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CERTAIN INTEGRALS FOR CLASSES OF p-VALENT MEROMORPHIC FUNCTIONS

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In this paper we introduce two classes, namely $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, of functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + a_{n+p-1}z^n + \ldots ,$$

regular and p-valent in $D - \{0\}$ where $D = \{z : |z| < 1\}$. We show that, for suitable choices of real constants α, β and γ , the integral operators of the form

$$F(z) = \left[\frac{\gamma-p(\alpha+\beta)+2}{z^{\gamma-p\beta+2}} \int_0^z t^{\gamma+1} f(t)^\alpha g(t)^\beta dt\right]^{1/\alpha}$$

map $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$, where $\Gamma_p^*(\rho)$ is the class of p-valent meromorphically starlike functions of order ρ , $0 \le \rho < 1$. For the classes $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, we obtain class preserving integral operators of the form

$$F(z) = \left[\frac{\gamma-p\alpha+1}{z^{\gamma+1}} \int_0^z t^\gamma f(t)^\alpha dt\right]^{1/\alpha} ,$$

with suitable restrictions on real constants α and γ .

Our results generalize almost all known results obtained so far in this direction.

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1. Introduction

Let I^+ denote the set of positive integers. We denote by Γ_p, $p \in I^+$, the set of the functions

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + a_{n+p-1}z^n + \ldots ,$$

regular and p-valent in $D - \{0\}$, where $D = \{z : |z| < 1\}$ and $f'(z) \neq 0$ there. A function f of Γ_p is said to belong to Γ_p^*, the class of p-valent meromorphically starlike functions, if and only if $\mathrm{Re}\{z(f'(z)/f(z))\} < 0$, $z \in D$. A function f of Γ_p is said to belong to $\Gamma_p^*(\rho)$, the class of p-valent meromorphically starlike functions of order ρ, $0 \leq \rho < 1$, if and only if $\mathrm{Re}\{z(f'(z)/f(z))\} < -p\rho$, $z \in D$. A function f of Γ_p is said to belong to $\Sigma_p(\rho)$, the class of p-valent meromorphically convex functions of order ρ, $0 \leq \rho < 1$, if and only if $\mathrm{Re}\{1+z(f''(z)/f'(z))\} < -p\rho$, $z \in D$. The class Σ_p of p-valent meromorphically convex functions is identified by $\Sigma_p \equiv \Sigma_p(0)$.

Now we define two subclasses, namely $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$, of Γ_p^* and Σ_p respectively.

A function f of Γ_p belongs to the class $\Gamma_p(m, M)$ if and only if $|z(f'(z)/f(z))+m| < M$, $z \in D$ where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\} .$$

A function f of Γ_p belongs to the class $\Sigma_p(m, M)$ if and only if $|1+z(f''(z)/f'(z))+m| < M$, $z \in D$, where

$$(m, M) \in E_p = \{(m, M) : |m-p| < M \leq m\} .$$

It is clear that $\Gamma_p(m, M) \subset \Gamma_p^*((m-M)/p) \subset \Gamma_p^* \subset \Gamma_p$ and $\Sigma_p(m, M) \subset \Sigma_p((m-M)/p) \subset \Sigma_p \subset \Gamma_p$. Also, a function f belongs to $\Sigma_p(m, M)$ if and only if $zf'(z)/-p$ belongs to $\Gamma_p(m, M)$.

In [1], [2], [3], the integral operators of the forms

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$$F(z) = \left[\frac{\gamma-2\beta+2}{z^{\gamma-\beta+2}} \int_0^z t^{\gamma+1} f(t)^\beta g(t)^\beta dt\right]^{1/\beta}$$

and

$$F(z) = \left[\frac{\gamma-\alpha+1}{z^{\gamma+1}} \int_0^z t^{\gamma} f(t)^\alpha dt\right]^{1/\alpha} ,$$

with suitable restrictions on the real constants α, β and γ, and for f and g belonging to some favoured classes of meromorphic functions have been studied. The purpose of this paper is to obtain the integral operators that are more general and transform $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$, $\Gamma_p(m, M)$ into $\Gamma_p(m, M)$, and $\Sigma_p(m, M)$ into $\Sigma_p(m, M)$. Our results generalize almost all known results obtained so far in this direction [1], [2], [3].

2. Preliminary lemmas

Let $S(m, M)$ be the class of functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and satisfying

(2.1) $$\left|z \frac{h'(z)}{h(z)} - m\right| < M \text{ in } D ,$$

where

(2.2) $$(m, M) \in E = \{(m, M) : m > \tfrac{1}{2},\ |m-1| < M \le m\} .$$

This class $S(m, M)$ was introduced by Jakubowski [5]. It is worth mentioning here that the requirement $m > \frac{1}{2}$ in (2.2) is superfluous and may be dropped since it follows directly from the inequality $|m-1| < m$ in (2.2).

The proof of the following lemma is based on the lines of a result of Silverman [7, Theorem 1], the only difference is that in the definition of $S(m, M)$ Silverman [7] has taken equality also in (2.1) and restricted m, M by the inequalities

(2.3) $$m + M \ge 1 , \quad M \le m \le M + 1$$

which are equivalent to the inequalities

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$$|m-1| \le M \le m .$$

However we follow the definition of $S(m, M)$ given by Jakubowski [5].

LEMMA 2.1. *The function* $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to* $S(m, M)$ *if and only if there exists a function* $w(z)$ *regular in* D *and satisfying* $w(0) = 0$, $|w(z)| < 1$ *for* z *in* D, *such that*

$$z\frac{h'(z)}{h(z)} = \frac{1+a'w(z)}{1-b'w(z)} , \quad z \in D , \tag{2.4}$$

where

$$a' = \frac{M^2-m^2+m}{M} \quad \text{and} \quad b' = \frac{m-1}{M} . \tag{2.5}$$

Proof. First suppose that $h \in S(m, M)$. Then

$$\left| z\frac{h'(z)}{Mh(z)} - \frac{m}{M} \right| < 1 .$$

Let $g(z) = z\left(h'(z)/Mh(z)\right) - m/M$ and

$$w(z) = \frac{g(z)-g(0)}{1-g(0)g(z)} = \frac{z\left(h'(z)/h(z)\right)-1}{M+\left((m-1)/M\right)\left\{z\left(h'(z)/h(z)\right)-m\right\}} . \tag{2.6}$$

Then $w(0) = 0$, $|w(z)| < 1$. Rearranging (2.6) and using (2.5) we get (2.4).

Conversely, suppose that $h(z)$ satisfies (2.4). Then

$$z\frac{h'(z)}{h(z)} - m = M\frac{\left((1-m)/M\right)+w(z)}{1+\left((1-m)/M\right)w(z)} \tag{2.7}$$

$$= MG(z) , \text{ say.}$$

In view of (2.2), $|(1-m)/M| < 1$. Thus the function

$$G(z) = \frac{\left((1-m)/M\right)+w(z)}{1+\left((1-m)/M\right)w(z)}$$

satisfies $|G(z)| < 1$. From (2.7), it follows now that $h \in S(m, M)$.

This completes the proof of the lemma.

LEMMA 2.2. *The function* f *is in* $\Gamma_p(m, M)$ *if and only if there exists a function* $w(z)$ *regular and satisfying* $w(0) = 0$, $|w(z)| < 1$, *for* z *in* D *such that*

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$$z \frac{f'(z)}{f(z)} = -p \frac{1+aw(z)}{1-bw(z)} , \quad z \in D ,$$

where $a = \left(M^2-m^2+mp\right)/Mp$ *and* $b = (m-p)/M$.

Proof. Since $f \in \Gamma_p(m, M)$, $z\left(f'(z)/f(z)\right) = -p + \ldots$. Let $\left\{z\left(f'(z)/f(z)\right)\right\}/-p = z\left(h'(z)/h(z)\right) = 1 + \ldots$, then from Lemma 2.1, for $|(m/p)-1| < M/p \le m/p$,

$$\left|\frac{z\left(f'(z)/f(z)\right)}{-p} - \frac{m}{p}\right| < \frac{M}{p}$$

if and only if

$$\frac{z\left(f'(z)/f(z)\right)}{-p} = \frac{1+aw(z)}{1-bw(z)}$$

where

$$a = \frac{(M/p)^2-(m/p)^2+(m/p)}{M/p} \text{ and } b = \frac{(m/p)-1}{M/p} ;$$

or, for $|m-p| < M \le m$,

$$\left|z \frac{f'(z)}{f(z)} + m\right| < M$$

if and only if

$$z \frac{f'(z)}{f(z)} = -p \frac{1+aw(z)}{1-bw(z)} ,$$

where

$$a = \frac{M^2-m^2+mp}{Mp} \text{ and } b = \frac{m-p}{M} .$$

LEMMA 2.3. *If the function* $w(z)$ *is regular for* $|z| \le r < 1$, $w(0) = 0$ *and* $\left|w\left(z_1\right)\right| = \max_{|z|=r} |w(z)|$, *then*

$$z_1 w'\left(z_1\right) = kw\left(z_1\right) , \quad k \ge 1 .$$

A proof of Lemma 2.3, which is due to Jack, may be found in [4].

LEMMA 2.4. *A function* f *belongs to* $\Gamma_p^*(\rho)$, $0 \le \rho < 1$, *if and only if there exists a function* $w(z)$ *regular and satisfying* $w(0) = 0$, $|w(z)| < 1$ *in* D *such that*

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$$z\,\frac{f'(z)}{f(z)} = -p\,\frac{1+(2\rho-1)w(z)}{1+w(z)}\ .$$

Proof. Let $P(z) = \big(\big(z\big(f'(z)/f(z)\big)/-p\big)-\rho\big)/(1-\rho) = 1 + \dots$; then $\operatorname{Re}\{P(z)\} > 0$ and hence by a well known result [6], $P(z)$ can be written as

$$P(z) = \frac{1-w(z)}{1+w(z)}\ ,$$

where $w(z)$ is regular and $w(0) = 0$, $|w(z)| < 1$, for z in D . Thus

$$\frac{\big(z\big(f'(z)/f(z)\big)/-p\big)-\rho}{1-\rho} = \frac{1-w(z)}{1+w(z)}$$

and hence the result follows.

3. Integral operators that map $\Gamma_p^*(\rho) \times \Gamma_p(m, M)$ into $\Gamma_p^*(\rho)$

THEOREM 3.1. *Let* α, β *and* γ *be real constants such that*

$$\alpha > 0\ ,\quad \beta \ge 0 \quad \text{and} \quad \gamma - p(\alpha+\beta) + 1 > -1\ .$$

If $f \in \Gamma_p^*(\rho)$ *and* $g \in \Gamma_p(m, M)$, $(m, M) \in E_p^* = \{(m, M) : |m-p| < M \le m^*\}$ *where* $m^* = \min\{m, (m-p)+\big(\alpha p(1-\rho)/2\beta(\gamma-p\beta-\alpha\rho p+2)\big)\}$, *then*

$$(3.1)\qquad F(z) = \left[\frac{\gamma-p(\alpha+\beta)+2}{z^{\gamma-p\beta+2}}\int_0^z t^{\gamma+1}f(t)^{\alpha}g(t)^{\beta}dt\right]^{1/\alpha} = \frac{1}{z^p} + \dots$$

also belongs to $\Gamma_p^*(\rho)$. *In* (3.1) *all powers are principal ones.*

Proof. Let us choose a function $w(z)$ such that

$$(3.2)\qquad z\,\frac{F'(z)}{F(z)} = -p\,\frac{1+(2\rho-1)w(z)}{1+w(z)}\ ,\quad w(0) = 0\ ,$$

and $w(z)$ is either regular or meromorphic in D .

From (3.1) and (3.2), we have

$$(3.3)\qquad z^{p\beta}\,\frac{f(z)^{\alpha}}{F(z)^{\alpha}}\,g(z)^{\beta} = \frac{1+(\xi/\delta)w(z)}{1+w(z)}\ ,$$

where $\xi = \gamma + p(\alpha-\beta) + 2(1-\alpha\rho p)$ and $\delta = \gamma - p(\alpha+\beta) + 2$.

Logarithmic differentiation of (3.3) yields

(3.4) $$z\frac{f'(z)}{f(z)} = \frac{\beta}{\alpha}(m-p) - \frac{\beta}{\alpha}\left\{z\frac{g'(z)}{g(z)} + m\right\} - p\frac{1+(2\rho-1)w(z)}{1+w(z)} + \frac{2p(1-\rho)zw'(z)}{\delta\{1+w(z)\}[1+(\xi/\delta)w(z)]} .$$

Let r^* be the distance from the origin of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|z| \le r$ $(r < r_0)$ there is a point z_0 such that

(3.5) $$z_0w'(z_0) = kw(z_0) , \quad k \ge 1 .$$

From (3.4) and (3.5) we have

(3.6) $$\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\}$$
$$\ge \frac{\beta}{\alpha}(m-p) - \frac{\beta}{\alpha}\left|z_0\frac{g'(z_0)}{g(z_0)} + m\right| - p\frac{\operatorname{Re}\left[\{1+(2\rho-1)w(z_0)\}\{1+\overline{w(z_0)}\}\right]}{|1+w(z_0)|^2} + \frac{2p(1-\rho)k\operatorname{Re}\left[w(z_0)\{1+\overline{w(z_0)}\}\{1+(\xi/\delta)\overline{w(z_0)}\}\right]}{\delta|1+w(z_0)|^2|1+(\xi/\delta)w(z_0)|^2}$$

$$> \frac{\beta}{\alpha}\{(m-p)-M\} - p\frac{1+2\rho\operatorname{Re}w(z_0)+(2\rho-1)|w(z_0)|^2}{1+2\operatorname{Re}w(z_0)+|w(z_0)|^2} + \frac{2p(1-\rho)k\operatorname{Re}\left[w(z_0)+(2\{\gamma-p\beta-\alpha\rho p+2\}/\delta)|w(z_0)|^2+(\xi/\delta)|w(z_0)|^2\overline{w(z_0)}\right]}{\delta\left[1+2\operatorname{Re}w(z_0)+|w(z_0)|^2\right]\left[1+(2\xi/\delta)\operatorname{Re}w(z_0)+(\xi/\delta)^2|w(z_0)|^2\right]} .$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some r $(r < r_0 \le 1)$. At the point z_0 where this occurred, we would have $|w(z)| = 1$. Then, from (3.6),

$$\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} > -p\rho + \frac{2\{\gamma-p\beta-\alpha\rho p+2\}[(1-\rho)p+(2\beta/\alpha)\{(m-p)-M\}\{\gamma-p\beta-\alpha\rho p+2\}]}{[\delta^2+2\delta\xi\operatorname{Re}w(z_0)+\xi^2]}$$
$$\ge -p\rho , \text{ provided } M \le (m-p) + \frac{\alpha p(1-\rho)}{2\beta(\gamma-p\beta-\alpha\rho p+2)} .$$

But this is contrary to the fact that $f \in \Gamma_p^*(\rho)$. So we cannot have $M(r, w) = 1$. Thus $|w(z)| \ne 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$

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is continuous in $|z| < r_0$ and $|w(z)| \neq 1$ there, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $|w(z)| < 1$ and $w(z)$ is regular in D.

Hence from (3.2) and Lemma 2.4, $F \in \Gamma_p^*(\rho)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 3.1:

(i) for $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\rho = 0$, a result of Bajpai [1];

(ii) for $\alpha = \beta = 1$, a result of Dwivedi [2];

(iii) for $\alpha = \beta$, a result of Dwivedi, Bhargava and Shukla [3].

THEOREM 3.2. *Let* β *and* γ *be real constants such that* $\beta \geq 0$ *and* $\gamma > 0$. *If* $f \in \Gamma_p^*(\rho)$ *and* $g \in \Gamma_p(m, M)$,

$$(m, M) \in E_p^{**} = \{(m, M) : |m-p| < M \leq m^*\}$$

where

$$m^* = \min\left\{m,\ (m-p) + \frac{p(1-\rho)}{2\beta\{\gamma+p(1-\rho)\}}\right\}.$$

Then

$$F(z) = \frac{\gamma}{z^{\gamma+p}} \int_0^z t^{\gamma+p(1+\beta)-1} f(t)\cdot g(t)^\beta dt = \frac{1}{z^p} + \dots \tag{3.7}$$

also belongs to $\Gamma_p^*(\rho)$. *In* (3.7) *all powers are principal ones.*

Proof. Let us choose a function $w(z)$ such that

$$z\frac{F'(z)}{F(z)} = -p\,\frac{1+(2\rho-1)w(z)}{1+w(z)},\quad w(0) = 0, \tag{3.8}$$

$w(z)$ is either regular or meromorphic in D.

From (3.7) and (3.8), we have

$$\frac{z^{p\beta} f(z) g(z)^\beta}{F(z)} = \frac{1+\left[\{\gamma+2p(1-\rho)\}/\gamma\right]w(z)}{1+w(z)}. \tag{3.9}$$

Logarithmic differentiation of (3.9) yields

(3.10) $$z\frac{f'(z)}{f(z)} = -p\beta - \beta z\frac{g'(z)}{g(z)} - p\frac{1+(2\rho-1)w(z)}{1+w(z)} + \frac{2p(1-\rho)zw'(z)}{[1+w(z)][\gamma+\{\gamma+2p(1-\rho)\}w(z)]} .$$

Let r^* be the distance from the origin of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for $|z| \le r$ $(r < r_0)$ there is a point z_0 such that

(3.11) $$z_0w'(z_0) = kw(z_0) , \quad k \ge 1 .$$

From (3.10) and (3.11), we have

$$\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} \ge \beta(m-p) - \beta\left|z_0\frac{g'(z_0)}{g(z_0)} + m\right| - p\frac{\operatorname{Re}\left[\{1+(2\rho-1)w(z_0)\}\{1+\overline{w(z_0)}\}\right]}{|1+w(z_0)|^2} + \frac{2p(1-\rho)k\operatorname{Re}\left[w(z_0)\{\gamma+(\gamma+2p(1-\rho))\overline{w(z_0)}\}\{1+\overline{w(z_0)}\}\right]}{|1+w(z_0)|^2|\gamma+\{\gamma+2p(1-\rho)\}w(z_0)|^2} ,$$

or

(3.12)
$$\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} > -\beta\{M-(m-p)\} - p\frac{1+2\rho\operatorname{Re}w(z_0)+(2\rho-1)|w(z_0)|^2}{1+2\operatorname{Re}w(z_0)+|w(z_0)|^2} + \frac{2p(1-\rho)k\operatorname{Re}\left[\gamma w(z_0)+2\{\gamma+p(1-\rho)\}|w(z_0)|^2+\{\gamma+2p(1-\rho)\}|w(z_0)|^2\overline{w(z_0)}\right]}{\left[1+2\operatorname{Re}w(z_0)+|w(z_0)|^2\right]\left[\gamma^2+2\gamma\{\gamma+2p(1-\rho)\}\operatorname{Re}w(z_0)+\{\gamma+2p(1-\rho)\}^2|w(z_0)|^2\right]} .$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)| = 1$ for some r $(r < r_0)$. At the point z_0 where this occurred we would have $|w(z_0)| = 1$. Then, from (3.12),

$$\operatorname{Re}\left\{z_0\frac{f'(z_0)}{f(z_0)}\right\} > -p\rho + \frac{2\{\gamma+p(1-\rho)\}[p(1-\rho)-2\beta\{M-(m-p)\}\{\gamma+p(1-\rho)\}]}{\left[\gamma^2+2\gamma\{\gamma+2p(1-\rho)\}\operatorname{Re}w(z_0)+\{\gamma+2p(1-\rho)\}^2\right]}$$
$$\ge -p\rho , \text{ provided } M \le (m-p) + \frac{p(1-\rho)}{2\beta\{\gamma+p(1-\rho)\}} .$$

But this is contrary to the fact that $f \in \Gamma_p^*(\rho)$. So we can not have $M(r, w) = 1$. Thus $|w(z)| \ne 1$ in $|z| < r_0$. Since $|w(0)| = 0$,

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$|w(z)|$ is continuous in $|z| < r_0$ and $|w(z)| \neq 1$ there, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $|w(z)| < 1$ and $w(z)$ is regular in D.

Hence from (3.8) and Lemma 2.4, $F \in \Gamma_p^*(\rho)$.

4. Class preserving integral operators for $\Gamma_p(m, M)$ and $\Sigma_p(m, M)$

THEOREM 4.1. *If* $f \in \Gamma_p(m, M)$ *and* $F(z)$ *is defined by*

$$F(z) = \left[\frac{\gamma-p\alpha+1}{z^{\gamma+1}}\int_0^z t^\gamma f(t)^\alpha dt\right]^{1/\alpha} = \frac{1}{z^p} + \ldots , \tag{4.1}$$

$\alpha > 0$. *Then* $F \in \Gamma_p(m, M)$ *if* $\gamma > \max\{(p\alpha(1+a)+b-1)/(1-b), p\alpha-1\}$.

In (4.1) all powers are principal ones.

Proof. From (4.1) we have

$$\alpha z\frac{F'(z)}{F(z)} + (\gamma+1) = (\gamma-p\alpha+1)\left\{\frac{f(z)}{F(z)}\right\}^\alpha . \tag{4.2}$$

Let us choose a function $w(z)$ such that

$$z\frac{F'(z)}{F(z)} = -p\,\frac{1+aw(z)}{1-bw(z)} , \quad w(0) = 0 , \tag{4.3}$$

and $w(z)$ is either regular or meromorphic in D. From (4.2) and (4.3), we have

$$(\gamma-p\alpha+1)\left\{\frac{f(z)}{F(z)}\right\}^\alpha = \frac{(\gamma-p\alpha+1)-(a\alpha p+b+b\gamma)w(z)}{1-bw(z)} . \tag{4.4}$$

Logarithmic differentiation of (4.4) yields

$$z\frac{f'(z)}{f(z)} + m = \frac{[(m-p)\kappa-\{(m-p)\theta+(ap+bm)\kappa\}w(z)+(ap+bm)\theta w^2(z)-(a+b)pzw'(z)]}{[\kappa-\{a\alpha p+b(2-p\alpha)+2b\gamma\}w(z)+b\theta w^2(z)]} \tag{4.5}$$

where $\kappa = \gamma - p\alpha + 1$ and $\theta = a\alpha p + b + b\gamma$.

Let r^* be the distance of the pole of $w(z)$ nearest the origin. Then $w(z)$ is regular in $|z| < r_0 = \min\{r^*, 1\}$. By Lemma 2.3, for

$|z| \le r$ $(r < r_0)$, there is a point z_0 such that

$$z_0 w'(z_0) = kw(z_0) , \quad k \ge 1 . \tag{4.6}$$

From (4.5) and (4.6), we have

$$z_0 \frac{f'(z_0)}{f(z_0)} + m = \frac{N(z_0)}{D(z_0)} , \tag{4.7}$$

where

$$N(z_0) = (m-p)(\gamma-p\alpha+1) - \{(m-p)(a\alpha p+b+b\gamma)+(ap+bm)(\gamma-p\alpha+1)+(a+b)pk\}w(z_0) + \{(ap+bm)(a\alpha p+b+b\gamma)\}w^2(z_0)$$

and

$$D(z_0) = (\gamma-p\alpha+1) - \{a\alpha p+b(2-p\alpha)+2b\gamma\}w(z_0) + b(a\alpha p+b+b\gamma)w^2(z_0) .$$

Now suppose that it were possible to have $M(r, w) = \max_{|z|=r} |w(z)|$ for some r $(r < r_0)$. At the point z_0 where this occurred we would have $|w(z_0)| = 1$. Then we have

$$|N(z_0)|^2 - M^2|D(z_0)|^2 = A - 2B \operatorname{Re}\{w(z_0)\} , \tag{4.8}$$

where

$$A = kp(a+b)[kp(a+b)+2M(\gamma-p\alpha+1)+2Mb(a\alpha p+b+b\gamma)]$$

and

$$B = kp(a+b)M[a\alpha p+b(2-p\alpha)+2b\gamma] .$$

From (4.8), we have

$$|N(z_0)|^2 - M^2|D(z_0)|^2 > 0 , \tag{4.9}$$

provided $A \pm 2B > 0$.

Now

$$A + 2B = kp(a+b)[kp(a+b)+2M(1+b)\{\gamma(1+b)+(1-p\alpha+b+ap\alpha)\}] > 0$$

provided $\gamma \ge (-ap\alpha-b+p\alpha-1)/(1+b)$ and

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$$A - 2B = kp(a+b)[kp(a+b)+2M(1-b)\{\gamma(1-b)+(1-p\alpha-b-ap\alpha)\}] > 0 ,$$

provided $\gamma \geq (ap\alpha+b+p\alpha-1)/(1-b)$.

Thus from (4.7) and (4.9), it follows that

$$\left| z_0 \frac{f'(z_0)}{f(z_0)} + m \right| > M , \text{ if } \gamma \geq \frac{\alpha p(1+a)+b-1}{1-b} .$$

But this is contrary to the fact that $f \in \Gamma_p(m, M)$. So we can not have $M(r, w) = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $|w(0)| = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, $w(z)$ can not have a pole at $|z| = r_0$. Therefore $w(z)$ is regular and $|w(z)| < 1$, for z in D .

Hence from (4.3) and Lemma 2.2, $F \in \Gamma_p(m, M)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 4.1:

(i) for $\alpha = \gamma = 1$, $m = M$ and $m \to \infty$, a result of Bajpai [1];

(ii) for $\alpha = 1$, a result of Dwivedi [2];

(iii) a result of Dwivedi, Bhargava and Shukla [3].

THEOREM 4.2. *If* $f \in \Sigma_p(m, M)$ *and* $F(z)$ *is defined by*

$$F(z) = \frac{\gamma-p+1}{z^{\gamma+1}} \int_0^z t^{\gamma} f(t)dt = \frac{1}{z^p} + \dots . \tag{4.10}$$

Then $F \in \Sigma_p(m, M)$ *if* $\gamma > \max\{(p(1+a)+b-1)/(1-b), p-1\}$.

In (4.10) all powers are principal ones.

Proof. We can write (4.10) as

$$zF'(z) = \frac{\gamma-p+1}{z^{\gamma+1}} \int_0^z t^{\gamma} tf'(t)dt . \tag{4.11}$$

Since $f \in \Sigma_p(m, M)$ if and only if $zf'(z)/-p \in \Gamma_p(m, M)$ and hence from

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Theorem 4.1, and (4.11), we get $zF'(z)/-p$ belongs to $\Gamma_p(m, M)$. So $F(z)$ belongs to $\Sigma_p(m, M)$.

REMARK. Taking $p = 1$, the undermentioned results follow from Theorem 4.2:

(i) for $m = M$ and $m \to \infty$, a result of Bajpai [1].

(ii) a result of Dwivedi [2].

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