SOME REMARKS CONCERNING (f, d_n) AND $[F, d_n]$ SUMMABILITY METHODS

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1. Introduction. In this paper we note a simple connection between the (f, d_n) method of summability (defined by Smith (**8**)) and a composition of $[F, d_n]$ (defined by Jakimovski (**4**)) and Sonnenschein methods (**9**; **10**). This connection is then used to supply some sufficient conditions for the regularity of (f, d_n) methods by using known regularity conditions for various $[F, d_n]$ and Sonnenschein methods. Finally, the connection is further exploited to obtain information about the Lebesgue constants for a certain class of $[F, d_n]$ methods by investigating related (f, d_n) methods.

2. Definitions and the regularity theorem. The (f, d_n) method of summability is defined by Smith (**8**) essentially as follows. Let $f(z)$ be a non-constant entire function satisfying $f(1) = 1$ and let $\{d_n\}$ be a sequence of complex numbers satisfying $d_i \neq -1$, $d_i \neq -f(0)$ $(i \geqq 1)$. Then the equations

$$a_{00} = 1, \qquad a_{0k} = 0 \quad (k \neq 0),$$

$$\prod_{i=1}^{n} \frac{f(z) + d_i}{1 + d_i} = \sum_{k=0}^{\infty} a_{nk} z^k \qquad (n \geqq 1) \tag{2.1}$$

define the elements of the sequence to sequence matrix (a_{nk}). Note that a slight notational change has been made in Smith's definition by requiring that $f(1) = 1$.

The $[F, d_n]$ method of summability is defined by Jakimovski (**4**) as the special case of the (f, d_n) method in which $f(z) = z$. Thus the $[F, d_n]$ summability matrix (P_{nj}) is defined by

$$P_{00} = 1, \qquad P_{0k} = 0 \quad (k \neq 0),$$

$$\prod_{j=1}^{n} \frac{z + d_j}{1 + d_j} = \sum_{j=0}^{\infty} P_{nj} z^j.$$

Since $f(z)$ is entire, so is $(f(z))^j$, and as such has an absolutely convergent power series expansion $(f(z))^j = \sum_{k=0}^{\infty} f_{jk} z^k$ valid for all complex z. We formally replace z^j in the Jakimovski definition by the power series expansion of $(f(z))^j$ to obtain $\sum_{j=0}^{n} P_{nj} \sum_{k=0}^{\infty} f_{jk} z^k$ as a double series representation of the left side of (2.1). This double series converges absolutely since $\sum_{k=0}^{\infty} f_{jk} z^k$

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is absolutely convergent for all j and the summation on j is finite. Thus it may be rearranged to form the power series

$$\sum_{k=0}^{\infty} \sum_{j=0}^{n} P_{nj} f_{jk} z^k = \prod_{i=1}^{n} \frac{f(z) + d_i}{1 + d_i}. \tag{2.2}$$

Then, comparing (2.1) and (2.2), the uniqueness of power series representations guarantees that $a_{nk} = \sum_{j=0}^{n} P_{nj} f_{jk}$ for all n and k.

The Sonnenschein method of summability was defined by Sonnenschein (**9**; **10**) as follows. Let $f(z)$ be analytic for $|z| < R$, where $R > 1$ and such that $f(1) = 1$. Then the element f_{jk} in the Sonnenschein summability matrix is defined to be the coefficient of z^k in the power series expansion of $(f(z))^j$, for $j \geqq 0$, $k \geqq 0$.

We thus note that the element a_{nk} in an (f, d_n) matrix can be obtained by formal matrix multiplication of the appropriate $[F, d_n]$ and Sonnenschein matrices.

In general, if A and B are regular summability matrices, then the A transform of the B transform of a sequence (herein denoted by the AB method) need not be the same as the method obtained by multiplying the A and B matrices together and then applying the resulting matrix to the given sequence (denoted by the $A \cdot B$ method). Indeed, for a given sequence and given A and B either one could exist or fail to exist; see (**1**). However, if A is, in addition, row-finite, we have the following theorem of Agnew (**1**, Theorem 10.2).

THEOREM 1 (Agnew). *If A and B are regular with A row-finite, then the convergence field of $A \cdot B$ includes the convergence field of AB.*

We may now state the following theorem concerning regularity of (f, d_n) methods.

THEOREM 2. *If (P_{nj}) is a regular $[F, d_n]$ matrix and (f_{jk}) is a regular Sonnenschein matrix, then the (f, d_n) method (a_{nk}), defined by $a_{nk} = \sum_{j=0}^{n} P_{nj} f_{jk}$, is regular.*

Proof. If the sequence $\{s_k\}$ converges to s, regularity of (f_{jk}) and (P_{nj}) guarantees convergence of the $[F, d_n]$ transform of the Sonnenschein transform to s. Then Agnew's theorem supplies convergence of the $[F, d_n]$ transform to s.

We note that if $f(z)$ is an entire function and the (f, d_n) method is applied to the geometric series, we have the following special result.

THEOREM 3. *If $f(z)$ is an entire function, then those values of z for which the (f, d_n) transform of the partial sums of the geometric series converge to $1/(1 - z)$ are precisely those values for which the related $[F, d_n]$ transform of the related Sonnenschein transform converges to $1/(1 - z)$.*

Proof. Since the partial sums of the geometric series can be written in the form $1/(1-z) - z^{n+1}/(1-z)$, it is sufficient to consider the various summability transforms of the sequence $\{z^n\}$. From the fact that $f(z)$ is entire,

$$A_n(z) = \sum_{j=0}^{n} P_{nj} \sum_{k=0}^{\infty} f_{jk} z^k$$

converges absolutely for all z. Thus we can interchange the order of summation and those values of z for which $\lim_n A_n(z) = 0$ are the same as those values of z for which $\lim_n B_n(z) = 0$, where

$$B_n(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} P_{nj} f_{jk} z^k.$$

For the convenience of the reader we now list various well-known sufficient conditions for regularity of $[F, d_n]$ and Sonnenschein methods. Choosing an $[F, d_n]$ method satisfying (i), (ii), or (iii) below and a Sonnenschein method satisfying (iv), (v), or (vi), the associated (f, d_n) method will be regular.

The $[F, d_n]$ method is regular if:

(i) $d_n \geqq 0$ for all n and $\sum_{n=1}^{\infty} 1/d_n = +\infty$ (**4**), or

(ii) $d_n = \rho_n e^{i\theta_n}$ and $\sum_{n=1}^{\infty} 1/\rho_n = +\infty$ and $\sum_{n=1}^{\infty} \theta_n^2/\rho_n < +\infty$ (**3**), or

(iii) $\{\lambda_m\}$ satisfies $\lambda_m \geqq 0$ for all m and $\sum_{m=1}^{\infty} 1/\lambda_m = +\infty$, $k \geqq 2$, is a fixed positive integer, and $d_n = \rho_n e^{i\theta_n}$, where $\rho_{k(m-1)+j} = \lambda_m^{1/k}$ for $j = 1, 2, \ldots, k$ and $\theta_{k(m-1)+j} = \exp\{j(k-1)\pi i/k\}$ for $j = 1, 2, \ldots, k$ (**6**).

The Sonnenschein method is regular provided:

(iv) (a) $f(z)$ is analytic in $|z| < R$, $R > 1$,

(b) $|f(z)| < 1$ for $|z| < 1$ except at a finite number of points ξ,

(c) the real part of $A_\xi \neq 0$, where A_ξ is defined by

$$h_\xi(z) - z^{h_\xi'(1)} = A_\xi i^p (z-1)^p + o(z-1)^p \quad \text{as } z \to 1$$

with $h_\xi(z) = f(\xi z)/f(\xi)$, and

(d) $f(1) = 1$ (**2**).

In the case of Karamata matrices which are those Sonnenschein matrices where $f(z) = [\alpha + (1-\alpha-\beta)z]/(1-\beta z)$, we have regularity provided:

(v) $\alpha = \beta = 0$, or

(vi) $1 - |\alpha|^2 > (1-\bar{\alpha})(1-\beta) > 0$ (**7**).

3. An application. The Lebesgue constants for the $[F, d_n]$ method with $d_n > 0$ have been shown to be unbounded by Lorch and Newman (**5**). However, their method will not apply directly to the $[F, d_n]$ methods with d_n complex. For those complex methods defined by Miracle (**6**) we have the following result.

THEOREM 4. *The Lebesgue constants for the* $[F, d_n]$ *methods defined by Miracle* (**6**) *are unbounded.*

Proof. Let P_{nk} denote the coefficients in the $[F, d_n]$ matrix as defined by Miracle. We note that for $n = mp$ we have

$$\prod_{k=1}^{n} \frac{z + d_k}{1 + d_k} = \prod_{k=1}^{m} \frac{z^p + \lambda_k}{1 + \lambda_k}.$$

Thus the elements with row subscript mp in the $[F, d_n]$ matrix are identical to the elements with row subscript m in the (z^p, λ_m) matrix defined by Smith (**8**). Hence, the Lebesgue constants for the (z^p, λ_n) method coincide with a subsequence of the Lebesgue constants for the $[F, d_n]$ method. Then the unboundedness of the subsequence will imply the unboundedness of the Lebesgue constants for the $[F, d_n]$ method. These constants are given by

$$L(z^p, \lambda_m) = \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sin t} \left| \operatorname{Im}\left\{ e^{it} \prod_{k=1}^{m} \frac{e^{2pit} + \lambda_k}{1 + \lambda_k} \right\} \right| dt. \tag{3.1}$$

Since $\lambda_m > 0$ we can apply the method of Lorch, and with only slight modifications. Analogous to their derivation, we make the following definitions.

$$S_m = 2p^2 \sum_{k=1}^{m} \frac{\lambda_k}{(1 + \lambda_k)^2} \quad \text{and} \quad U_m = 1 + 2p \sum_{k=1}^{m} \frac{1}{1 + \lambda_k}.$$

We then obtain the following estimates:

$$\prod_{k=1}^{m} \frac{e^{2pit} + \lambda_k}{1 + \lambda_k} = \exp\{(U_m - 1)it - S_m t^2\} + O(S_m t^3), \tag{3.2}$$

$$\left| \frac{e^{2pit} + \lambda_k}{1 + \lambda_k} \right| \leqq \exp\left\{ \frac{-p^2 \lambda_k t^2}{2(1 + \lambda_k)^2} \right\} \quad \text{for } 0 \leqq t \leqq \pi/2p, \tag{3.3}$$

and

$$\prod_{k=1}^{m} \frac{e^{2pit} + \lambda_k}{1 + \lambda_k} = O(\exp\{-\tfrac{1}{4} S_m t^2\}) \quad \text{for } 0 \leqq t \leqq \pi/2p. \tag{3.4}$$

It should be noted that estimates (3.3) and (3.4) are not valid for $0 \leqq t \leqq \pi/2$ since $p \geqq 2$. However, the following additional estimate, though much cruder, will fill the gap:

$$\prod_{k=1}^{m} \frac{e^{2pit} + \lambda_k}{1 + \lambda_k} = O(1). \tag{3.5}$$

We note that if $p = 1$, these estimates coincide with those of Lorch and Newman. Since they are obtained in an analogous manner, their derivation will not be supplied here.

Next we introduce the quantity ξ, $0 < \xi < \pi/2p$, to be fixed later. We use formula (3.2) to estimate the portion of the integral (3.1) from 0 to ξ, then formula (3.4) for the portion from ξ to $\pi/2p$, and formula (3.5) for the portion from $\pi/2p$ to $\pi/2$. We extend the interval of integration in the approximating exponential of formula (3.2), thereby introducing an additional error which can, however, be absorbed in the error arising from formula (3.4).

Thus we obtain

$$(3.6)\quad L(z^p, \lambda_m) = \frac{2}{\pi}\int_0^{\pi/2} \exp\{-S_m t^2\}\,\frac{|\sin U_m t|}{\sin t}\,dt + O(S_m \xi^3) + O(\xi^{-2}\exp\{-\tfrac{1}{4}S_m\xi^2\}) + O(1).$$

Now if S_m is bounded, we choose ξ to be fixed; if S_m is unbounded, we choose $\xi = S_m^{-3/8}$, reducing the error terms to $O(1)$ in either case. Formula (3.6) differs from that of Lorch and Newman (**5**, formula (3.6)) only by $O(1)$, hence from this point on their method can be followed exactly to show that the Lebesgue constants for the (z^p, λ_m) method, and hence also for the $[F, d_n]$ method, are unbounded.

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