. Condition (iv) follows on choosing $\mathcal{I}$ to be the collection of intervals

$$[kM^{-1} - \exp(-M)/2, kp^{-1} + \exp(-M)/2] \text{ with } 1 \leq k \leq M.$$

Finally, using a sequence of inequalities that should be compared to the corresponding sequence in Lemma 2.3,

$$\begin{aligned}\sum_{r\neq 0} \frac{|\hat{g}(r)|^2}{|r|} &= \sum_{k\neq 0} \frac{|\hat{g}(kM)|^2}{|kM|} = \frac{1}{M}\sum_{k\neq 0} \frac{|\hat{u}_{\exp(M)}(k)|^2}{|k|} \\ &\leq \frac{1}{M}\left(\sum_{|k|\leq \exp(M)} \frac{1}{|k|} + \frac{M^2 2\beta_1^2}{M} \sum_{|k|>\exp(M)} \beta_1^2\big(\exp(M)\big)^2 \frac{1}{|k|^3}\right) \\ &\leq \frac{C_1}{M} \log\big(\exp(M)\big) + C_2 \leq C\end{aligned}$$

for appropriate constants C_1, C_2 and C, so (iii) holds. □

The proof of Theorem 3.1 now follows the same path as that of Theorem 1.5.

4. Proof of Theorem 1.6

4.1. The building blocks

Our proof of Theorem 1.6 requires a more complicated building block. We start with another version of Lemma 2.3.

Lemma 4.1. *Let $1 > \alpha > 0$. Then if the conditions of Theorem 1.6 hold, there exist constants $A = A(\alpha)$ and $B = B(\alpha)$ such that the following is true. If $p \geq 2$, there exists a positive function $g_p \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{g}_p(0) = \dfrac{1}{pL(p)}$.
(ii) $\hat{g}_p(r) = 0$ *if r is not divisible by p.*
(iii) $|\hat{g}_p(r)| \leq B|r|^{-2}$ *for $|r| \geq p^{9/\alpha}$.*
(iv) $|\hat{g_p}(r)| \leq \dfrac{A}{pL(p)}$ *for all $r \neq 0$.*
(v) *There is a finite collection of intervals $\mathcal{I}_p$ such that*

$$\bigcup_{I \in \mathcal{I}_p} I \supseteq \operatorname{supp} g_p, \ \textit{but} \ \sum_{I \in \mathcal{I}_p} h(|I|) \leq \frac{1}{pL(p)}.$$

Proof. We set $R_p = k^{-1}(p^2 L(p))$ and

$$g_p(t) = \frac{1}{PL(p)} u_{R_p} * \frac{\delta_0 + \delta_{1/p} + \delta_{2/p} + \ldots + \delta_{(p-1)/p}}{p},$$

with u_R as in Lemma 2.2.

Conditions (i) and (ii) are immediate. Condition (v) follows on choosing $\mathcal{I}_p$ to be the collection of intervals

$$[jp^{-1} - h^{-1}(p^{-2}L(p)^{-1})/2, jp^{-1} + h^{-1}(p^{-2}L(p)^{-1})/2] \text{ with } 1 \leq j \leq p.$$

Before looking at conditions (iv) and (iii), we note that

$$\hat{g}_p(r) = \begin{cases} \frac{1}{pL(p)} \hat{u}_{R_p}(r) & \text{if } p \text{ divides } r, \\ 0 & \text{otherwise} \end{cases}.$$

Condition (iii) follows from condition (ii) of Lemma 2.2, which yields

$$\begin{aligned} |\hat{g}_p(r)| &\leq \frac{1}{pL(p)} |\hat{u}_{R_p}(r)| \leq \frac{\beta_3}{pL(p)} R_p^3 |r|^{-3} \\ &= \frac{\beta_3}{pL(p)} (k^{-1}(p^2 L(p)))^3 |r|^{-3} \leq \frac{\beta_3}{pL(p)} k^{-1}(p^3)^3 |r|^{-3} \\ &\leq \frac{\beta_3}{p} p^{9/\alpha} |r|^{-3} \leq Bp^{-1}|r|^{-2} \leq Br^{-2} \end{aligned}$$

for $|r| \geq p^{9/\alpha}$, where B is some appropriate constant.

To check condition (iv), we first observe that, if $1 \leq |r| \leq R$, then

$$\begin{aligned} k(|r|)L(|r|) &\leq k(R_p)L(R_p) = p^2 L(p) L\big(k^{-1}(p^2 L(p))\big) \leq p^2 L(p) L(k^{-1}(p^3)) \\ &\leq p^2 L(p) L(p^{3/\alpha}) \leq Cp^2 L(p^2) \end{aligned}$$

so that

$$\frac{C}{k(|r|)L(|r|)} \geq \frac{1}{p^2 L(p)^2} \geq |\hat{g}(r)|^2.$$

On the other hand, if $r \geq R$, then choosing $q > (1+\alpha^{-1})/2$ using condition (ii) of Lemma 2.2, we have

$$|\hat{g}(r)|^2 \leq \beta_q^2 R_p^{2q} |r|^{-2q}$$

and

$$k(|r|)L(|r|) \leq \frac{|r|^{1+\alpha^{-1}}}{R_p^{1+\alpha^{-1}}} k(R_p)L(R_p)$$

so

$$\frac{C\beta_q^2}{k(|r|)L(|r|)} \geq |\hat{g}(r)|^2.$$

Thus condition (iv) holds for an appropriate A. □

We shall make use of the following result.

Lemma 4.2. *Let $L : [1,\infty) \to [1,\infty)$ be a a continuous increasing function, and let P be the set of prime numbers. Then*

$$\int_1^\infty \frac{1}{xL(x)\log x}\, dx = \infty, \text{ implies } \sum_{p\in P} \frac{1}{pL(p)} = \infty.$$

Proof. By the prime number theorem or the much more easily obtained Chebychev inequality (see, for example, [7]), there exists a strictly positive constant K such that if n is large enough, the number of elements in

$$P(n) = \{p \in P : 2^n \leq p < 2^{n+1}\}$$

exceeds $Kn^{-1}2^n$. Thus, if n is large enough,

$$\begin{aligned}\sum_{p\in P(n)} \frac{1}{pL(p)} &\geq K\frac{2^n}{n} \times \frac{1}{2^{n+1}L(2^{n+1})} \geq \frac{K}{2} \times \frac{2^{n+1}}{(n+1)2^{n+1}L(2^{n+1})}\\ &\geq \frac{K}{2}\int_{2^{n+1}}^{2^{n+2}} \frac{1}{xL(x)\log x}\, dx,\end{aligned}$$

and the result follows. □

We can now prove our central step.

Lemma 4.3. *Let $\alpha > 0$. Then there exists a constant $A' = A'(\alpha)$ such that the following is true.*

Suppose that the conditions of Theorem 1.6 hold. Then there exists an M_1 such that, given any $M \geq M_1$, there exists exists a positive function $g \in C^\infty(\mathbb{T})$ with the following properties:

(i) $\hat{g}(0) = 1$.
(ii) $\hat{g}(r) = 0$ *for* $1 \leq |r| \leq M$.
(iii) $|\hat{g}(r)| \leq \dfrac{A'}{\big(k(r)L(r)\big)^{1/2}}$ *for all* $r \neq 0$.
(iv) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} g, \text{ but } \sum_{I\in\mathcal{I}} h(|I|) \leq 2.$$

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Proof. It is sufficient to prove the result with (i) replaced by

(i)′ $\hat{g}(0) \geq 1$.

To this end, we choose M_1 such that

$$\frac{1}{rL(r)} \leq \frac{1}{2}.$$

If $M \geq M_1$, we consider the sequence of consecutive primes $p(1), p(2), \ldots$ starting with $p(1)$, the smallest prime greater than M.

By Lemma 4.2 and our choice of M_1, we can find an N such that $1 \leq \sum_{j=1}^{N} \frac{1}{p(j)L\big(p(j)\big)} \leq 2$. We set $g = \sum_{j=1}^{N} g_{p(j)}$, where the $g_{p(j)}$ are as in Lemma 4.1. Conditions (i)′, (ii) and (iv) can be read off directly from conditions (i), (ii) and (v) of that lemma.

We now wish to bound $|\hat{g}(r)|$. We observe that

$$|\hat{g}(r)| \leq \sum_{p(j) \leq |r|^{\alpha/9}} |\hat{g}_{p(j)}(r)| + \sum_{|r|^{\alpha/9} \leq p(j) \leq |r|} |\hat{g}_{p(j)}(r)|$$

and bound the two sums separately. By condition (iii) of Lemma 4.1,

$$\begin{aligned}\sum_{p(j) \leq r^{\alpha/9}} |\hat{g}_{p(j)}(r)| &\leq \sum_{p(j) \leq r^{\alpha/9}} B|r|^{-2} \leq Br^{-1} \\ &= B\left(\frac{k(|r|)}{|r|}\right)^{1/2}\left(\frac{L(|r|)}{|r|}\right)^{1/2} \frac{1}{\sqrt{k(|r|)L(|r|)}} \leq B_1 \frac{1}{\sqrt{k(|r|)L(|r|)}}\end{aligned}$$

for an appropriate B_1 since $k(|r|)|r|^{-1}$ and $L(|r|)|r|^{-1}$ decrease with $|r|$.

On the other hand, there can be at most $9/\alpha$ distinct primes $r \geq p(j) \geq r^{\alpha/9}$ with $\hat{g}_{p(j)}(r) \neq 0$ (since the $p(j)$ must divide r, we write $p(j)|r$ when this happens), so, using condition (iv) of Lemma 4.1,

$$\begin{aligned}\sum_{|r| \geq p(j) > |r|^{\alpha/9}} |\hat{g}_{p(j)}(r)| &= \sum_{r \geq p(j) > r^{\alpha/9}, p(j)|r} |\hat{g}_{p(j)}(r)| \\ &\leq \sum_{r \geq p(j) > r^{\alpha/9}, r|p(j)} \frac{A}{\big(k(|r|)L(|r|)\big)^{1/2}} \leq \frac{9A\alpha^{-1}}{\big(k(|r|)L(|r|)\big)^{1/2}}\end{aligned}$$

and we are done. □

4.2. *Completion of the proof*

The remainder of the proof is straightforward.

Lemma 4.4. *Suppose that the conditions of Theorem 1.6 hold. Then given any $\eta > 0$, there exists an $M_0(\eta)$ such that, if $M \geq M_0(\eta)$, we can find a $g \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{g}(0) = 1$.

(ii) $\hat{g}(r) = 0$ *for* $1 \leq |r| \leq M$.

(iii) $|\hat{g}(r)| \leq \dfrac{\eta}{\big(k(|r|)L(|r|)\big)^{1/2}}$ *for all* $r \neq 0$.

(iv) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I \in \mathcal{I}} I \supseteq \operatorname{supp} g, \;\; but \;\; \sum_{I \in \mathcal{I}} h(|I|) \leq \eta.$$

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Proof. We can choose $\tilde{h} = A'\eta^{-1}h$ and $\tilde{L}$ such that $\tilde{L}(x)/L(x) \to 0$ so that the conditions of Theorem 1.6 still hold. We now apply Lemma 4.3 with and h replaced by $\tilde{h}$ and L by $\tilde{L}$. □

The next result corresponds to Lemma 2.8 but is little simpler.

Lemma 4.5. *Suppose the conditions of Theorem 1.6 hold. Then if $f \in C^\infty(\mathbb{T})$ is a positive function with $\hat{f}(0) = 1$, $\epsilon > 0$ and $Q \geq 1$, we can find a positive function $F \in C^\infty(\mathbb{T})$ with the following properties:*

(i) $\hat{F}(0) = 1$.
(ii) $|\hat{F}(r) - \hat{f}(r)| \leq \epsilon\big(k(|r|)L(|r|)\big)^{-1/2}$.
(iii) *There is a finite collection of intervals $\mathcal{I}$ such that*

$$\bigcup_{I\in\mathcal{I}} I \supseteq \operatorname{supp} F, \ \textit{but} \ \bigcup_{I\in\mathcal{I}} h(|I|) \leq \eta.$$

(iv) $\operatorname{supp} F \subseteq \operatorname{supp} f$.

Proof. As in the first paragraph of the proof of Lemma 2.8, we observe that we can replace condition (i) by the weaker condition $|\hat{F}(0) - 1| < \epsilon$.

Now let g be as in Lemma 4.4, with η and M to be determined. Set $F(t) = f(t)g(t)$. Conditions (iii) and (iv) are immediate. We know that there exists a C such that $|\hat{f}(r)| \leq Cr^{-4}$, so, provided N is large enough,

$$|\hat{f}(r)| \leq \frac{\epsilon}{2}\big(k(|r|)L(|r|)\big)^{-1/2}$$

whenever $|r| \geq N$.

We also have

$$\begin{aligned}|\hat{F}(r) - \hat{f}(r)| = \left|\sum_{j\neq 0}\hat{f}(j)\hat{g}(r-j)\right| &\leq \sum_{j\neq 0}|\hat{f}(j)||\hat{g}(r-j)| \\ &\leq \sum_{j\neq 0, |r-j|\geq M} Cj^{-4} \to 0\end{aligned}$$

as $M \to \infty$ for each r, so, provided we take M large enough, we will have

$$|\hat{F}(r) - \hat{f}(r)| \leq \epsilon\big(k(|r|)L(|r|)\big)^{-1/2}$$

for all $1 \leq |r| \leq N$ and in addition $|\hat{F}(0) - 1| < \epsilon$.

Combining the results of the two previous paragraphs, we see that the required result will hold, provided we can ensure that

$$|\hat{F}(r) - \hat{f}(r)| \leq \frac{\epsilon}{2}\big(k(|r|)L(|r|)\big)^{-1/2}$$

for all $|r| \geq N$. We have

$$\begin{aligned}|\hat{F}(r) - \hat{f}(r)| &\leq \sum_{j\neq 0}|\hat{f}(j)||\hat{g}(r-j)| \\ &\leq \sum_{j\neq 0, |r-j|\leq r/2} Cj^{-4}|\hat{g}(r-j)| + \sum_{|r-j|>|r|/2} Cj^{-4}.\end{aligned}$$

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We estimate the two sums separately:

$$\sum_{|r-j|>|r|/2} j^{-4} \leq C_1|r|^{-3} \leq \frac{\epsilon}{4}\big(k(|r|)L(|r|)\big)^{-1/2}$$

(with C_1 an appropriate constant) for all $|r| \geq N$, provided only that we have taken N large enough.

On the other hand, we know that there is a constant C_2 such that

$$k(|r|)L(|r|) \leq C_2^2 k(|j|)L(|j|)$$

for all $|r-j| \leq |r|/2$, so

$$\begin{aligned}\sum_{j\neq 0, |r-j|\leq |r|/2} Cj^{-4}|\hat{g}(r-j)| &\leq \sum_{j\neq 0, |r-j|\leq |r|/2} Cj^{-4}\frac{\eta}{\big(k(|r-j|)L(|r-j|)\big)^{1/2}}\\ &\leq \sum_{j\neq 0, |r-j|\leq |r|/2} C_2Cj^{-4}\frac{\eta}{\big(k(|r|)L(|r|)\big)^{1/2}}\\ &\leq \frac{\eta C_2 C}{\big(k(|r|)L(|r|)\big)^{1/2}}\\ &\leq \frac{\epsilon}{4\big(k(|r|)L(|r|)\big)^{1/2}},\end{aligned}$$

provided only that we have choose η sufficiently small. Condition (ii) follows. □

The rest of the argument is standard.

Proof of Theorem 1.4. Take $f_0 = 1$. By Lemma 4.5, we can find a sequence of positive function $f_n \in C^\infty(\mathbb{T})$ with the following properties:

(i)$_n$ $\hat{f}_n(0) = 1$.

(ii)$_n$ $|\hat{f}_n(r) - \hat{f}_{n-1}(r)| \leq \dfrac{2^{-n}}{\big(k(|r|)L(|r|)\big)^{1/2}}$.

(iii)$_n$ There is a finite collection of intervals $\mathcal{I}_n$ such that

$$\bigcup_{I\in\mathcal{I}_n} I \supseteq \operatorname{supp} f_n, \text{ but } \bigcup_{I\in\mathcal{I}_n} h(|I|) \leq 2^{-n}.$$

(iv)$_n$ $\operatorname{supp} f_n \subseteq \operatorname{supp} f_{n-1}$.

Standard theorems now tell us that the measures $f_n m$ (where m is Lebesgue measure) converge weakly to a probability measure μ with $\operatorname{supp}\mu \subseteq \operatorname{supp} f_n$ and that μ has the properties we require. □

The results and proofs of this paper go over with appropriate modifications to $\mathbb{T}^n$ and $\mathbb{R}^n$.

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Conflicts of Interest. None.

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