

Dear Editor,

*A note on the GI/M/1 queue with Poisson negative arrivals*

The equilibrium  $M/M/1$  queue with negative arrivals (Harrison and Pitel (1993)) has simple solutions for queue length and sojourn time distributions, but the generalization to  $M/G/1$  (Harrison and Pitel (1996)) is surprisingly complex. In this note, we demonstrate that the generalization to  $GI/M/1$  is much simpler because the embedded Markov chain method works easily, in contrast to the  $M/G/1$  case. The crucial difference is as follows:

- (i) In the  $GI/M/1$  case, during the time between successive arrivals, state transitions arise either from service completions or from negative arrivals. These both decrease the queue length and can be superposed easily.
- (ii) In the  $M/G/1$  case, during the time between successive departures, there is a Poisson arrival process of positive customers increasing the queue length and another Poisson arrival process of negative customers decreasing it. The system therefore behaves like a transient  $M/M/1$  queue during this period.

We consider two types of Poisson negative arrivals under FCFS service discipline: a RCE removes one customer from the end of the waiting line; and a DST (disaster) removes the whole queue (Jain and Sigman (1996)).

**Remark 1.** The inclusion of RCH, which removes one customer from the head, is straightforward since it just ‘helps’ the server. Details about the inclusion of RCH and about the preemptive LCFS case can be found in Yang and Chae (1998).

The usual assumptions are as follows:

**Assumption 1.** Two independent Poisson processes generate RCEs and DSTs with respective rates  $\varepsilon$  and  $\delta$ . These processes are also independent of both arrival and service processes.

**Assumption 2.** RCE and DST cause no effect if the system is empty.

**Assumption 3.** Interarrival times are i.i.d. random variables with mean  $1/\lambda$ , and service times are i.i.d. exponential with rate  $\mu$ . Interarrival times are independent of service times.

Denote by  $N_n^A$  the number of customers in the system as seen by the  $n$ th arrival of a (positive) customer. Let  $g(t)$  and  $G^*(\theta)$  denote, respectively, the interarrival time density and its Laplace transform. Then, the embedded Markov chain  $\{N_n^A, n \geq 1\}$  has the following transition probabilities:

$$P_{i,i+1-j} = \int_0^\infty e^{-\delta t} \frac{\{(\mu + \varepsilon)t\}^j e^{-(\mu+\varepsilon)t}}{j!} g(t) dt, \quad j = 0, 1, \dots, i,$$

$$P_{i,0} = 1 - \sum_{j=0}^i P_{i,i+1-j}.$$

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The usual procedure yields the following steady-state probability distribution for the Markov chain:

$$\pi_k = (1 - \sigma)\sigma^k, \quad k \geq 0, \quad (1)$$

where  $\sigma$ ,  $0 < \sigma < 1$ , is the unique root of the equation  $x = G^*\{(\mu + \varepsilon)(1 - x) + \delta\}$ .

**Proposition 1.** Let  $P_k$  denote the long-run proportion of time that there are  $k$  customers in the system. Then,

$$\lambda\pi_{k-1} = (\mu + \varepsilon)P_k + \delta \sum_{n=k}^{\infty} P_n, \quad k \geq 1, \quad (2)$$

$$P_k = (1 - P_0)\pi_{k-1} = (1 - P_0)(1 - \sigma)\sigma^{k-1}, \quad k \geq 1, \quad (3)$$

$$1 - P_0 = \frac{\lambda(1 - \sigma)}{(\mu + \varepsilon)(1 - \sigma) + \delta}. \quad (4)$$

*Proof.* Equation (2) is obtained by the level-crossing analysis (see El-Taha and Stidham (1999, p. 98)), (3) by substituting (1) into (2) and solving for  $P_k$ , and (4) by normalization. (A rigorous proof can be found in Yang and Chae (1998).)

**Remark 2.** A sufficient stability condition is that  $\delta \neq 0$ . When  $\delta = 0$ , however, we require that  $\lambda > \mu + \varepsilon$ .

**Remark 3.** Let the number of customers in the (steady-state) system be denoted by  $N$ . Then its expected value is given as

$$E[N] = \sum_{k=1}^{\infty} kP_k = \frac{\lambda}{(\mu + \varepsilon)(1 - \sigma) + \delta}. \quad (5)$$

Next, we find the sojourn time distribution (as a Laplace transform) for an arriving customer, called the tagged customer (TC). TC will either depart the system after receiving his service or be removed by RCE or by DST. It is convenient, however, to regard RCE and DST as single servers: RCE serves the customer at the end of the queue at a rate  $\varepsilon$ ; and DST serves the whole queue together at a rate  $\delta$  (allowing new arrivals to join the service). Note that the service discipline of RCE is preemptive LCFS, whereas that of the 'real' server is FCFS.

Let  $T_{RE}$  denote TC's sojourn time in the fictitious setting such that DST is deactivated upon TC's arrival. Also, let  $T_D$  be an exponential random variable with a rate  $\delta$ . Then TC's sojourn time, denoted by  $W$ , is the minimum of  $T_{RE}$  and  $T_D$ . We can consider  $T_D$  as TC's sojourn time in another fictitious setting such that the real server and RCE are deactivated upon TC's arrival.

Now we analyze  $T_{RE}$ . Note that, during  $T_{RE}$ , the real server does not have a chance to serve those who arrive after TC, and RCE does not have a chance to serve those who arrived before TC. Moreover, by Assumption 1, the two service processes are independent; and by Assumption 3, customers' arrival epochs after TC's arrival are independent of those before TC's arrival. As a result, we can express  $T_{RE}$  as the minimum of  $T_R$  and  $T_E$ , where  $T_R$  denotes TC's sojourn time in the fictitious setting that RCE and DST are deactivated and  $T_E$  denotes TC's sojourn time in the fictitious setting that the real server and DST are deactivated, both upon TC's arrival. Consequently,  $W$  becomes the minimum of  $T_R$ ,  $T_E$ , and  $T_D$ .

**Remark 4.** We have  $T_E = \infty$  if  $\varepsilon = 0$ , and  $T_D = \infty$  if  $\delta = 0$ .

**Proposition 2.** Let  $T_R^*(\theta)$ ,  $T_E^*(\theta)$  and  $T_D^*(\theta)$  denote respective Laplace transforms of the density functions of  $T_R$ ,  $T_E$  and  $T_D$ . Then

$$T_R^*(\theta) = \frac{\mu(1 - \sigma)}{\theta + \mu(1 - \sigma)}, \tag{6}$$

$$T_E^*(\theta) = \frac{\varepsilon - \varepsilon B^*(\theta)}{\theta + \varepsilon - \varepsilon B^*(\theta)}, \tag{7}$$

$$T_D^*(\theta) = \frac{\delta}{\theta + \delta}, \tag{8}$$

where  $B^*(\theta)$  is the smallest root of the equation  $x = G^*(\theta + \varepsilon - \varepsilon x)$ .

*Proof.* Suppose RCE and DST are deactivated upon arrival of TC who sees  $N^A$  customers in the system. By (1),  $N^A + 1$  is distributed geometrically with a ‘success’ probability  $1 - \sigma$ . Thus,  $T_R$ , being the sum of  $N^A + 1$  i.i.d. exponential service times, is itself distributed exponential with a rate  $\mu(1 - \sigma)$ . This proves (6). Next, suppose we make only RCE active. Then  $T_E$  is identical to the busy period of the  $GI/M/1$  queue with a service rate  $\varepsilon$  since RCE serves customers as if the service discipline were preemptive LCFS. Thus we have (7) according to Takács (1962, p. 122). Finally, we have (8) since  $T_D$  is exponential with a rate  $\delta$ .

**Proposition 3.** The variables  $T_R$ ,  $T_E$ , and  $T_D$  are independent.

*Proof.* Let  $PP^b(\omega)$  and  $PP^a(\omega)$  denote Poisson processes with a rate  $\omega$  ‘before’ and ‘after’ TC’s arrival, respectively. Due to the property of independent increments (Wolff (1989), p. 70),  $PP^b(\omega)$  and  $PP^a(\omega)$  are independent. Also, let  $A^b$  and  $A^a$  denote sets of interarrival times (of real customers) ‘before’ and ‘after’ TC’s arrival, respectively. Since, by Assumption 3, interarrival times are i.i.d. random variables,  $A^b$  and  $A^a$  are independent. First,  $T_R$  is determined by  $PP^a(\mu)$  and  $N^A$ , and  $N^A$  is determined by  $PP^b(\mu)$ ,  $PP^b(\varepsilon)$ ,  $PP^b(\delta)$  and  $A^b$ . Note that  $N^A$  is determined by the past evolution of the system before RCE and DST are deactivated upon TC’s arrival. That is,  $N^A$  is determined by every factor that influences the past evolution. Next,  $T_E$  is determined only by  $PP^a(\varepsilon)$  and  $A^a$  since RCE’s service discipline is preemptive LCFS. Note that, by Assumptions 1 and 3, both  $PP^a(\varepsilon)$  and  $A^a$  are independent of each and every factor that determines  $T_R$ . Thus  $T_R$  and  $T_E$  are independent. Finally,  $T_D$  is independent of  $T_R$  and  $T_E$  since  $T_D$  is solely determined by  $PP^a(\delta)$ , which is independent of everything else.

Since  $T_R$  and  $T_D$  are independent exponential random variables, the minimum of the two, denoted by  $T_{RD}$ , is also exponential. Let  $\alpha$  denote the rate of  $T_{RD}$ , then

$$\alpha = \mu(1 - \sigma) + \delta. \tag{9}$$

**Remark 5.** It is well known that the following three have the same distribution;  $T_{RD}$ ,  $T_{RD}$  given that  $T_R < T_D$ , and  $T_{RD}$  given that  $T_R > T_D$ .

Now  $W$  becomes the minimum of  $T_E$  and  $T_{RD}$ . It is then straightforward to compute the distribution of  $W$  (as a Laplace transform  $W^*(\theta)$ ), since  $T_{RD}$  is distributed exponential and is independent of  $T_E$ .

**Proposition 4.** We have

$$W^*(\theta) = \frac{\alpha + \theta T_E^*(\theta + \alpha)}{\theta + \alpha}, \tag{10}$$

where  $T_E^*(\cdot)$  and  $\alpha$  are as given in (7) and (9).

*Proof.* Since  $W^*(\theta) = E[e^{-\theta \min\{T_E, T_{RD}\}}]$ , we can express  $W^*(\theta)$  as follows:

$$W^*(\theta) = P(T_E < T_{RD}) E[e^{-\theta T_E} | T_E < T_{RD}] + P(T_{RD} < T_E) E[e^{-\theta T_{RD}} | T_{RD} < T_E]. \tag{11}$$

Let  $T_{RD}^*(\theta) = E[e^{-\theta T_{RD}}] = \alpha/(\theta + \alpha)$ , then we obtain (10) by substituting the following into (11):

$$P(T_E < T_{RD}) = T_E^*(\alpha), \tag{12}$$

$$P(T_{RD} < T_E) = 1 - T_E^*(\alpha), \tag{13}$$

$$P(T_E < T_{RD}) E[e^{-\theta T_E} | T_E < T_{RD}] = T_E^*(\theta + \alpha), \tag{14}$$

$$\begin{aligned} P(T_{RD} < T_E) E[e^{-\theta T_{RD}} | T_{RD} < T_E] &= T_{RD}^*(\theta) - P(T_{RD} > T_E) E[e^{-\theta T_{RD}} | T_{RD} > T_E] \\ &= T_{RD}^*(\theta) - T_E^*(\theta + \alpha) T_{RD}^*(\theta). \end{aligned} \tag{15}$$

Note that (15) is obtained by the memoryless property of exponential  $T_{RD}$  and by (14).

**Remark 6.** It can be shown that (see Yang and Chae (1998)):

$$\begin{aligned} P(T_E < T_{RD}) &= P(T_E = \min\{T_R, T_E, T_D\}) = \frac{\varepsilon(1 - P_0)}{\lambda}, \\ P(T_R = \min\{T_R, T_E, T_D\}) &= \frac{\mu(1 - P_0)}{\lambda}, \\ P(T_D = \min\{T_R, T_E, T_D\}) &= \frac{\delta E[N]}{\lambda} = \delta E[W], \end{aligned}$$

where  $E[N]$  is as given in (5) and ‘ $E[N] = \lambda E[W]$ ’ confirms Little’s law.

**Remark 7.** By arguments similar to Remark 5 and by (12), (13) and (15),

$$\begin{aligned} E[e^{-\theta W} | T_R = \min\{T_R, T_E, T_D\}] &= E[e^{-\theta W} | T_D = \min\{T_R, T_E, T_D\}] \\ &= E[e^{-\theta W} | T_{RD} < T_E] \\ &= T_{RD}^*(\theta) \frac{1 - T_E^*(\theta + \alpha)}{1 - T_E^*(\alpha)}. \end{aligned}$$

Moreover, these conditional Laplace transforms turn out to be identical to the Laplace transform of the equilibrium density function of  $W$ . That is,

$$E[e^{-\theta W} | T_{RD} < T_E] = \frac{1 - W^*(\theta)}{\theta E[W]}. \tag{16}$$

Perhaps (16) is a natural consequence of the PASTA property (Wolff (1982)), yet it was unexpected and is interesting. If we knew (16), it could have been easier to obtain (10).

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