Can. J. Math., Vol. XXX, No. 6, 1978, pp. 1166–1173

A NOTE ON A CLASS OF SLIT CONFORMAL MAPPINGS

W. E. KIRWAN AND R. W. PELL

1. Introduction. We denote by S the class of functions, $f(z)$, that are analytic and univalent in $U = \{z\colon |z| < 1\}$ and have the normalization

$$f(z) = z + a_2 z^2 + \dots \quad (|z| < 1).$$

Of particular interest in studying extremal problems in S is the subclass M, consisting of functions that map U onto the complement of a single Jordan arc extending from some finite point to ∞ and along which $|w|$ is strictly increasing. Indeed, L. Brickman [**3**] has shown that if $f \in S$ is an extreme point of S, then $f \in M$. Furthermore, A. Pfluger [**9**] (see also L. Brickman and D. Wilken [**4**]) has shown that if $f_0 \in S$ and

$$\max_{f \in S} \operatorname{Re} L(f) = \operatorname{Re} L(f_0)$$

for some non-trivial continuous linear functional on S (i.e., f_0 is a support point of S), then f_0 satisfies the so-called $\pi/4$ property which implies in particular that $f_0 \in M$.

Recently, W. Hengartner and G. Schober [**5**] have shown that extreme points, $f(z)$, of S have the remarkable property that $f(z)/z$ is $1 - 1$ in U. In § 3 of this note we generalize this result considerably. We show that for each $f \in M$ there is a family of linear fractional transformations of the disc, $L_\lambda(z)$ $(\lambda \geqq 1)$, such that

$$f_\lambda(z) = \frac{f(z)}{z} L_\lambda(z) \quad (\lambda \geqq 1)$$

is $1 - 1$ and analytic in U. If $\lambda \to +\infty$, $L_\lambda(z) \to$ constant, and so we obtain as a special case the Hengartner-Schober result. Consideration of the family of functions $\{f_\lambda(z)\}$ enables us to embed each $f \in M$ in an explicit subordination chain (see [**10**] and [**11**, chapter 6]). Through an application of C. Pommerenke's version of the Löwner equation, we obtain in § 4 some coefficient relations for functions in M which parallel some recent results of Hengartner and Schober [**5**] and [**6**].

Finally, we note that if $f \in M$, then f has a continuous extension to $\gamma = \{z\colon z = e^{i\theta}, 0 \leqq \theta \leqq 2\pi\}$ as a mapping from $U \cup \gamma$ to the Riemann sphere and $f(\gamma) = \text{boundary} f(U)$. We denote by p and q the points on γ which correspond under $w = f(z)$ to the finite endpoint of $f(\gamma)$ and ∞ respectively.

Received June 15, 1977 and in revised form, March 17, 1978. This research was supported in part by a National Science Foundation Grant.

2. A univalence criterion. A classical criterion for the univalence of a function $h(z)$ that is analytic in U states that if in addition h is continuous in $U \cup \gamma$ and $h(\gamma)$ is a Jordan curve, then h is $1 - 1$ in U. It would be natural to assume that this result remains true if, rather than being analytic, one allows f to have a simple pole in U (see [**8**, p. 139] where in fact such a result is stated). However, the function

$$\pi(z) = \frac{z - \epsilon}{1 - \bar{\epsilon} z} \frac{(1 - z)^2}{z} \quad (|\epsilon| < 1)$$

shows that this is not the case. Indeed a simple computation shows that π is $1 - 1$ on γ but clearly π cannot be $1 - 1$ in U since $\pi(\epsilon) = 0 = \pi(1)$. A correct generalization of the above univalence criterion was proved by L. A. Aksent'ev [**1**] (see also [**2**]). We state a special case of Aksent'ev's theorem in the form we will need it in § 3.

THEOREM 1 (Aksent'ev). *Let $g(z)$ be analytic in U except for a simple pole at $z_0 \in U$. Suppose $g(z)$ is continuous in $(U \cup \gamma) - \{z_0\}$ and $1 - 1$ on γ. If the Jordan curve $g(\gamma)$ has negative orientation, then g is $1 - 1$ in U.*

3. A family of univalent functions. Before proving the main theorem of this section, we recall some elementary facts concerning linear fractional transformations.

Let p and q be two distinct points on the unit circle. We need to determine the family of linear fractional transformations each of which fixes p and q, and has p as a *repulsive* fixed point and q as an *attractive* fixed point. For this purpose it is convenient to map U onto the upper half-plane sending p to 0 and q to ∞. Such a map is given by

$$S(z) = A \frac{z - p}{z - q} \quad \text{where } A = \sqrt{q\bar{p}}$$

with the root chosen so that $\operatorname{Im} \sqrt{q\bar{p}} < 0$.

The family of linear fractional transformations that fixes 0 and ∞ and preserve the upper half-plane is given by $w \to \lambda w$ with $\lambda > 0$. Such a transformation $(w \to \lambda w)$ has $w = 0$ as a repulsive fixed point and $w = \infty$ as an attractive fixed point if and only if $\lambda \geqq 1$. Thus the family of linear fractional transformations that preserves the disc U, fixes p and q, and has p and q as repulsive and attractive fixed points respectively is given by

$$L_\lambda(z) = S^{-1}[\lambda S(z)] \quad (\lambda \geqq 1).$$

A simple calculation yields

$$(3.1) \quad L_\lambda(z) = \frac{p - q\lambda}{p\lambda - q} \left[\frac{z + \dfrac{pq(\lambda - 1)}{p - q\lambda}}{1 + \dfrac{\lambda - 1}{q - p\lambda} z} \right].$$

Since

$$\left|\frac{p - q\lambda}{p\lambda - q}\right| = \left|\frac{p\bar{q} - \lambda}{\lambda - q\bar{p}}\right| = 1,$$

$$\left|\frac{pq(\lambda - 1)}{p - q\lambda}\right| = \frac{\lambda - 1}{|p\bar{q} - \lambda|} \leqq \frac{\lambda - 1}{\lambda - |p\bar{q}|} = 1, \text{ and}$$

$$\left[\overline{\frac{pq(\lambda - 1)}{p - q\lambda}}\right] = \frac{\bar{p}\bar{q}(\lambda - 1)}{\bar{p} - \bar{q}\lambda} = \frac{\lambda - 1}{q - p\lambda},$$

$L_\lambda(z)$ is in the standard form,

$$L_\lambda(z) = A\,\frac{z + \epsilon}{1 + \bar{\epsilon}z}$$

with $|A| = 1$ and $|\epsilon| < 1$, for linear fractional transformations that preserve U.

Let γ_1 be the open arc of γ moving counterclockwise from p to q and let γ_2 be the open arc obtained by moving counterclockwise from q to p.

Observe that $L_\lambda(z)$ moves points on γ_1 in a counterclockwise direction and points on γ_2 in a clockwise direction. Furthermore, if we fix z_1 on γ_1 (or γ_2) and let λ increase, it is clear that $\arg L_\lambda(z_1)$ increases on γ_1 and decreases on γ_2.

We now state and prove

THEOREM 2. *Let $f \in M$ and let $L_\lambda(z)$ be given by* (3.1) *where p and q are the distinct points on the unit circle such that $f(p)$ is the finite endpoint of $f(\gamma)$ and $f(q) = \infty$. Assume also that $\lambda \geqq 1$. Then*

$$f_\lambda(z) = \frac{f(z)}{z}\,L_\lambda(z)$$

is $1 - 1$ and analytic in U.

Proof. The function $f_\lambda(z)$ is clearly analytic in U. We will show that $f_\lambda(z)$ is $1 - 1$ on γ ($\lambda \neq 1$) and will then apply Theorem 1 to the function $g_\lambda(z) = 1/f_\lambda(z)$ in order to show that $g_\lambda(z)$ (and therefore $f_\lambda(z)$) is $1 - 1$ in U.

Suppose $f_\lambda(z_1) = f_\lambda(z_2)$ with $|z_1| = 1 = |z_2|$ and $z_1 \neq z_2$, i.e., we assume

(3.2) $\quad f(z_2)/f(z_1) = L_\lambda(z_1)z_2/L_\lambda(z_2)z_1.$

The right hand side of (3.2) has modulus one and so $|f(z_1)| = |f(z_2)|$. But, the monotonic property of $f(\gamma)$ then implies that $f(z_1) = f(z_2)$, that neither z_1 nor z_2 can equal p or q, and that z_1 and z_2 cannot both lie on the same arc γ_1 or γ_2. For definiteness suppose that $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$. Then since $f(z_1) = f(z_2)$, we have from (3.2)

$$L_\lambda(z_1)z_2 = L_\lambda(z_2)z_1$$

and so

(3.3) $\quad \arg z_2 - \arg z_1 = \arg L_\lambda(z_2) - \arg L_\lambda(z_1)$

where we have chosen a branch of arg that is continuous and strictly increasing as z moves in the counterclockwise direction on $\gamma - \{p\}$. However, (3.3) is impossible since $L_\lambda(z)$ moves points on γ_1 in a counterclockwise direction and points on γ_2 in a clockwise direction. We have obtained a contradiction to (3.2) and so f_λ is $1 - 1$ on γ.

Next consider $g_\lambda(z)$. This function is analytic in U except for a simple pole at $z = pq(1 - \lambda)/p - q\lambda$ and is continuous on γ. Therefore, to show that $g_\lambda(z)$ is $1 - 1$ in U it is enough by Theorem 1 to show that $g_\lambda(\gamma)$ is negatively oriented. To see that the Jordan curve $g_\lambda(\gamma)$ is negatively oriented, we consider the Jordan curve $\Gamma_\lambda = f_\lambda(\gamma)$ which passes through the point at ∞. We show that for $|\xi| = 1$, $\xi \neq p, q$, 0 and $f(\xi)$ lie in opposite components of Γ_λ and that Γ_λ is negatively oriented with respect to the points $f(\xi)$. We then obtain upon inversion that the plane Jordan curve $g_\lambda(\gamma)$ has negative orientation with respect to the points $g(\xi) = 1/f(\xi)$ which lie in its interior. As noted above, an application of Theorem 1 will then complete the proof.

First note that $|L_\lambda(\xi)/\xi| = 1$ for $\xi \in \gamma$ and that the end points $f(p)$ and $f(q)$ are fixed under the transformation $w = f_\lambda(\xi)$. Also recall that $L_\lambda(\xi)$ moves points counterclockwise on γ_1 and clockwise on γ_2. Now for each $\xi \in \gamma$

$$f_\lambda(\xi) = f(\xi)\,\frac{L_\lambda(\xi)}{\xi}$$

and hence $f_\lambda(\xi)$ is obtained from $f(\xi)$ upon multiplication of $f(\xi)$ by the complex number $L_\lambda(\xi)/\xi$ whose modulus is one. On γ_1,

$$0 < \arg L_\lambda(\xi) - \arg \xi < \arg q - \arg p$$

and on γ_2,

$$\arg q - \arg p - 2\pi < \arg L_\lambda(\xi) - \arg \xi < 0.$$

That is, for ξ on γ_1, points $f_\lambda(\xi)$ on Γ_λ are obtained by rotating the points $f(\xi)$ in a counterclockwise direction by an amount less than $\arg q - \arg p$, whereas, for $\xi \in \gamma_2$ points $f_\lambda(\xi)$ on Γ_λ are obtained by rotating the points $f(\xi)$ in a clockwise direction by an amount less than $2\pi - (\arg q - \arg p)$. It follows that for $|\xi| = 1$, $\xi \neq p, q$, $f(\xi)$ and 0 are in opposite components of the Jordan curve Γ_λ and that as ξ traverses γ starting say from $\xi = p$, the points $f(\xi)$ lie to the right of $f_\lambda(\xi)$, i.e., Γ_λ is negatively oriented with respect to the points $f(\xi)$. As noted in the previous paragraph, this fact suffices to complete the proof of the theorem.

Note. For future reference we observe that

$$\arg L_\lambda(\xi) - \arg \xi \quad (\lambda \geqq 1)$$

is monotonically increasing as a function of λ for each fixed $\xi \in \gamma_1$ and monotonically decreasing in λ for each fixed $\xi \in \gamma_2$. It follows that if $1 \leqq \lambda_1 < \lambda_2$, $f_{\lambda_1}(U) \supset f_{\lambda_2}(U)$. This observation will be important in the next section.

It follows immediately from the definition of $L_\lambda(z)$, (3.1), that as $\lambda \to \infty$, $L_\lambda(\xi) \to q$. This observation establishes the following

COROLLARY. *If* $f \in M$, *then* $f(z)/z$ *is* $1 - 1$ *in* U.

As noted above, this result was first proved by Hengartner and Schober [**5**].

4. Applications. Before applying Theorem 2 we need to recall the definition of a subordination chain [**10**].

Definition. Let I be a real finite or semi-finite interval. Then $f(z, t)$, $z \in U$, $t \in I$, is called a *subordination chain* if

(i) $f(z, t)$ is analytic and univalent for $z \in U$ and for each fixed $t \in I$;
(ii) $f'(0, t)$ is continuous as a function of $t \in I$; and
(iii) $t_1, t_2 \in I$, $t_1 \leqq t_2$ implies $f(z, t_1)$ is subordinate to $f(z, t_2)$.

Pommerenke [**10**] introduced the concept of a subordination chain and used it to develop the following version of a theorem of K. Löwner [**7**].

THEOREM 3. (Pommerenke). *Let* $f(z, t)$ *be a subordination chain on* I. *Then there exists a function* $p(z, t)$ *analytic in* z *for* $z \in U$, $t \in I$ *with* Re $p(z, t) \geqq 0$ *for all* $z \in U$, $t \in I$, *measurable for* $t \in I$ *and such that*

$$\frac{\partial f(z, t)}{\partial t} = zp(z, t)f'(z, t)$$

for each $z \in U$ *and almost all* $t \in I$.

It should be noted here that we are intentionally not assuming that the subordination chain is normalized, i.e., that $f'(0, t) = e^t$. For normalized chains one knows that $p(0, t) = 1$ for almost all $t \in I$. For general chains, one only has Re $p(0, t) \geqq 0$ for almost all $t \in I$.

We now show how Theorem 2 enables us to construct an explicit subordination chain for functions in M.

As pointed out in the note at the end of Theorem 2

(4.1) $\quad f_{\lambda_1}(U) \supset f_{\lambda_2}(U) \quad$ if $1 \leqq \lambda_1 < \lambda_2$.

Composing $f_\lambda(z)$ with $L_\lambda^{-1}(z)$ we obtain

$$f_\lambda[L_\lambda^{-1}(z)] = z\frac{f(L_\lambda^{-1}(z))}{L_\lambda^{-1}(z)}, \quad \text{and } f_\lambda \circ L_\lambda^{-1}(0) = 0.$$

By (4.1)

$$f_{\lambda_2} \circ L_{\lambda_2}^{-1}(U) \subset f_{\lambda_1} \circ L_{\lambda_1}^{-1}(U) \quad (1 \leqq \lambda_1 < \lambda_2),$$

i.e., $f_{\lambda_2} \circ L_{\lambda_2}^{-1}$ is subordinate to $f_{\lambda_1} \circ L_{\lambda_1}^{-1}$. Set $t = 1/\lambda$ and define

(4.2) $\quad \hat{f}(z, t) = f_\lambda \circ L_\lambda^{-1}(z)$.

It follows that $\hat{f}(z, t)$ is a subordination chain. Using the definition of $f_\lambda(z)$

we may write (4.2) in the form

$$\hat{f}(z, t) = \frac{f(L_\lambda^{-1}(z))}{L_\lambda^{-1}(z)} z \quad (\lambda = 1/t)$$

where

$$L_\lambda(z) = A(t)\frac{z + \epsilon(t)}{1 + \bar{\epsilon}(t)z} \quad (\lambda = 1/t),$$

$$A(t) = \frac{q - pt}{qt - p},$$

$$\epsilon(t) = \frac{pq(t-1)}{q - pt}, \quad \text{and}$$

$$L_\lambda^{-1}(z) = \frac{\bar{A}(t)z - \epsilon(t)}{1 - \bar{\epsilon}(t)\bar{A}(t)z}.$$

If we set $f(z, t) = \hat{f}(A(t)z, t)$, then $f(z, t)$ is also a subordination chain on $0 \leqq t \leqq 1$ with

$$f(z, 0) = \lim_{t \to 0} \hat{f}(A(t)z, t) = \frac{f(p)q}{-p^2} z, f(z, 1) = f(z), \quad \text{and}$$

$$(4.3) \quad f(z, t) = \frac{f\left[\dfrac{z - \epsilon(t)}{1 - \bar{\epsilon}(t)z}\right]}{\left[\dfrac{z - \epsilon(t)}{1 - \bar{\epsilon}(t)z}\right]} A(t)z.$$

By Pommerenke's theorem (Theorem 3),

$$(4.4) \quad \frac{\partial f(z, t)}{\partial t} = zp(z, t)f'(z, t)$$

where $p(z, t)$ is analytic in z with positive real part for $z \in U$ and $t \in [0, 1]$.

Computing $\partial f(z, t)/\partial t$ from (4.3), solving for $p(z, t)$ in (4.4), setting $t = 1$, and using the fact that $\operatorname{Re}\, (p + q)/(p - q) = 0$ we obtain the following property of functions in M.

THEOREM 4. *Let $f(z) \in M$. Let p and q be defined as in Theorem 2. Then*

$$\operatorname{Re} p(z, 1) = \operatorname{Re} \frac{(z - p)(z - q)}{(p - q)z}\left[1 - \frac{f(z)}{zf'(z)}\right] > 0$$

for all $z \in U$.

COROLLARY. *Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots \in M$. Let p and q be defined as in Theorem 2. Then*

1) $\operatorname{Re}\left(\dfrac{a_2pq}{p - q}\right) > 0,$ *and*

2) $|a_2(p + q) + 2pq(a_2^2 - a_3)| \leqq 2|a_2|.$

Proof. Let $p(z, 1) = p_0 + p_1 z + p_2 z^2 + \ldots$. Since $\operatorname{Re} p(z, 1) > 0$, a classical result of Carathéodory says that $\operatorname{Re} p_0 \geqq 0$ and $|p_n| \leqq 2|p_0|$ for $n \geqq 1$. We can express p_0 and p_1 in terms of a_2 and a_3 if we observe that

$$\frac{f(z)}{zf'(z)} = 1 - a_2 z + 2(a_2^2 - a_3)z^2 + \ldots .$$

Thus

$$p_0 = \frac{a_2 pq}{p - q}, \quad \text{and}$$

$$p_1 = \frac{-a_2(p + q)}{(p - q)} - \frac{2pq(a_2^2 - a_3)}{(p - q)}.$$

If we multiply p_0 and p_1 by $(p - q)$, the corollary follows. We have been unable to determine whether or not inequalities 1) and 2) are sharp.

Notes. 1) Theorem 4 and the Corollary are remarkably similar to recent results of Hengartner and Schober [**5**] and [**6**] which were obtained by different methods. It is hoped that their results may be combined with ours to produce further results showing the strong connection between the coefficients of functions in M on the one hand and the points p and q on the other.

2) If $a_2 > 0$, then inequality 1) yields the interesting relation $\operatorname{Re} q > \operatorname{Re} p$.

3) If $\operatorname{Re} (a_2 pq)/(p - q) = 0$, then $p(z, 1) \equiv Ai$ where A is real and constant. A straightforward integration of (4.4) shows that $f(z)$ must be a polynomial and therefore not in the class M. In particular this says that $a_2 \neq 0$ for any function $f \in M$.

4) We feel that the idea of constructing explicit subordination chains for functions in M (or say for support points of S) is potentially quite useful and should be explored further.

References

1. L. A. Aksent'ev, *Application of the argument principle to the study of schlichtness conditions II*, Izv. Vysš. Učebn. Zaved, Matematika No. *12* (1969), 3.15.

2. F. G. Avkhadiev and L. A. Aksent'ev, *The main results on sufficient conditions for an analytic function to be schlicht*, Russian Math. Surveys *30:4* (1975), 1–64.

3. L. Brickman, *Extreme points of the set of univalent functions*, Bull. Amer. Math. Soc. *76* (1970), 372–374.

4. L. Brickman and D. Wilken, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc. *42* (1974), 523–528.

5. W. Hengartner and G. Schober, *Extreme points for univalent functions*, Trans. Amer. Math. Soc. *185* (1973), 265–270.

6. ——— *Some new properties of support points for compact families of univalent functions in the unit disc*, Mich. Math. J. *23* (1976), 207–216.

7. K. Löwner, *Untersuchungen über schlicte konforme abbildungen des Einheitskresses*, Math. Ann. *89*, 102–121.

8. A. I. Markushevich, *Theory of functions of a complex variable, Vol. II* (Prentice Hall, Englewood Cliffs, New Jersey, 1965).

https://doi.org/10.4153/CJM-1978-097-1 Published online by Cambridge University Press

9. A. Pfluger, *Linear Extremal problems bei schlichten Functionen*, Ann. Acad. Sci. Fenn. Ser. AI *489* (1971).

10. Chr. Pommerenke, *Über die Subordination analytischer Functionen*, J. Reine Angew. Math. *218* (1965), 159–173.

11. ——— *Univalent functions* (Vandenhoech and Ruprecht, Göttingen, 1973).

University of Maryland,
College Park, Maryland

https://doi.org/10.4153/CJM-1978-097-1 Published online by Cambridge University Press