Proceedings of the Edinburgh Mathematical Society (1997) 40, 375–381 ©

A REMARK ON THE TENSOR PRODUCT OF TWO MAXIMAL OPERATOR SPACES

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(Received 20th September 1995)

Given a Banach space E, let us denote by Max(E) the largest operator space structure on E. Recently Paulsen–Pisier and, independently, Junge proved that for any Banach spaces E, F, $\mathrm{Max}(E) \overset{h}{\otimes} \mathrm{Max}(F) = \mathrm{Max}(E) \check{\otimes} \mathrm{Max}(F)$ isomorphically where $\overset{h}{\otimes}$ and $\check{\otimes}$ respectively denote the Haagerup tensor product and the spatial tensor product of operator spaces. In this paper we show that, in general, this equality does not hold completely isomorphically.

1991 *Mathematics subject classification:* 46B28, 46M05.

Let E be a Banach space. Among all the operator space structures on E, there is a largest one Max(E) usually called the maximal operator space structure on E. It can be defined as follows. For any $n, m \geq 1$ and any $n \times m$ matrix $x = [x_{ij}]$ with entries in E,

$$\|x\|_{M_{n,m}(\mathrm{Max}(E))} = \sup\{\|[T(x_{ij})]\|_{M_{n,m}(B(H))} / H \text{ is a Hilbert space, } T : E \to B(H) \text{ is a linear contraction}\} \tag{1}$$

Similarly there is a least operator space structure Min(E) on the space E defined by:

$$\|x\|_{M_{n,m}(\mathrm{Min}(E))} = \sup\{\|[\xi(x_{ij})]\|_{M_{n,m}} / \xi \in E^*, \|\xi\| \leq 1\} \tag{2}$$

These extremal operator space structures were introduced and then investigated in the papers [**3**, **2**, **8**]. Recently Paulsen–Pisier in a joint work included in [**9**] and, independently, Junge (unpublished) proved that for any Banach spaces E, F:

$$\mathrm{Max}(E) \overset{h}{\otimes} \mathrm{Max}(F) = \mathrm{Max}(E) \check{\otimes} \mathrm{Max}(F) \qquad \text{isomorphically} \tag{3}$$

where $\overset{h}{\otimes}$ and $\check{\otimes}$ respectively denote the Haagerup tensor product and the spatial tensor product of operator spaces. More precisely for any $u \in \mathrm{Max}(E) \otimes \mathrm{Max}(F)$,

$$\|u\|_{\mathrm{Max}(E)\check{\otimes}\mathrm{Max}(F)} \leq \|u\|_{\mathrm{Max}(E)\overset{h}{\otimes}\mathrm{Max}(F)} \tag{4(i)}$$

$$\|u\|_{\mathrm{Max}(E)\overset{h}{\otimes}\mathrm{Max}(F)} \leq 2\|u\|_{\mathrm{Max}(E)\check{\otimes}\mathrm{Max}(F)} \tag{4(ii)}$$

(The proof in [9] yields this constant 2 whereas Junge's proof yields a worse constant.) Since $\mathrm{Max}(E) \overset{h}{\otimes} \mathrm{Max}(F)$ and $\mathrm{Max}(E) \check{\otimes} \mathrm{Max}(F)$ can be regarded as operator spaces, it is natural to consider the inequalities (4) at the matrix level. The resulting problem, which was pointed out by Paulsen in [9], reads as follows:

$$\text{Does (3) hold completely isomorphically?} \tag{5}$$

that is, does there exist two constants $C_1, C_2 \geq 1$ such that for any $n \geq 1$ and any $u \in M_n \otimes \mathrm{Max}(E) \otimes \mathrm{Max}(F)$,

$$\|u\|_{M_n\left(\mathrm{Max}(E)\check{\otimes}\mathrm{Max}(F)\right)} \leq C_1\|u\|_{M_n\left(\mathrm{Max}(E)\overset{h}{\otimes}\mathrm{Max}(F)\right)} \tag{6(i)}$$

$$\|u\|_{M_n\left(\mathrm{Max}(E)\overset{h}{\otimes}\mathrm{Max}(F)\right)} \leq C_2\|u\|_{M_n\left(\mathrm{Max}(E)\check{\otimes}\mathrm{Max}(F)\right)}. \tag{6(ii)}$$

In fact the first inequality (6(i)) holds as (4(i)) with $C_1 = 1$. This is the consequence of the general fact that for all operator spaces X, Y, we have a completely contractive inclusion

$$X \overset{h}{\otimes} Y \to X \check{\otimes} Y \tag{7}$$

Thus the significant part of problem (5) is the existence of C_2 such that (6(ii)) is satisfied.

The aim of this note is to provide the following negative answer to (5).

Theorem. *Let E and F be two Banach spaces. Then the following are equivalent.*

(i) $\mathrm{Max}(E) \overset{h}{\otimes} \mathrm{Max}(F) = \mathrm{Max}(E) \check{\otimes} \mathrm{Max}(F)$ *completely isomorphically.*

(ii) *E or F is finite-dimensional.*

We will assume the reader is familiar with the basic notions of the theory of operator spaces and use traditional notation. We especially denote by $\| \|_{cb}$ the completely bounded norm of a linear map between operator spaces and by $R_n(= M_{1,n})$ and $C_n(= M_{n,1})$ the row and column operator space structures on the n-dimensional Hilbert space ℓ_2^n. We recall that given two operator spaces X, Y and $u \in X \otimes Y$, we have by definition:

$$\|u\|_{X\overset{h}{\otimes}Y} = \inf\left\{ \|(x_1, \ldots, x_N)\|_{R_N(X)} \left\| \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \right\|_{C_N(Y)} \right\} \tag{8}$$

where the infimum runs over all $N \geq 1$ and all families $(x_j)_{j\leq N}$ in X, $(y_j)_{j\leq N}$ in Y satisfying $u = \sum_{j=1}^N x_j \otimes y_j$.

We also recall that for any Banach space E, $(\mathrm{Max}(E))^* = \mathrm{Min}(E^*)$ in the sense of the duality theory of operator spaces [2, 3, 6]. It then follows from the self-duality of the Haagerup tensor product [7] that given two Banach spaces E, F:

$$\mathrm{Min}(E^*) \overset{h}{\otimes} \mathrm{Min}(F^*) \subset (\mathrm{Max}(E) \overset{h}{\otimes} \mathrm{Max}(F))^* \quad \text{completely isometrically} \tag{9}$$

For undefined notions and terminology we refer the reader to [1, 2, 3, 6, 7]. The proof of our theorem will be mainly based on the following

Proposition. *Let ℓ_2^n denote the n-dimensional Hilbert space. Let $\tau_n : \mathrm{Min}(\ell_2^n) \overset{h}{\otimes} \mathrm{Min}(\ell_2^n) \to \mathrm{Min}(\ell_2^n) \overset{h}{\otimes} \mathrm{Min}(\ell_2^n)$ be the flip isomorphism defined by $\tau_n(x \otimes y) = y \otimes x$ where $x, y \in \ell_2^n$. Then: $\|\tau_n\|_{cb} \geq \sqrt{n}$.*

Proof of proposition. We shall use tensor products of Banach spaces. Following the usual notation (see e.g. [10]), we denote by $\overset{\vee}{\otimes}$ the injective tensor product and by $\overset{\gamma_2}{\otimes}$ the Grothendieck tensor product corresponding to factorization through Hilbert space. Namely let A, B be two Banach spaces and let $u \in A \otimes B$. Then:

$$\|u\|_{A \overset{\gamma_2}{\otimes} B} = \inf\{\|\alpha\| \, \|\beta\|\}$$

where the infimum runs over all linear maps $\alpha : \ell_2 \to A, \beta : \ell_2 \to B$ such that the composed map $\beta\alpha^* : A^* \to B$ represents u.

We begin with an identification lemma. It should be noticed that in the case $n = m = 1$, the following statement reduces to [3, Proposition 4.1].

Lemma. *Let E, F be Banach spaces. Given integers $n, m \geq 1$, let us denote by $(E_{ij})_{\substack{i \leq n \\ j \leq m}}$ the canonical basis of $M_{n,m}$. We also denote by $(e_i)_{i \geq 1}$ the canonical basis of ℓ_2. Then:*

$$M_{n,m}(\mathrm{Min}(E) \overset{h}{\otimes} \mathrm{Min}(F)) = (\ell_2^n \overset{\vee}{\otimes} E) \overset{\gamma_2}{\otimes} (F \overset{\vee}{\otimes} \ell_2^m) \quad \textit{isometrically} \tag{10}$$

under the identification:

$$E_{ij} \otimes x \otimes y \longleftrightarrow (e_i \otimes x) \otimes (y \otimes e_j) \quad \textit{with } x \in E, y \in F \tag{11}$$

Proof of lemma. As mentioned above, (10) was proved with $n = m = 1$ in [3] thus:

$$\mathrm{Min}(E_1) \overset{h}{\otimes} \mathrm{Min}(F_1) = E_1 \overset{\gamma_2}{\otimes} F_1$$

isometrically for any Banach spaces E_1, F_1. Applying this with $E_1 = \ell_2^n \overset{\vee}{\otimes} E$ and $F_1 = F \overset{\vee}{\otimes} \ell_2^m$, we obtain:

$$(\ell_2^n \overset{\vee}{\otimes} E) \overset{\gamma_2}{\otimes} (F \overset{\vee}{\otimes} \ell_2^m) = \mathrm{Min}(\ell_2^n \overset{\vee}{\otimes} E) \overset{h}{\otimes} \mathrm{Min}(F \overset{\vee}{\otimes} \ell_2^m). \tag{12}$$

Furthermore recall that we have an isometric identification

$$M_{n,m}(X) = C_n \overset{h}{\otimes} X \overset{h}{\otimes} R_m \tag{13}$$

valid for any operator space X. Applying this with X equal to $\mathrm{Min}(E)$, $\mathrm{Min}(F)$ or $\mathrm{Min}(E) \overset{h}{\otimes} \mathrm{Min}(F)$ we see that (11) gives rise to:

$$M_{n,m}(\mathrm{Min}(E) \overset{h}{\otimes} \mathrm{Min}(F)) = C_n(\mathrm{Min}(E)) \overset{h}{\otimes} R_m(\mathrm{Min}(F)). \tag{14}$$

In view of (8), (12) and (14) it then suffices to check that for any $N \geq 1$:

$$R_N(C_n(\mathrm{Min}(E))) = R_N(\mathrm{Min}(\ell_2^n \overset{\vee}{\otimes} E)), \tag{15}$$

$$C_N(R_m(\mathrm{Min}(F))) = C_N(\mathrm{Min}(F \overset{\vee}{\otimes} \ell_2^m)). \tag{16}$$

Now we have isometrically:

$$\begin{aligned} R_N(C_n(\mathrm{Min}(E))) &= M_{n,N}(\mathrm{Min}(E)) \quad \text{by (13)} \\ &= M_{n,N} \overset{\vee}{\otimes} E \quad \text{by (2)}. \end{aligned}$$

Writing $M_{n,N} = \ell_2^N \overset{\vee}{\otimes} \ell_2^n$, we thus obtain $R_N(C_n(\mathrm{Min}(E))) = \ell_2^N \overset{\vee}{\otimes} (\ell_2^n \overset{\vee}{\otimes} E)$ whence (15) by (2) again. The proof of (16) is similar. □

End of the proof of proposition. Let us apply the lemma with $n = m$, $E = F = \ell_2^n$. We thus have:

$$M_n(\mathrm{Min}(\ell_2^n) \overset{h}{\otimes} \mathrm{Min}(\ell_2^n)) = (\ell_2^n \overset{\vee}{\otimes} \ell_2^n) \overset{\gamma_2}{\otimes} (\ell_2^n \overset{\vee}{\otimes} \ell_2^n). \tag{17}$$

Let us denote by Z_n the Banach space in the right hand of (17). Under the identification (17), the tensor map $Id_{M_n} \otimes \tau_n$ corresponds to the linear map:

$$\theta_n : Z_n \to Z_n, \qquad \theta_n(e_i \otimes e_j \otimes e_k \otimes e_\ell) = e_i \otimes e_k \otimes e_j \otimes e_\ell.$$

Let $t = \sum_{1 \leq i,j \leq n} t_{ij} e_i \otimes e_j$ (with $t_{ij} \in \mathbb{C}$) and let $T = \sum_{1 \leq i,j \leq n} t_{ij} e_i \otimes e_j \otimes e_i \otimes e_j \in Z_n$. We claim that:

$$\|T\|_{Z_n} \leq \|t\|_{\ell_\infty^n \overset{\vee}{\otimes} \ell_\infty^n}, \tag{18}$$

https://doi.org/10.1017/S0013091500023816 Published online by Cambridge University Press

$$\|\theta_n(T)\|_{Z_n} = \|t\|_{\ell_\infty^n \overset{\gamma_2}{\otimes} \ell_\infty^n}. \tag{19}$$

It therefore follows that

$$\|\theta_n\| \geq \|Id : \ell_\infty^n \overset{\vee}{\otimes} \ell_\infty^n \to \ell_\infty^n \overset{\gamma_2}{\otimes} \ell_\infty^n\|. \tag{20}$$

It is well-known that $c_0 \overset{\vee}{\otimes} c_0 \neq c_0 \overset{\gamma_2}{\otimes} c_0$ hence the right hand side of (20) tends to $+\infty$ when n grows to $+\infty$. More specifically, the description of $\ell_\infty^n \overset{\gamma_2}{\otimes} \ell_\infty^n$ as the Banach space of Schur multipliers on M_n (see e.g. [**11**, Theorem 5.1]) shows that the norm $\|Id : \ell_\infty^n \overset{\vee}{\otimes} \ell_\infty^n \to \ell_\infty^n \overset{\gamma_2}{\otimes} \ell_\infty^n\|$ is the unconditional constant of the canonical basis $(E_{ij})_{\substack{i\leq n\\ j\leq n}}$ of M_n. This constant is known to be $\sqrt{n}$, see e.g. [**10**, Chapter 8]. Since $\|\tau_n\|_{cb} \geq \|\theta_n\|$, this yields the result. It thus remains to check (18) and (19).

Since $\ell_2^n \overset{\vee}{\otimes} \ell_2^n = M_n$, we can write $Z_n = M_n \overset{\gamma_2}{\otimes} M_n$. In this setting, T corresponds to the Schur product $\sigma : (M_n)^* \to M_n$ defined by $\sigma([a_{ij}]) = [t_{ij}a_{ij}]$. Let HS_n be the vector space M_n endowed with the Hilbert–Schmidt norm $\|[a_{ij}]\|_{HS_n} = (\sum_{1\leq i,j\leq n} |a_{ij}|^2)^{1/2}$. Then HS_n is a Hilbert space and:

$$\|\sigma : HS_n \to HS_n\| = \text{Sup}\{|t_{ij}| / 1 \leq i, j \leq n\}. \tag{21}$$

Since the identity maps $M_n^* \to HS_n$ and $HS_n \to M_n$ are contractive, the equality (21) leads to:

$$\|T\|_{Z_n} \leq \text{Sup}\{|t_{ij}| / 1 \leq i, j \leq n\}.$$

Since $\ell_\infty^{n^2} = \ell_\infty^n \overset{\vee}{\otimes} \ell_\infty^n$, this proves (18).

We now turn to (19). By definition

$$\theta_n(T) = \sum_{1\leq i,j\leq n} t_{ij} e_i \otimes e_i \otimes e_j \otimes e_j.$$

Observe that the mapping $e_i \mapsto e_i \otimes e_i$ induces an isometry from ℓ_∞^n into $M_n = \ell_2^n \overset{\vee}{\otimes} \ell_2^n$ which maps ℓ_∞^n onto the space of diagonal matrices. Hence (19) follows from the injectivity of the tensor norm $\overset{\gamma_2}{\otimes}$. □

Proof of theorem. (i) $\Rightarrow$ (ii): Assume (i). As an early result of the theory of operator spaces it has been proved that the spatial tensor product is commutative [**3**]. This means that for any operator spaces X, Y, the flip map $x \otimes y \mapsto y \otimes x$ is a completely isometric isomorphism from $X \check{\otimes} Y$ onto $Y \check{\otimes} X$. Applying this with $X = \text{Max}(E)$ and $Y = \text{Max}(F)$, our assumption yields a complete isomorphism between $\text{Max}(E) \otimes_h \text{Max}(F)$ and $\text{Max}(F) \otimes_h \text{Max}(E)$. Let τ be the flip map on $\text{Min}(F^*) \otimes \text{Min}(E^*)$. By duality (see (9)),

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$$\tau : \text{Min}(F^*) \overset{h}{\otimes} \text{Min}(E^*) \to \text{Min}(E^*) \overset{h}{\otimes} \text{Min}(F^*)$$

is a completely bounded map. Since the Haagerup tensor product $\overset{h}{\otimes}$ is injective, we thus deduce that there is a constant $K > 0$ such that for any pair of subspaces $A \subset E^*$, $B \subset F^*$, the restriction map $\tau_{B\otimes A}$ of τ to the subspace $B \otimes A$ satisfies:

$$\|\tau_{B\otimes A} : \text{Min}(B) \overset{h}{\otimes} \text{Min}(A) \to \text{Min}(A) \overset{h}{\otimes} \text{Min}(B)\|_{cb} \leq K. \tag{22}$$

Now assume that E and F are infinite-dimensional. Then by Dvoretsky's theorem [**4**], E^* contains subspaces of arbitrary dimension which are uniformly isomorphic to Hilbert spaces. More precisely, for any $n \geq 1$, there exist $A_n \subset E^*$ with $\dim A_n = n$ and an isomorphism $\alpha_n : A_n \to \ell_2^n$ which satisfies, say, $\|\alpha_n\| \, \|\alpha_n^{-1}\| \leq 2$. Similarly, there are subspaces $B_n \subset F^*$ and isomorphisms $\beta_n : B_n \to \ell_2^n$ with $\|\beta_n\| \, \|\beta_n^{-1}\| \leq 2$. Then the inequality (22) and an obvious composition of maps yields:

$$\|\tau_n : \text{Min}(\ell_2^n) \overset{h}{\otimes} \text{Min}(\ell_2^n) \to \text{Min}(\ell_2^n) \overset{h}{\otimes} \text{Min}(\ell_2^n)\| \leq K\|\alpha_n\| \, \|\alpha_n^{-1}\| \, \|\beta_n\| \, \|\beta_n^{-1}\| \leq 4K.$$

This contradicts our proposition, whence the result.

(ii) $\Rightarrow$ (i); this is the easy implication. Note that by (7), we only have to check that the identity map $Id : \text{Max}(E) \overset{\vee}{\otimes} \text{Max}(F) \to \text{Max}(E) \overset{h}{\otimes} \text{Max}(F)$ is completely bounded. We will use the fact that for any operator space X and any $n \geq 1$:

$$\text{The identity map } Id : R_n \overset{h}{\otimes} X \to X \overset{h}{\otimes} R_n \text{ is completely contractive} \tag{23}$$

(see [**7**, Theorem 4.3]).

Now assume for instance that F is finite dimensional. Then observe that there is a constant $C > 0$ (only depending on the dimension of F) such that for any operator space X

$$\|Id : X \overset{\vee}{\otimes} \text{Max}(F) \to X \overset{h}{\otimes} \text{Max}(F)\| \leq C. \tag{24}$$

Let $n \geq 1$. The following identity maps are either isometries or bounded maps with norms not depending on n.

$$\begin{aligned} M_n(\text{Max}(E) \overset{\vee}{\otimes} \text{Max}(F)) &= M_n(\text{Max}(E)) \overset{\vee}{\otimes} \text{Max}(F) \quad \text{by associativity} \\ &\to M_n(\text{Max}(E)) \overset{h}{\otimes} \text{Max}(F) \quad \text{by (24)} \\ &= C_n \overset{h}{\otimes} \text{Max}(E) \overset{h}{\otimes} R_n \overset{h}{\otimes} \text{Max}(F) \quad \text{by (13)} \\ &\to C_n \overset{h}{\otimes} \text{Max}(E) \overset{h}{\otimes} \text{Max}(F) \overset{h}{\otimes} R_n \quad \text{by (23)} \\ &= M_n(\text{Max}(E) \overset{h}{\otimes} \text{Max}(F)) \quad \text{by (13).} \end{aligned}$$

This completes the proof. □

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