ON TWO-DIMENSIONAL BERNSTEIN POLYNOMIALS

P. L. BUTZER

1. Introduction. Let the function of two real variables $f(x, y)$ be given over the square

$$S: 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 1.$$

Then the Bernstein polynomial in two variables x and y, corresponding to the function $f(x, y)$ is defined as

$$B^f_{n_1 n_2}(x, y) = \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} f\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) p_{\nu_1, n_1}(x)\, p_{\nu_2, n_2}(y). \tag{1}$$

where

$$p_{\nu, n}(u) = \binom{n}{\nu} u^\nu (1 - u)^{n-\nu}.$$

Obviously

$$\sum_{\nu=0}^{n} p_{\nu, n}(u) = 1.$$

The main purpose of this paper will be to prove the following result.

THEOREM. *If the function $f(x, y)$ is bounded in the closed square S, then*

$$\lim_{n_1, n_2 \to \infty} \frac{\partial^p}{\partial x_0^q\, \partial y_0^{p-q}} B^f_{n_1 n_2}(x_0, y_0) = \frac{\partial^p}{\partial x_0^q\, \partial y_0^{p-q}} f(x_0, y_0) \tag{2}$$

at every point (x_0, y_0) belonging to the open square $0 < x < 1$, $0 < y < 1$ where the pth total differential of $f(x, y)$ exists, provided n_1, n_2 tend to infinity in such a way that

$$0 < r \leqslant \frac{n_1 + 1}{n_2 + 1} \leqslant s < +\infty. \tag{3}$$

If, moreover, the partial derivatives of $f(x, y)$ of the first p orders exist and are continuous in S, then the relation (2) holds uniformly in S as n_1, n_2 approach infinity in any manner whatsoever. This latter result, which is used in establishing our theorem, was recently proven by Kingsley [**1**], but we shall deduce it in a more direct manner.

2. Preliminary results. If $\delta \geqslant n_1^{-\alpha}$, $0 < \alpha < \frac{1}{2}$ then for every $k > 0$ there is a constant C (depending only on α and k) such that

Received March 5, 1952. This paper forms a portion of the author's thesis, *On Bernstein Polynomials*, written under the direction of Professor G. Lorentz and accepted for a Ph.D. degree at the University of Toronto in November, 1951. The author gratefully acknowledges the assistance received from the National Research Council, Ottawa.

https://doi.org/10.4153/CJM-1953-014-2 Published online by Cambridge University Press

$$\sum_{\left|\frac{\nu}{n}-u\right|>\delta} p_{\nu,n}(u) \leqslant Cn^{-k}, \qquad 0 \leqslant u \leqslant 1. \tag{4}$$

Furthermore, for a given $\gamma \geqslant 0$ and any $0 \leqslant u \leqslant 1$,

$$\sum_{\nu=0}^{n} |\nu - nu|^{\gamma} p_{\nu,n}(u) \leqslant C_1 n^{\frac{1}{2}\gamma} \tag{5}$$

where C_1 is a constant depending only on γ.

For the proofs of these inequalities see [**2**].

We now recall several facts on differentials. By definition, the pth total differential of $f(x, y)$ exists [**3**] at the point (x_0, y_0) if

$$f(x, y) = P(x, y) + g(x, y) \tag{6}$$

where $P(x, y)$ represents a polynomial of degree p, and

$$g(x, y) = \sum_{i=0}^{p} \alpha_i(x, y)(x - x_0)^{p-i}(y - y_0)^{i} = \sum_{i=0}^{p} \beta_i \tag{7}$$

for the sake of brevity, where the $\alpha_i(x, y)$ have zero as a unique double limit as $x \to x_0$, $y \to y_0$.

We now define

$$\Delta^0_{(x,h)} f(x, y) = f(x, y), \quad \Delta^1_{(x,h)} f(x, y) = f(x + h, y) - f(x, y)$$

and in general

$$\Delta^p_{(x,h)} f(x, y) = \sum_{i=1}^{p} (-1)^{p-i} \binom{p}{i} f(x + ih, y), \tag{8}$$

and similarly

$$\Delta^q_{(y,k)} f(x, y) = \sum_{j=1}^{q} (-1)^{q-j} \binom{q}{j} f(x, y + jk). \tag{9}$$

From (8) and (9) it follows that

$$\begin{aligned} \Delta^p_{(x,h)} \Delta^q_{(y,k)} f(x, y) &= \Delta^q_{(y,k)} \Delta^p_{(x,h)} f(x, y) \\ &= \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{p+q-i-j} \binom{p}{i} \binom{q}{j} f(x + ih, y + jk). \end{aligned} \tag{10}$$

If the partial derivatives of $f(x, y)$ of order $\leqslant p + q$ all exist at (x_0, y_0) and are continuous in S, it can be shown that

$$\Delta^p_{(x_0,h)} \Delta^q_{(y_0,k)} f(x_0, y_0) = h^p k^q \frac{\partial^{p+q}}{\partial x_0^q \, \partial y_0^{p-q}} f(x_0 + p\theta_1 h, y + q\theta_2 k) \tag{11}$$

where $\theta_i = \theta_i(x_0, y_0; h, k)$ $(0 < \theta_i < 1 \; ; \; i = 1, 2)$.

Applying (10), by induction we can show for $q = 0, 1, 2, \ldots, p$,

$$\begin{aligned} &\frac{\partial^p}{\partial x^q \, \partial y^{p-q}} B^f_{n_1 n_2}(x, y) \\ &= n_1(n_1 - 1) \ldots (n_1 - q + 1) n_2(n_2 - 1) \ldots (n_2 - p + q + 1) \\ &\qquad \cdot \sum_{\nu_1=0}^{n_1-q} \sum_{\nu_2=0}^{n_2-p+q} \Delta^q_{(x,n_1^{-1})} \Delta^{p-q}_{(y,n_2^{-1})} f\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) p_{\nu_1,n_1-q}(x)\, p_{\nu_2,n_2-p+q}(y). \end{aligned} \tag{12}$$

https://doi.org/10.4153/CJM-1953-014-2 Published online by Cambridge University Press

3. The main theorem. We state Bernstein's Theorem for functions of two variables.

LEMMA 1. *If $f(x, y)$ is bounded in the square S, then at every point of continuity (x, y),*

$$\lim_{n_1,\, n_2 \to \infty} B^f_{n_1\, n_2}(x, y) = f(x, y);$$

the result holding uniformly in x and y if $f(x, y)$ is continuous in S.

The proof follows simply by extending the proof of the one-dimensional Bernstein's Theorem to two dimensions.

LEMMA 2. *If all the partial derivatives of $f(x, y)$ of order $\leqslant p$ exist and are continuous in S, then*

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1\, n_2}(x, y) \to \frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y) \tag{13}$$

uniformly in S as n_1, n_2 approach infinity in any manner whatever.

Proof. Applying the relations (12) and (11) we obtain

$$\begin{aligned}(14)\quad &\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1\, n_2}(x, y)\\ &= n_1(n_1 - 1) \ldots (n_1 - q + 1) n_2(n_2 - 1) \ldots (n_2 - p + q + 1)\\ &\cdot \left(\frac{1}{n_1}\right)^q \left(\frac{1}{n_2}\right)^{p-q} \sum_{\nu_1=0}^{n_1-q} \sum_{\nu_2=0}^{n_2-p+q} \frac{\partial^p}{\partial x^q \partial y^{p-q}} f\left(\frac{\nu_1}{n_1} + q\,\frac{\theta_{\nu_1\nu_2}}{n_1}, \frac{\nu_2}{n_2} + (p - q)\,\frac{\phi_{\nu_1\nu_2}}{n_2}\right)\\ &\qquad\qquad \cdot\, p_{\nu_1, n_1-q}(x)\, p_{\nu_2, n_2-p+q}(y),\end{aligned}$$

where

$$0 < \theta_{\nu_1\nu_2} < 1, \quad 0 < \phi_{\nu_1\nu_2} < 1.$$

Now, in the relation (14), the product of the factors outside the double sum will approach 1 as n_1, n_2 approach infinity. We denote $\partial^p f/\partial x^q \partial y^{p-q}$ by $h(x, y)$ and observe that the difference between the double sum in (14) and

$$B^h_{n_1-q, n_2-p+q}(x, y)$$

approaches zero uniformly in x and y as $n_1, n_2 \to \infty$. But by the previous lemma, the last expression approaches

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y)$$

uniformly in x and y. Hence the limit relation (13) follows, proving the lemma.

LEMMA 3.

$$\frac{d^l}{du^l}\,[u^\nu(1 - u)^{n-\nu}] = Q(u) u^{\nu-l}(1 - u)^{n-\nu-l}$$

where

$$Q(u) = \sum_{i,j} n^i(\nu - nu)^j h^l_{i,j}(u), \qquad i, j \geqslant 0,\ 2i + j \leqslant l$$

and the $h^l_{i,j}(u)$ *are polynomials in* u *independent of* ν *and* n.

The proof of this lemma follows by induction, see [**2**].

We are now in a position to consider the theorem stated in the introduction.

Proof of Theorem. Since the pth total differential of $f(x, y)$ exists at (x_0, y_0), we have the representaion (6) with $P(x, y)$ a polynomial of degree p and $g(x, y)$ as defined by (7). We therefore obtain

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^f_{n_1 n_2}(x_0, y_0) = \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^P_{n_1 n_2}(x_0, y_0) + \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^g_{n_1 n_2}(x_0, y_0)$$

and by Lemma 2 we only need to show that the second term on the right-hand side approaches zero. Using the definition of $g(x, y)$ in (7), it is sufficient to show that

$$\text{(15)} \qquad \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\beta_k}_{n_1 n_2}(x_0, y_0) \to 0, \qquad k = 0, 1, 2, \ldots, p.$$

By Lemma 3, we find that

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\beta_k}_{n_1 n_2}(x_0, y_0)$$
$$= \sum_{2i_1+j_1 \leqslant q} h^q_{i_1, j_1}(x_0) \sum_{2i_2+j_2 \leqslant p-q} h^{p-q}_{i_2, j_2}(y_0) \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right)\left(\frac{\nu_1}{n_1} - x_0\right)^{p-k}$$
$$\cdot \left(\frac{\nu_2}{n_2} - y_0\right)^k n_1^{i_1}(\nu_1 - n_1 x_0)^{j_1} n_2^{i_2}(\nu_2 - n_2 y_0)^{j_2} \binom{n_1}{\nu_1} x_0^{\nu_1 - q}(1 - x_0)^{n_1 - \nu_1 - q} \binom{n_2}{\nu_2}$$
$$\cdot\, y_0^{\nu_2 - p + q}(1 - y_0)^{n_2 - \nu_2 - p + q}$$

and we can rewrite this in the form

$$\text{(16)} \quad \frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B^{\beta_k}_{n_1 n_2}(x_0, y_0) = \sum_{2i_1+j_1 \leqslant q} h^*_{i_1, j_1}(x_0) \sum_{2i_2+j_2 \leqslant p-q} h^*_{i_2, j_2}(y_0)$$
$$\cdot \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right)\left(\frac{\nu_1}{n_1} - x_0\right)^{p-k}\left(\frac{\nu_2}{n_2} - y_0\right)^k n_1^{i_1}(\nu_1 - n_1 x_0)^{j_1}$$
$$\cdot\, n_2^{i_2}(\nu_2 - n_2 y_0)^{j_2} p_{\nu_1, n_1}(x_0) p_{\nu_2, n_2}(y_0)$$

where the $h^*{}_{i_1, j_1}(x_0)$ are independent of ν_1 and n_1 and the $h^*{}_{i_2, j_2}(y_0)$ are independent of ν_2 and n_2. Hence it is sufficient to show that the inner double sum in (16), which we denote by σ, approaches zero under the restriction (3).

Now for any $\epsilon > 0$ there is a δ such that

$$\text{(17)} \quad |\alpha_k(x_1, y_1)| < \epsilon, \quad |x_1 - x_0| < \delta, \quad |y_1 - y_0| < \delta; \qquad k = 0, 1, 2, \ldots, p$$

(the choice of δ is made more precise later on), and there also exists an M such that $|\alpha_k(x_1, y_1)| \leqslant M$ for $0 \leqslant x_1 \leqslant 1$, $0 \leqslant y_1 \leqslant 1$.

In absolute value the part of σ in (16) corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| < \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| < \delta$$

does not exceed

$$n_1^{i_1-p+k} n_2^{i_2-k} \epsilon \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} |\nu_1 - n_1 x_0|^{p-k+j_1} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_1,n_1}(x_0) p_{\nu_2,n_2}(y_0)$$

and by (5), this last expression does not exceed

$$C_2 \epsilon n_1^{i_1-p+k+\frac{1}{2}(p-k+j_1)} n_2^{i_2-k+\frac{1}{2}(k+j_2)}$$
$$\leqslant C_2 \epsilon (\sqrt{n_1})^{2i_1+j_1-(p-k)} (\sqrt{n_2})^{2i_2+j_2-k}.$$

As $2i_1 + j_1 \leqslant q$, $2i_2 + j_2 \leqslant p - q$, we have $2i_1 + 2i_2 + j_1 + j_2 - p \leqslant 0$, and so this last expression does not exceed $C_3\epsilon$ for all sufficiently large n_1, n_2 which satisfy (3).

In absolute value the part of σ in (16) corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| < \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| \geqslant \delta$$

does not exceed

$$n_1^{i_1-p+k} n_2^{i_2-k} M \sum_{\nu_1=0}^{n_1} |\nu_1 - n_1 x_0|^{p-k+j_1} p_{\nu_1 n_1}(x_0) \sum_{\left|\frac{\nu_2}{n_2} - y_0\right| \geqslant \delta} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_2,n_2}(y_0).$$

Using (3) and the relations $|\nu - nu| \leqslant n$ and

$$\sum_{\nu_1=0}^{n_1} p_{\nu_1,n_1}(x_0) = 1,$$

we conclude that the above expression does not exceed

$$n_2^{i_1-p+k} n_2^{i_2-k} M_1 n_2^{p-k+j_1} n_2^{k+j_2} \sum_{\left|\frac{\nu_2}{n_2} - y_0\right| \geqslant \delta} p_{\nu_2,n_2}(y_0). \tag{18}$$

Provided n_1, n_2 are sufficiently large and $0 < \alpha < \frac{1}{2}$, the number $\delta = n_1^{-\alpha} + n_2^{-\alpha}$ satisfies the condition (17). Using the inequality (4) with this δ, we see that the expression (18) does not exceed

$$n_2^{i_1-p+k}\, n_2^{i_2-k}\, M_1\, n_2^{p-k+j_1}\, n_2^{k+j_2}\, C_4 / n_2^{i_1+i_2+j_1+j_2+1},$$

which approaches zero as n_2 tends to infinity.

A similar conclusion holds for the part of σ corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| \geqslant \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| < \delta.$$

Finally, the part of σ corresponding to

$$\left|\frac{\nu_1}{n_1} - x_0\right| \geqslant \delta, \quad \left|\frac{\nu_2}{n_2} - y_0\right| \geqslant \delta$$

is in absolute value not greater than

https://doi.org/10.4153/CJM-1953-014-2 Published online by Cambridge University Press

$$n_1^{i_1-p+k} n_2^{i_2-k} M \sum_{\left|\frac{\nu_1}{n_1}-x_0\right|\geqslant\delta} |\nu_1 - n_1 x_0|^{p-k+j_1} p_{\nu_1,n_1}(x_0) \sum_{\left|\frac{\nu_2}{n_2}-y_0\right|\geqslant\delta} |\nu_2 - n_2 y_0|^{k+j_2} p_{\nu_2,n_2}(y_0)$$

and by (4) this does not exceed

$$n_1^{i_1-p+k} n_2^{i_2-k} M C_5 \frac{n_1^{p-k+j_1}}{n_1^{i_1+j_1+1}} \frac{n_2^{k+j_2}}{n_2^{i_2+j_2+1}},$$

which approaches zero as $n_1, n_2 \to \infty$. Hence

$$\frac{\partial^p}{\partial x_0^q \partial y_0^{p-q}} B_{n_1 n_2}^{\beta_k}(x_0, y_0)$$

can be made arbitrarily small by taking n_1, n_2 large enough, satisfying (3). The theorem is now established.

We shall now show that the limit relation (2) also holds at the four corners (0, 0), (0, 1), (1, 0) and (1, 1) of the square S. It will be sufficient to discuss the corner (0, 0).

Writing

$$D_{\nu_1\nu_2}(x, y) = \frac{d^q}{dx^q} x^{\nu_1}(1-x)^{n_1-\nu_1} \frac{d^{p-q}}{dy^{p-q}} y^{\nu_2}(1-y)^{n_2-\nu_2},$$

we have $D_{\nu_1\nu_2}(0, 0) = 0$ for $\nu_1 \geqslant q+1$ or $\nu_2 \geqslant p-q+1$. For $\nu_1 \leqslant q$ and $\nu_2 \leqslant p-q$ by Leibniz's theorem, we have

$$\begin{aligned} D_{\nu_1\nu_2}(0, 0) &= \binom{q}{\nu_1} \frac{d^{\nu_1}}{dx^{\nu_1}} x^{\nu_1} \frac{d^{q-\nu_1}}{dx^{q-\nu_1}} (1-x)^{n_1-\nu_1} \binom{p-q}{\nu_2} \frac{d^{\nu_2}}{dy^{\nu_2}} y^{\nu_2} \\ &\qquad \cdot \frac{d^{p-q-\nu_2}}{dy^{p-q-\nu_2}} (1-y)^{n_2-\nu_2} \Big|_{x=y=0} \\ &= (-1)^{q-\nu_1} \frac{q!}{(q-\nu_1)!} (n_1-\nu_1)(n_1-\nu_1-1) \ldots (n_1-q+1) \\ &\qquad \cdot (-1)^{p-q-\nu_2} \frac{(p-q)!}{(p-q-\nu_2)!} (n_2-\nu_2) \ldots (n_2-p+q+1) \end{aligned}$$

$$\simeq C_6\, n_1^{q-\nu_1} n_2^{p-q-\nu_2}, \qquad n_1, n_2 \to \infty. \tag{19}$$

Again we have to show that the limit relation (15) holds, but now at the point (0, 0) instead of (x_0, y_0).

We have

$$\begin{aligned} &\frac{\partial^p}{\partial x^q \partial y^{p-q}} B_{n_1 n_2}^{\beta_k}(x, y) \Big|_{x=y=0} \\ &= \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \left(\frac{\nu_1}{n_1} - x\right)^{p-k} \left(\frac{\nu_2}{n_2} - y\right)^k \binom{n_1}{\nu_1} \binom{n_2}{\nu_2} D_{\nu_1\nu_2}(x, y) \Big|_{x=y=0} \end{aligned}$$

and by (19), this in absolute value is

$$\simeq C_7 \left| \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \left(\frac{1}{n_1}\right)^{p-k} \left(\frac{1}{n_2}\right)^k n_1^{\nu_1} n_2^{\nu_2} n_1^{q-\nu_1} n_2^{p-q-\nu_2} \right|$$

and since n_1 and n_2 satisfy the inequality (3), the last expression does not exceed

$$C_8 \sum_{\nu_1=0}^{q} \sum_{\nu_2=0}^{p-q} \left| \alpha_k\left(\frac{\nu_1}{n_1}, \frac{\nu_2}{n_2}\right) \right|,$$

which approaches zero as $n_1, n_2 \to \infty$ because $\alpha_k(x, y) \to 0$ as $x \to 0, y \to 0$. Hence we have the relation

$$\frac{\partial^p}{\partial x^q \partial y^{p-q}} B^f_{n_1 n_2}(x, y) \bigg|_{x=y=0} \to \frac{\partial^p}{\partial x^q \partial y^{p-q}} f(x, y) \bigg|_{x=y=0}$$

if the pth total differential of $f(x, y)$ exists at (0, 0), provided n_1 and n_2 satisfy (3).

We now consider an example to show that the relation (2) does not necessarily hold at a boundary point of S different from one of the four corners. Consider the point $(x_0, 0)$, $0 < x_0 < 1$ and the function

$$f(x, y) = \epsilon(y)|x - x_0|, \qquad 0 \leqslant x \leqslant 1$$

where $\epsilon(y)$ is defined for $0 \leqslant y \leqslant 1$ and has the properties $\epsilon(y) \to 0$ and $y^{-\frac{1}{2}}\epsilon(y) \to +\infty$ as $y \to 0$. Take $n_1 = n_2 = n$. Obviously

$$\frac{d}{dy} p_{\nu_2,n}(y) \bigg|_{y=0} = \begin{cases} n & \text{for } \nu_2 = 1, \\ 0 & \text{for } \nu_2 = 2, 3, \ldots, n. \end{cases}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial y} B_{nn}(x, y) \bigg|_{(x_0, 0)} &= \sum_{\nu_1=0}^{n} \left| \frac{\nu_1}{n} - x \right| p_{\nu_1,n}(x) \sum_{\nu_2=0}^{n} \epsilon\left(\frac{\nu_2}{n}\right) \frac{d}{dy} p_{\nu_2,n}(y) \bigg|_{(x_0, 0)} \\ &= \frac{1}{n} \sum_{\nu_1=0}^{n} \left| \nu_1 - n x_0 \right| p_{\nu_1,n}(x_0) \epsilon\left(\frac{1}{n}\right) n \\ &\geqslant C_9 \sqrt{n}\, \epsilon\left(\frac{1}{n}\right), \end{aligned}$$

which approaches $+\infty$ as $n \to \infty$ because it is known [2] that for every $0 < x < 1$ there exists a constant $C_9 > 0$ such that

$$\sum_{\nu=0}^{n} |\nu - nx| \, p_{\nu,n}(x) \geqslant C_9 \sqrt{n}, \qquad n \to \infty.$$

References

1. E. H. Kingsley, *Bernstein polynomials for functions of two variables of class* $C^{(k)}$, Proc. Amer. Math. Soc., *2* (1951), 64–71.
2. G. G. Lorentz, *Bernstein polynomials*, to be published by the University of Toronto Press.
3. W. H. Young, *The fundamental theorems of the differential calculus* (Cambridge Tracts, no. 11, 1910).

University of Toronto