# VI
# Some boundary properties of solutions

In this chapter we will explore certain boundary properties of the solutions of locally integrable vector fields. In the first section we present a growth condition that ensures the existence of a distribution boundary value for a solution of a locally integrable complex vector field in $\mathbb{R}^N$. This condition extends the well-known tempered growth condition for holomorphic functions which we will recall in Theorem VI.1.1 below. Section VI.2 considers the pointwise convergence of solutions of planar, locally integrable vector fields to their boundary values. Sections VI.3 and VI.4 explore the class of vector fields in the plane for which Hardy space-like properties are valid. The chapter concludes with applications to the boundary regularity of solutions. The boundary variant of the Baouendi–Treves approximation theorem, namely, Theorem II.4.12, will be crucial for the results in Sections VI.2 and VI.4.

## VI.1 Existence of a boundary value

Suppose $L$ is a smooth complex vector field,

$$L = \sum_{j=1}^{N} a_j(x) \frac{\partial}{\partial x_j}$$

defined on a domain $\Omega \subseteq \mathbb{R}^N$ and $u \in C(\Omega)$ is such that $Lu = 0$ in $\Omega$. Assume $\partial\Omega$ is smooth. We would like to explore conditions on $u$ that guarantee that $u$ will have a distribution boundary value on $\partial\Omega$. Theorem V.2.6 showed us that when $u$ is holomorphic on a domain $D \subseteq \mathbb{C}^n$, then $u$ has a boundary value if

$$|u(z)| \leq \frac{C}{\mathrm{dist}(z, \partial D)^k} \tag{VI.1}$$

for some $C, k > 0$. Conversely, it is well known that if a holomorphic function on $\Omega$ has a distribution trace on $\partial D$, then $u(z)$ has a tempered growth as in (VI.1). For simplicity, we recall here a precise version in the planar case:

THEOREM VI.1.1 (Theorems 3.1.11, 3.1.14 [**H2**].). *Let $A, B > 0$, $Q = (-A, A) \times (0, B)$ and $f$ holomorphic on $Q$.*

(i) *If for some integer $N \geq 0$ and $C > 0$,*

$$|f(x+iy)| \leq Cy^{-N}, \quad x+iy \in Q,$$

*then there exists $bf \in D'(-A, A)$ of order $N+1$ such that*

$$\lim_{y \to 0^+} \int f(x+iy)\psi(x)\mathrm{d}x = \langle bf, \psi \rangle \quad \forall \psi \in C_0^{N+1}(-A, A).$$

(ii) *If $\lim_{y \to 0^+} f(\cdot + iy)$ exists in $D'^k(-A, A)$, then for any $0 < A' < A$, and $0 < B' < B$, there exists $C'$ such that*

$$|f(x+iy)| \leq C' y^{-k-1}, \quad x+iy \in (-A', A') \times (0, B').$$

Because of the local equivalence of $L^1$ and sup norms for solutions in the elliptic (Cauchy–Riemann) case, the preceding theorem asserts that a holomorphic function $f$ on $Q$ has a trace at $y = 0$ if and only if for some integer $N > 0$,

$$\iint_Q |f(x+iy)| y^N \, \mathrm{d}x\mathrm{d}y < \infty.$$

It is natural to investigate generalizations of this theorem for nonelliptic vector fields. It turns out that the tempered growth condition (VI.1) is sufficient to ensure the existence of a boundary value for a general nonvanishing vector field that may not be locally integrable. Indeed, we have:

THEOREM VI.1.2 (Theorem 1.1 [**BH4**]). *Let $L$ be a $C^\infty$ complex vector field in a domain $\Omega \subseteq \mathbb{R}^n$, $f \in C(\Omega)$, $Lf = 0$ in $\Omega$. Suppose*

$$|f(x)| \leq C \operatorname{dist}(x, \partial\Omega)^{-N}$$

*for some $C, N > 0$. If $\Sigma \subseteq \partial\Omega$ is open, smooth and noncharacteristic for $L$, then $f$ has a distribution boundary value on $\Sigma$.*

The preceding result suggests that for a locally integrable vector field, in general, one should seek a growth condition that is weaker than a tempered growth expressed in terms of dist $(x, \partial\Omega)$.

As a motivation, suppose $Z = x + i\varphi(x, y)$ is smooth in a neighborhood of the origin in $\mathbb{R}^2$, $\varphi$ real-valued. Then $Z$ is a first integral for

$$L = \frac{\partial}{\partial y} - \frac{i\varphi_y}{1 + i\varphi_x}\frac{\partial}{\partial x}.$$

Assume that $\varphi(x, y) > 0$ when $y > 0$ and $\varphi(x, 0) = 0$, for all $x$. Then for any integer $N > 0$, since the holomorphic function $\frac{1}{(x+iy)^N}$ has a boundary value as $y \to 0^+$, it is not hard to see that

$$u_N(x, y) = \frac{1}{Z(x, y)^N}$$

also has the same boundary value.

Note that $Lu_N = 0$ when $y > 0$, $|u_N(0, y)| = \frac{1}{|\varphi(0,y)|^N}$, while

$$|u_N(x, y)| \le \frac{1}{|\varphi(x, y)|^N} = \frac{1}{|Z(x, y) - Z(x, 0)|^N}.$$

Observe that $\varphi$ may be chosen so that $u_N(x, y)$ is not bounded by any power of $y$ as $y \to 0^+$. In general, if $L$ is locally integrable, $Z$ is a first integral of $L$ near the origin and $Lu = 0$ in the region $y > 0$, then the growth condition

$$|u(x, y)||Z(x, y) - Z(x, 0)|^N \le C < \infty \tag{VI.2}$$

is sufficient for $u$ to have a distribution boundary value at $y = 0$. When $L$ is real-analytic, (VI.2) is also a necessary condition for the existence of a boundary trace at $y = 0$ (see [**BH5**]). Before we state the main result of this section, as a motivation for its proof, we review the classical case of holomorphic functions. Consider a holomorphic function $f$ on the rectangle $Q = (-A, A) \times (-B, B)$ satisfying the growth condition

$$|f(x + iy)\, y^N| \le C < \infty.$$

We wish to show that $f$ has a boundary value at $y = 0$. Let $\psi \in C_0^\infty(-A, A)$. Fix $0 < T < B$. For each integer $m \ge 0$, choose $\psi_m(x, y) \in C^\infty((-A, A) \times [0, B])$ such that

(i) $\psi_m(x, 0) = \psi(x)$ and
(ii) $|\bar{\partial}\psi_m(x, y)| \le Cy^m$

where $C$ depends only on the size of the derivatives of $\psi$ up to order $m + 1$. Indeed, if we define

$$\psi_m(x, y) = \sum_{k=0}^{m} \frac{\psi^{(k)}(x)}{k!}(iy)^k,$$

then it is easy to see that (i) and (ii) hold. Note that since $f$ is holomorphic, for any $0 < \epsilon < T$, and $g \in C_0^1(-A, A)$, integration by parts gives:

$$\int_{-A}^{A} f(x+i\epsilon)g(x,\epsilon)\,\mathrm{d}x = \int_{-A}^{A} f(x+iT)g(x,T)\mathrm{d}x \\ +2i\int_{-A}^{A}\int_{\epsilon}^{T} f(x+iy)\bar{\partial}g(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Plugging $g(x,y) = \psi_N(x, y-\epsilon)$ in the preceding formula yields

$$\int_{-A}^{A} f(x+i\epsilon)\psi(x)\,\mathrm{d}x = \int_{-A}^{A} f(x+iT)\psi_N(x, T-\epsilon)\mathrm{d}x \\ +2i\int_{-A}^{A}\int_{\epsilon}^{T} f(x+iy)e(x,y,\epsilon)\,\mathrm{d}x\mathrm{d}y$$

where $|e(x,y,\epsilon)| \le C|y-\epsilon|^N$. Since $|f(x+iy)\,y^N| \le C$, as $y \to 0$, the right-hand side in the formula converges. This proves that $f(x+iy)$ has a boundary value at $y = 0$.

We will prove now the sufficiency of (VI.2) in a more general set-up. Let $L$ be a smooth, locally integrable vector field defined near the origin in $\mathbb{R}^{m+1}$. In appropriate coordinates $(x,t)$ we may assume that $L$ possesses $m$ smooth first integrals of the form $Z_j(x,t) = A_j(x,t) + iB_j(x,t)$, $j = 1,\dots,m$ defined on a neighborhood of the closure of the cylinder $Q = B_r(0)\times(-T,T)$ where $B_r(0)$ is a ball in $x$ space $\mathbb{R}^m$ and $Z_x(0,0)$ is invertible. Thus, after multiplication by a nonvanishing factor, $L$ may be written as

$$L = \frac{\partial}{\partial t} - \sum_{k=1}^{m} \frac{\partial Z_k}{\partial t} M_k \tag{VI.3}$$

where the $M_k$ are the vector fields in $x$ space satisfying $M_k Z_j = \delta_{kj}$, $1 \le k, j \le m$. The next theorem gives, in particular, a sufficient condition for the existence of a boundary value of a continuous function $f$ when $f$ is a solution of $Lf = 0$.

THEOREM VI.1.3. *Let $L$ be as above and let $f$ be continuous on $Q^+ = B_r(0)\times(0,T)$. Suppose*

(i) $Lf \in L^1(Q^+)$;
(ii) *there exists $N \in \mathbb{N}$ such that*

$$\int_0^T \int_{B_r(0)} |Z(x,t) - Z(x,0)|^N |f(x,t)|\mathrm{d}x\mathrm{d}t < \infty.$$

*Then* $\lim_{t\to 0^+} f(x,t) = bf$ *exists in* $D'(B_r(0))$ *and it is a distribution of order* $N+1$.

Proof. Note first that by taking complex, linear combinations of the $Z_j$'s, we may assume that $Z_x(0,0) = \mathrm{Id}$, the identity matrix. This will not affect hypothesis (ii) in the theorem. Let $\psi \in C_0^\infty(B_r(0))$. For each integer $k \geq 0$, we will show that there exists $\psi_k(x,t) \in C^\infty(B_r(0) \times [0,T])$ such that

(i) $\psi_k(x,0) = \psi(x)$ and
(ii) $|L\psi_k(x,t)| \leq C\,|Z(x,t) - Z(x,0)|^k$

where $C$ depends only on the size of $D^\alpha\psi(x)$ for $|\alpha| \leq k+1$. To get $\psi_k(x,t)$ with these properties, we will use a smooth function $u_k = u_k(x,y)$ defined near $0 \in \Sigma = \{Z(x,0)\}$ in $\mathbb{C}^m$ and satisfying:

(a) $u_k(Z(x,0)) = \psi(x)$ and
(b) $|(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})u_k(x,y)| \leq C\,\mathrm{dist}((x,y),\Sigma)^k$ for $j = 1,\dots,m$.

Assuming for the moment that such a $u_k$ with these properties exists, we set

$$\psi_k(x,t) = u_k(A(x,t), B(x,t))$$

where

$$A(x,t) = (A_1(x,t),\dots,A_m(x,t)), \quad B(x,t) = (B_1(x,t),\dots,B_m(x,t)).$$

Then $\psi_k(x,0) = \psi(x)$ so that (i) above holds. To check (ii), observe that from the equations

$$L(Z_j) = L(A_j + iB_j) = 0, \quad j = 1,\dots,m,$$

we have

$$L(\psi_k) = \sum_{j=1}^m \left( \frac{\partial u_k}{\partial x_j} L(A_j) + \frac{\partial u_k}{\partial y_j} L(B_j) \right) = 2\sum_{j=1}^m L(A_j)\frac{\partial u_k}{\partial \bar{z}_j}. \tag{VI.4}$$

It follows that

$$\begin{aligned} |\,L(\psi_k)(x,t)\,| &\leq C_1\,|\,\bar\partial u_k(A(x,t),B(x,t))\,| \\ &\leq C_2\,\mathrm{dist}(A(x,t)+iB(x,t),\Sigma)^k \\ &\leq C_2\,|\,Z(x,t)-Z(x,0)\,|^k\,. \end{aligned}$$

Thus if $u_k$ satisfies (a) and (b), then $\psi_k(x,t)$ will satisfy (i) and (ii). We will next write a formula for the $u_k$. Since the map $x \mapsto A(x,0)$ is invertible, there is a smooth map $G = (G_1,\dots,G_m)$ such that

$$\Im(Z(x,0)) = B(x,0) = G(A(x,0)).$$

This and some of what follows may require decreasing the neighborhood around the origin. Note that since $\mathrm{d}B(0,0) = 0$, and $\mathrm{d}A(0,0) \neq 0$, $\mathrm{d}G(0,0) = 0$.

Let $V_j$ be the vector fields satisfying $V_j(x_s + iG_s(x)) = \delta_{js}$, $1 \le j, s \le m$. For each $k = 1, 2, \dots$ define

$$u_k(x, y) = \sum_{|\alpha| \le k} \frac{i^{|\alpha|}}{\alpha!} V^\alpha \tilde{\psi}(x)(y - G(x))^\alpha$$

where by definition, $\tilde{\psi}(x) = \psi(A(x, 0)^{-1})$. Clearly, $u_k(Z(x, 0)) = \psi(x)$. We claim that for each $j = 1, \dots, m$,

$$2\frac{\partial u_k}{\partial \overline{z_j}} = i^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\partial}{\partial x_j} \left( V^\alpha \tilde{\psi}(x) \right) (y - G(x))^\alpha. \tag{VI.5}$$

In particular, the claim implies property (b) for $u_k$. Indeed, after contracting the neighborhood of the origin, we may assume that $\Sigma = \{x + iG(x)\}$. Since $\mathrm{d}G(0, 0) = 0$, it follows that

$$|y - G(x)| \le \operatorname{dist}((x, y), \Sigma)$$

which gives (b). The claim will be proved by induction. We have:

$$\frac{\partial u_1}{\partial y_j}(x + iy) = iV_j(\tilde{\psi}(x))$$

and

$$\frac{\partial u_1}{\partial x_j}(x + iy) = \frac{\partial \tilde{\psi}}{\partial x_j} - i\sum_{s=1}^m V_s(\tilde{\psi}) \frac{\partial G_s}{\partial x_j} + i\sum_{s=1}^m \frac{\partial}{\partial x_j} \left( V_s(\tilde{\psi}) \right) (y_s - G_s(x)).$$

Next observe that

$$\frac{\partial}{\partial x_j} = i\sum_{s=1}^m \frac{\partial G_s(x)}{\partial x_j} V_s + V_j \tag{VI.6}$$

which can be seen by applying both sides to the $m$ linearly independent functions $x_1 + iG_1(x), \dots, x_m + iG_m(x)$. Hence

$$\frac{\partial u_1}{\partial x_j} + i\frac{\partial u_1}{\partial y_j} = i\sum_{s=1}^m \frac{\partial}{\partial x_j} \left( V_s(\tilde{\psi}) \right) (y_s - G_s(x))$$

which proves the claim for $k = 1$. Assume next that (VI.5) holds for $k - 1$, $k \ge 1$. We can write

$$u_k(x, y) = u_{k-1}(x, y) + E_k(x, y) \tag{VI.7}$$

where

$$E_k(x, y) = i^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \left( V^\alpha \tilde{\psi}(x) \right) (y - G(x))^\alpha.$$

For any $1 \le j \le m$, by the induction assumption, we have

$$2\frac{\partial u_{k-1}}{\partial \overline{z_j}} = i^{k-1} \sum_{|\beta|=k-1} \frac{1}{\beta!} \frac{\partial}{\partial x_j} \left( V^\beta \tilde{\psi} \right) (y - G(x))^\beta. \tag{VI.8}$$

Observe that

$$\frac{\partial E_k}{\partial x_j}(x, y) = i^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \left( \frac{\partial}{\partial x_j} V^\alpha \tilde{\psi} \quad (y - G(x))^\alpha + V^\alpha(\tilde{\psi}) \frac{\partial}{\partial x_j} (y - G(x))^\alpha \right) \tag{VI.9}$$

and

$$\frac{\partial E_k}{\partial y_j}(x, y) = i^k \sum_{|\alpha|=k} \frac{V^\alpha(\tilde{\psi})}{\alpha!} \frac{\partial}{\partial y_j} (y - G(x))^\alpha. \tag{VI.10}$$

Using the expression for $\frac{\partial}{\partial x_j}$ from (VI.6), (VI.8) can be written as

$$\begin{aligned} 2\frac{\partial u_{k-1}}{\partial \overline{z_j}} =& i^k \sum_{|\beta|=k-1} \sum_{s=1}^{m} \frac{1}{\beta!} \frac{\partial G_s}{\partial x_j}(x) V_s \left( V^\beta \tilde{\psi} \right) (y - G(x))^\beta \\ &+ i^{k-1} \sum_{|\beta|=k-1} \frac{1}{\beta!} V_j \left( V^\beta \tilde{\psi} \right) (y - G(x))^\beta. \end{aligned} \tag{VI.11}$$

From (VI.7), (VI.9), (VI.10) and (VI.11), we get

$$2\frac{\partial u_k}{\partial \overline{z_j}} = i^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\partial}{\partial x_j} \left( V^\alpha \tilde{\psi}(x) \right) (y - G(x))^\alpha$$

which establishes property (b) for $u_k$. Hence for each $k$ we have $\psi_k$ which satisfies (i) and (ii) and has the form

$$\psi_k(x, t) = \sum_{|\alpha| \le k} (P_\alpha(x, t, D_x) \tilde{\psi}(A(x, t)))(B(x, t) - G(A(x, t)))^\alpha \tag{VI.12}$$

where $P_\alpha(x, t, D_x)$ is a differential operator of order $|\alpha|$ involving differentiations only in $x$. Observe next that if $g(x, t)$ is a $C^1$ function, the differential of the $m$ form $g(x, t)\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m$ where $Z_j = A_j(x, t) + iB_j(x, t)$ is given by

$$\mathrm{d}(g\, \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m) = Lg\, \mathrm{d}t \wedge \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m.$$

This observation and integration by parts lead to:

$$\begin{aligned} \int_{B_r(0)} f(x, \epsilon)\psi_N(x, \epsilon)\mathrm{d}Z(x, \epsilon) =& \int_{B_r(0)} f(x, T)\psi_N(x, T)\, \mathrm{d}Z(x, T) \\ &+ \int_{B_r(0)} \int_\epsilon^T f(x, t) L\psi_N(x, t)\, \mathrm{d}t \wedge \mathrm{d}Z \\ &+ \int_{B_r(0)} \int_\epsilon^T Lf(x, t)\psi_N(x, t)\, \mathrm{d}t \wedge \mathrm{d}Z \end{aligned} \tag{VI.13}$$

where $dZ = dZ_1 \wedge dZ_2 \wedge \cdots \wedge dZ_m$. Now by the hypotheses on $f(x,t)$ and property (ii) of $\psi_N(x,t)$, $|f(x,t)L\psi_N(x,t)| \in L^1$ and so the second integral on the right in (VI.13) has a limit as $\epsilon \to 0$. The third integrand on the right is in $L^1$ since $Lf$ is. Therefore,

$$\lim_{\epsilon\to 0}\int_{B_r(0)} f(x,\epsilon)\psi_N(x,\epsilon)\,dZ(x,\epsilon) \quad \text{exists.} \tag{VI.14}$$

We can clearly modify $\psi_n$ by dropping the tilde in its definition and use (VI.14) to conclude:

$$\lim_{\epsilon\to 0}\int_{B_r(0)} f(x,\epsilon)\Psi_N(x,\epsilon)\,dZ(x,\epsilon) \quad \text{exists} \tag{VI.15}$$

where for any smooth function $\psi(x)$,

$$\Psi_n(x,t) = \sum_{|\alpha|\le n} (P_\alpha(x,t,D_x)\psi(A(x,t)))(B(x,t) - G(A(x,t)))^\alpha.$$

Let $P(x,t) = B(x,t) - G(A(x,t))$. For $g(x,t) \in C^\infty(B_r(0)\times(-T,T))$ whose $x$-support is contained in a fixed compact set independent of $t$, and $n$ a non-negative integer, define

$$T_n g(x,t) = \sum_{|\alpha|\le n} P_\alpha(x,t,D_x)(g(x,t))P(x,t)^\alpha, \quad T_0 g(x,t) = g(x,t). \tag{VI.16}$$

Using (VI.15), we will show next that in fact,

$$\lim_{t\to 0}\int_{B_r(0)} f(x,t)T_N g(x,t)\,dZ(x,t) \quad \text{exists} \tag{VI.17}$$

for any $g = g(x,t)$. To see this, for $\psi = \psi(x)$, we change variables $y = A(x,t)$ in (VI.15) to write

$$\int f(x,t)\Psi_N(x,t)\,dZ(x,t) = \int f(H(y,t),t)Q(y,t,D_y)\psi(y)\,dy$$

where $Q$ is a differential operator (with differentiation only in $y$) and $y \mapsto H(y,t)$ is the inverse of $x \mapsto A(x,t)$. Since

$$\lim_{t\to 0}\int f(H(y,t),t)Q(y,t,D_y)\psi(y)\,dy \quad \text{exists,}$$

it follows that

$$\lim_{t\to 0}\int f(H(y,t),t)Q(y,t,D_y)\psi(y,t)\,dy \quad \text{exists,}$$

for any smooth $\psi(y,t)$ with a fixed compact support in $y$. Going back to the $x$ coordinates, we have shown that

$$\lim_{t\to 0}\int_{B_r(0)} f(x,t)S_N g(x,t)\,dZ(x,t) \quad \text{exists} \tag{VI.18}$$


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where by definition

$$S_n g(x,t) = \sum_{|\alpha|\le n} (P_\alpha(x,t,D_x)g(A(x,t),t))P(x,t)^\alpha$$

for any smooth $g = g(x,t)$. Observe that the integral in (VI.18) can be written in the form

$$\int u(x,t)g(A(x,t),t)\,\mathrm{d}x$$

where this latter integral denotes the action of a distribution $u(.,t)$ on the smooth function $x \mapsto g(A(x,t),t)$. Now since $(x,t) \mapsto (A(x,t),t)$ is a diffeomorphism near the origin, any function $\psi(x,t)$ is of the form $g(A(x,t),t)$ for some $g = g(x,t)$. We can therefore use (VI.18) to conclude that for any $g(x,t)$,

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_N g(x,t)\,\mathrm{d}Z(x,t) \quad \text{exists,} \tag{VI.19}$$

which proves (VI.17). For $\psi(x,t) \in C^\infty(B_r(0)\times(-T,T))$ whose $x$-support is contained in a fixed compact set and a given multi-index $\beta$ with $|\beta| = N$, plug $g(x,t) = \psi(x,t)P(x,t)^\beta = \psi(x,t)(B(x,t) - G(A(x,t)))^\beta$ in (VI.19). Note that we may write

$$T_N(\psi P^\beta)(x,t) = \psi P^\beta + \psi \sum_{|\alpha|=N} e_\alpha(x,t)P^\alpha + \sum_{|\gamma|>N} h_\gamma(x,t)P^\gamma \tag{VI.20}$$

where the $h_\gamma$ and $e_\alpha$ are smooth functions and

$$\lim_{t\to 0} D_x^{\alpha'} e_\alpha(x,t) = 0 \quad \forall \alpha, \alpha'.$$

Observe that for each $\gamma$ with $|\gamma| > N$,

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t)h_\gamma(x,t)P(x,t)^\gamma\,\mathrm{d}Z(x,t) \quad \text{exists.} \tag{VI.21}$$

Indeed, this follows from applying the integration by parts formula (VI.13) to the $m$-form $f(x,t)h_\gamma(x,t)P(x,t)^\gamma\,\mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_m$, using the hypotheses on $f$, and the bound $|P(x,t)| \le |Z(x,t) - Z(x,0)|$. From (VI.19) and (VI.21) we conclude that

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) \left( \psi P^\beta + \psi \sum_{|\alpha|=N} e_\alpha(x,t)P^\alpha \right) \mathrm{d}Z(x,t) \quad \text{exists.} \tag{VI.22}$$

We can plug $\psi_\beta$ for $\psi$ in (VI.22) and sum over $\beta$ with $|\beta| = N$ to conclude

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) \sum_{|\beta|=N} P^\beta \left( \psi_\beta + \left( \sum_{|\alpha|=N} \psi_\alpha \right) E_\beta(x,t) \right) \mathrm{d}Z(x,t) \quad \text{exists} \tag{VI.23}$$

where all order derivatives of the $E_\beta$ go to zero as $t \to 0$. Observe that given $\{\psi_\beta\}_{|\beta|=N}$ as above, we can find $\{\phi_\beta\}_{|\beta|=N}$ such that

$$\sum_{|\beta|=N} P^\beta \left( \phi_\beta + \left( \sum_{|\alpha|=N} \phi_\alpha \right) E_\beta \right) = \sum_{|\beta|=N} P^\beta \psi_\beta.$$

It follows that

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) \sum_{|\beta|=N} \psi_\beta P^\beta \, \mathrm{d}Z(x,t) \quad \text{exists} \tag{VI.24}$$

whenever the functions $\psi_\beta(x,t) \in C^\infty(B_r(0) \times (-T,T))$ have their $x$-support contained in a fixed compact set independent of $t$. We now return to a general $g(x,t) \in C^\infty(B_r(0) \times (-T,T))$ with $x$-support contained in a fixed compact set independent of $t$. From (VI.19) and (VI.24) we conclude that

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_{N-1} g(x,t) \, \mathrm{d}Z(x,t) \quad \text{exists} \tag{VI.25}$$

for any $g(x,t) \in C^\infty(B_r(0) \times (-T,T))$ with $x$-support contained in a fixed compact set independent of $t$. We will prove by descending induction that for any such $g(x,t)$ and $0 \le k \le N$,

$$\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_k g(x,t) \, \mathrm{d}Z(x,t) \quad \text{exists,}$$

which for $k=0$ and $g(x,t) = \psi(x) \in C_c^\infty(B_r(0))$ gives us the desired limit. To proceed by induction, suppose $1 \le k \le N$ and assume that for any multi-index $\beta$ with $|\beta| = k$, the limits

$$\begin{aligned} &\lim_{t\to 0} \int_{B_r(0)} f(x,t) P^\beta(x,t) g(x,t) \, \mathrm{d}Z(x,t) \quad \text{and} \\ &\lim_{t\to 0} \int_{B_r(0)} f(x,t) T_{k-1} g(x,t) \, \mathrm{d}Z(x,t) \end{aligned} \tag{VI.26}$$

both exist for any $g(x,t) \in C^\infty(B_r(0) \times (-T,T))$ with $x$-support contained in a fixed compact set independent of $t$. We have already seen that (VI.26) is true for $k=N$ as follows from (VI.24) and (VI.25). Fix $\beta'$ with $|\beta'| = k-1$. Plug $g(x,t) = \psi(x,t) P(x,t)^{\beta'}$ in the limit on the right in (VI.26) and observe that $T_{k-1} g$ may be written as

$$T_{k-1} g(x,t) = \psi P^{\beta'} + \psi \sum_{|\alpha|=k-1} e_\alpha(x,t) P^\alpha + \sum_{|\gamma|\ge k} h_\gamma(x,t) P^\gamma \tag{VI.27}$$

where the $e_\alpha$ and $h_\gamma$ are smooth, the $x$-supports of the $h_\gamma(x,t)$ are contained in a compact set that is independent of $t$, and all order derivatives of the $e_\alpha$

go to zero as $t \to 0$. From the existence of the two limits in (VI.26) we derive that

$$\lim_{t\to 0} \int_{B_r(0)} f(x, t) \left( \psi P^{\beta'} + \psi \sum_{|\alpha|=k-1} e_\alpha(x, t) P^\alpha \right) \mathrm{d}Z(x, t) \tag{VI.28}$$

exists. We now argue as before by replacing $\psi$ by $\psi_{\beta'}$ and summing over $|\beta'| = k-1$ to conclude that

$$\lim_{t\to 0} \int_{B_r(0)} f(x, t) P(x, t)^\beta \psi(x, t)\, \mathrm{d}Z(x, t) \quad \text{exists} \tag{VI.29}$$

for all $\beta$ with $|\beta| = k-1$ and $\psi(x, t) \in C^\infty(B_r(0) \times (-B, B))$ with $x$-support contained in a fixed compact set independent of $t$. Hence, taking account of (VI.26) and (VI.29) we conclude that

$$\lim_{t\to 0} \int_{B_r(0)} f(x, t) T_{k-2} g(x, t)\, \mathrm{d}Z(x, t) \quad \text{exists.} \tag{VI.30}$$

We have thus proved that (VI.26) holds for $k-1$, completing the inductive step. Therefore,

$$\lim_{\epsilon\to 0} \int_{B_r(0)} f(x, \epsilon) \psi(x)\, \mathrm{d}Z(x, \epsilon) \quad \text{exists} \tag{VI.31}$$

and thus $bf = \lim_{t\to 0} f(., t)$ exists. Moreover, since the functions

$$x \longmapsto \psi_N(x, \epsilon) - \psi(x) \quad \text{and} \qquad x \longmapsto Z(x, \epsilon) - Z(x, 0)$$

and all their $x$-derivatives converge to zero as $\epsilon \to 0$, (VI.13), (VI.14), and (VI.31) imply the following formula for $bf$:

$$\begin{aligned} \langle Z_x(x, 0) bf, \psi \rangle = &\int_{B_r(0)} f(x, T) \psi_N(x, T)\, \mathrm{d}Z \\ &+ \int_{B_r(0)} \int_0^T f(x, t) L\psi_N(x, t)\, \mathrm{d}t \wedge \mathrm{d}Z \\ &+ \int_{B_r(0)} \int_0^T Lf(x, t) \psi_N(x, t)\, \mathrm{d}t \wedge \mathrm{d}Z. \end{aligned} \tag{VI.32}$$

This formula shows that $bf$ is a distribution of order $N+1$. □

## VI.2 Pointwise convergence to the boundary value

Suppose $L$ is a locally integrable vector field in a planar domain $\Omega$ with a smooth boundary. Let $f \in L^1_{\mathrm{loc}}(\Omega)$, and assume that $f$ has a weak trace $bf$ which is in $L^1_{\mathrm{loc}}(\partial\Omega)$. In this section we will discuss the pointwise convergence


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of $f$ to $bf$. It is classical that when $L$ is the Cauchy–Riemann operator, the holomorphic function $f$ converges nontangentially to $bf(p)$ for almost all $p$ in $\partial\Omega$. In general, this approach region cannot be relaxed. Indeed, we recall:

Theorem VI.2.1. (Theorem 7.44 in [**Zy**].) *Let $C_0$ be any simply closed curve passing through $z = 1$ situated, except for that point, totally inside the circle $|z| = 1$, and tangent to the circle at that point. Let $C_\theta$ be the curve $C_0$ rotated around $z = 0$ by the angle $\theta$. There is a Blaschke product $B(z)$ which, for almost all $\theta_0$, doesn't tend to any limit as $z \mapsto \exp(i\theta_0)$ inside $C_{\theta_0}$.*

This theorem shows us that for nonelliptic vector fields, we can't expect nontangential convergence. Indeed, by the theorem, if

$$L_k = \frac{\partial}{\partial t} - i(k+1)t^k \frac{\partial}{\partial x} \quad (k = 1, 2, 3, \dots)$$

then for each $k$, we can get a bounded solution $f_k = F_k(x + it^{k+1})$ of $L_k$ with $F_k$ holomorphic in a semidisk in the upper half-plane, $bf_k(x) = bF_k(x) \in L^1(-1, 1)$, but each $f_k(x, t)$ doesn't converge nontangentially on a subset of $(-1, 1)$ of positive measure. It suffices to take $F_k$ holomorphic and bounded on the semidisk $\{z : |z| < 1, \Im z > 0\}$ such that on a set of full measure in $(-1, 1)$, $F_k$ has no limit in certain appropriate regions. By considering the $L_k$ with $k$ even, we see that nontangential convergence may fail even for vector fields that are $C^\infty$ and analytic hypoelliptic. Note that for each $k$, and for almost all $p \in (-1, 1)$, there is an open region $\Gamma_k(p)$ with $p \in \overline{\Gamma}_k(p)$ such that $f_k(x, t)$ converges to $bf_k(p)$ in $\Gamma_k(p)$. On the other hand, if we take the real vector field $\frac{\partial}{\partial t}$, and the solution $u(x, t) \equiv bu(x) = \chi$, the characteristic function of a Cantor set $C$ of positive measure in $(-1, 1)$, the only sets of approach for which $u(x, t) \to bu(x), \quad x \in C$, are the vertical segments. Thus for a general locally integrable vector field, we cannot get approach sets for convergence larger than curves. Suppose now $L = X + iY$ is a smooth, locally integrable vector field near the closure of a planar domain $\Omega$. Assume $\Sigma \subseteq \partial\Omega$ is a smooth curve that is noncharacteristic for $L$, $f \in L^1_{\mathrm{loc}}(\Omega)$, $Lf = 0$ and $f$ has a trace $bf \in L^1(\Sigma)$. Multiplying by $i$ if necessary, we may assume that $X$ is not tangent to $\Sigma$ anywhere and that it points toward $\Omega$. For each $p \in \Sigma$, let $\gamma_p$ be the integral curve of $X$ through $p$ and set $\gamma_p^+ = \gamma_p \cap \Omega$. We shall classify the points of $\Sigma$ into two types:

(I) A point $p \in \Sigma$ is a type I point if the vector fields $X$ and $Y$ are linearly dependent on an arc $\{\gamma_p^+(s) : 0 < s < \epsilon\}$ for some $\epsilon > 0$.

(II) A point $q \in \Sigma$ is a type II point if there is a sequence $q_k \in \gamma_p^+$ converging to $q$ such that $L$ is elliptic at each $q_k$.

THEOREM VI.2.2. *Let $Lu = 0$ in $\Omega$, $u \in L^1_{loc}(\Omega)$, $bu \in L^1(\Sigma)$, and $\Sigma$ is noncharacteristic for $L$. Assume $L$ is locally integrable in a neighborhood of $\Sigma$. For each $p \in \Sigma$, there is an approach set $\Gamma(p) \subseteq \Omega$ such that:*

(i) *$p \in \overline{\Gamma}(p)$ and if $q \in \Sigma \cap \overline{\Gamma}(p)$, then $q = p$;*
(ii) *$\gamma_p^+ \subseteq \Gamma(p)$;*
(iii) *for a.e. $p \in \Sigma$, $\lim_{\Gamma(p)\ni q \to p} u(q) = u(p)$;*
(iv) *if $p$ is a type II point, $\Gamma(p)$ is an open set, otherwise $\Gamma(p) = \gamma_p^+$.*

PROOF. Since the problem is local, we may assume that we are in coordinates $(x,t)$ where $\Omega = (-1,1) \times (0,1)$, $\Sigma = (-1,1) \times \{0\}$, and $Z(x,t) = x + i\varphi(x,t)$ is a first integral of $L$ with $\varphi$ real, $\varphi(0,0) = 0$ and $\varphi_x(0,0) = 0$. Modulo a nonvanishing factor,

$$L = \frac{\partial}{\partial t} - i\frac{\varphi_t}{1+i\varphi_x}\frac{\partial}{\partial x}$$

and so

$$X = \frac{\partial}{\partial t} - \left(\frac{\varphi_t\varphi_x}{1+\varphi_x^2}\right)\frac{\partial}{\partial x}, \quad Y = \frac{-\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x}.$$

Observe that $L$ is elliptic, i.e., $X$ and $Y$ are linearly independent precisely at the points where $\varphi_t \neq 0$. Assume now that $0 \in \Sigma$ is a type II point. Then $t \mapsto \varphi(0,t)$ can't vanish on any interval $[0,\epsilon]$, $\epsilon > 0$. Indeed, otherwise, we would conclude that $L = X$ on $\{0\} \times [0,\epsilon)$—contradicting the hypothesis that 0 is a type II point. For $\delta > 0$ small, define

$$m(x) = \inf_{0\le t\le\delta} \varphi(x,t), \quad M(x) = \sup_{0\le t\le\delta} \varphi(x,t).$$

Then since $m(0) < M(0)$, we may choose $A > 0$ so that $m(x) < M(x)$ for $|x| \le A$. After decreasing $A$ and $\delta$, by the boundary version of the Baouendi–Treves approximation theorem in Chapter II (Theorem II.4.12), there is a sequence of entire functions $F_k$ satisfying:

(a) $F_k(Z(x,t)) \to u(x,t)$ pointwise a.e. on $(-A,A) \times (0,\delta)$;
(b) $F_k(Z(x,0)) \to bu(x)$ a.e. on $(-A,A)$.

Set

$$\Omega_A = \{\zeta = \xi + i\eta : |\xi| < A,\ m(\xi) < \eta < M(\xi)\}.$$

We may assume that the sequence $F_k$ converges uniformly on compact subsets of $\Omega_A$ to a holomorphic function $F$ and $u(x,t) = F(Z(x,t))$ for $(x,t) \in Z^{-1}(\Omega_A)$. Indeed, this is clearly true if $u(x,t)$ is continuous for $t > 0$. In general, we can use the fact that we can express $u$ as $Qh$ where $h$ is a continuous solution and $Q$ is an elliptic differential operator that maps solutions to solutions. The operator $Q$ can be taken to be a convenient power of the operator $D$ defined in Section IV.2. Since 0 is a type II point, by theorem 3.1 in [**BH1**] and [**BCT**] (page 465), for some $0 < A_1 < A$, $0 < \delta_1 < \delta$, there is a holomorphic function $G$ of tempered growth defined on the region $\Omega_1 = \{Z(x,0) + iZ_x(x,0)v : |x| < A_1,\ 0 < v < \delta_1\}$ such that for every $\psi \in C_0^\infty(-A_1, A_1)$,

$$\langle bu, \psi \rangle = \lim_{v \downarrow 0} \int G(Z(x,0) + iZ_x(x,0)v)\psi(x)\mathrm{d}x.$$

Since $bu \in L^1$, the holomorphic function $G(z)$ converges nontangentially to $bu(x)$ a.e. in $(-A_1, A_1)$. We may assume that $A_1$ and $\delta_1$ are small enough so that $\Omega_1 \subseteq \Omega_A$. We will show that $G = F$ on $\Omega_1$. Define the subsets of $[-A_1, A_1]$:

$$\begin{aligned}
E_1 &= \{x : \varphi(x,t) = \varphi(x,0),\ t \in [0,\tau] \quad \text{for some}\ \tau > 0\},\\
E_2 &= \{x : \varphi(x,t) \geq \varphi(x,0),\ t \in [0,\tau] \quad \text{for some}\ \tau > 0\},\\
E_3 &= \{x : \varphi(x,t) \leq \varphi(x,0),\ t \in [0,\tau] \quad \text{for some}\ \tau > 0\},\\
E_4 &= \{x : \text{for some} \quad t_j \to 0, \quad s_j \to 0,\ \varphi(x,s_j) < \varphi(x,0) < \varphi(x,t_j)\}.
\end{aligned}$$

Observe that $[-A_1, A_1] = E_1 \cup E_2 \cup E_3 \cup E_4$. If $x_0 \in E_4$, then by theorem 3.1 in [**BH1**], there is a holomorphic function $H$ defined in a neighborhood of $Z(x_0, 0)$ such that $u(x,t) = H(Z(x,t))$ for $(x,t)$ in a neighborhood of $(x_0, 0)$, $t > 0$. Hence in this case, $F(z)$ has a holomorphic extension to a neighborhood of $Z(x_0,0)$ and since $u(x,t) = F(Z(x,t))$ for $t > 0$, we have $F(Z(x,0)) = bu(x) = bG(Z(x,0))$. Therefore, by theorem 2.2 in [**Du**], $F(z) = G(z)$ on $\Omega_1$. We may therefore assume that $E_4 = \emptyset$. Each of the other three sets $E_1$, $E_2$, and $E_3$ can be written as a countable union of closed sets as follows: $E_1 = \bigcup_{j=1}^\infty E_{1j}$, where $E_{1j} = \{x \in [-A_1, A_1] : \varphi(x,t) = \varphi(x,0), t \in [0, \frac{1}{j}]\}$; $E_2 = \bigcup_{j=1}^\infty E_{2j}$, where $E_{2j} = \{x \in [-A_1, A_1] : \varphi(x,t) \geq \varphi(x,0),\ t \in [0, \frac{1}{j}]\}$; and $E_3 = \bigcup_{j=1}^\infty E_{3j}$, where $E_{3j} = \{x \in [-A_1, A_1] : \varphi(x,t) \leq \varphi(x,0),\ t \in [0, \frac{1}{j}]\}$. Thus the interval $[-A_1, A_1]$ is a countable union of the closed sets $E_{ij}$ and hence by Baire's Category Theorem, one of these sets contains an interval with nonempty interior.

**Case 1**: Suppose $\varphi(x,t) = \varphi(x,0)$ on $[A_2, A_3] \times [0,T]$ for some $T > 0$, $A_2 < A_3$. Then $L = \frac{\partial}{\partial t}$ on $[A_2, A_3] \times [0,T]$ and so $u(x,t) = bu(x)$ on this rectangle. This implies that $F(z)$ extends as a continuous function in $\Omega_1$ up to the boundary piece $\{Z(x,0) : A_2 < x < A_3\}$ and therefore $bF(Z(x,0)) = bu(x)$ for $x \in (A_2, A_3)$. But then $F \equiv G$ in $\Omega_1$.

**Case 2**: Suppose $\varphi(x,t) \geq \varphi(x,0)$ on $[A_2, A_3] \times [0,T]$, for some $T > 0$, $A_2 < A_3$. For $\epsilon > 0$ sufficiently small, define

$$u_\epsilon(x,t) = G(Z(x,t) + i\epsilon), \quad (x,t) \in (A_2, A_3) \times (0,T).$$

Observe that $Lu_\epsilon = 0$. Recall that $G$ is holomorphic on the region $\Omega_1 = \{Z(x,0) + iZ_x(x,0)v : |x| < A_1,\ 0 < v < \delta_1\}$. Let $\Omega_2 = \{Z(x,0) + iZ_x(x,0)v : |x| < A_1,\ 0 < v < \delta_2\}$ for some $0 < \delta_2 < \delta_1$, and for each $p = Z(x,0)$, $|x| < A_1$, define the nontangential approach region

$$\Gamma(p) = \{z \in \Omega_2 : |z - p| < 2\,\mathrm{dist}(z, \partial\Omega_2)\}.$$

Denote by $G^*(x)$ the nontangential maximal function of $G(z)$, that is,

$$G^*(x) = \sup\{|G(z)| : z \in \Gamma(Z(x,0))\}.$$

We have:

$$|u_\epsilon(x,t)| \leq G^*(x) \in L^1(A_2, A_3).$$

Let

$$\begin{aligned} w(x,t) &= \lim_{\epsilon \to 0} u_\epsilon(x,t) \quad \text{(the pointwise limit)} \\ &= \begin{cases} G(x + i\varphi(x,t)), & \text{if} \quad \varphi(x,t) > \varphi(x,0) \\ bu(x), & \text{if} \quad \varphi(x,t) = \varphi(x,0). \end{cases} \end{aligned}$$

Then $u_\epsilon \to w$ in $L^1((A_2, A_3) \times (0,T))$ and so $Lw = 0$ in $(A_2, A_3) \times (0,T)$. Since

$$|G(x + i\varphi(x,t))| \leq G^*(x) \quad \text{and a.e.} \quad G(x + i\varphi(x,t)) \to bu(x) \ \text{ as } \ t \to 0,$$

we conclude that

$$w(x,t) \to bu(x) \ \text{ in } \ L^1(A_2, A_3) \ \text{ as } \ t \to 0.$$

Therefore $u(x,t) = w(x,t)$ in a neighborhood of $(A_2, A_3) \times \{0\}$, $t > 0$. In particular, since we may assume that

$$\{(x,t) \in (A_2, A_3) \times (0,T) : \varphi(x,t) > \varphi(x,0)\}$$

is not empty (otherwise, we would be placed under Case 1), $F(z) \equiv G(z)$ on $\Omega_1$.

**Case 3**: Suppose $\varphi(x,t) \leq \varphi(x,0)$ on $[A_2, A_3] \times [0,T]$, $T > 0$, $A_2 < A_3$. We may assume that there exists $x_0 \in (A_2, A_3)$ and $s_j \to 0$ such that $\varphi(x_0, s_j) < \varphi(x_0, 0)$. Indeed, otherwise, matters will reduce to Case 1. By theorem 3.1

in [**BH1**] and [**BCT**] (page 465), after decreasing $[A_2, A_3] \times [0, T]$, we get a tempered holomorphic function $G_1(z)$ defined on the region

$$\Omega_1' = \{Z(x,0) + iZ_x(x,0)v : A_2 < x < A_3, \ -T < v < 0\}$$

such that for every $\psi \in C_0^\infty(A_2, A_3)$,

$$\langle bu, \psi \rangle = \lim_{v \to 0} \int G_1(Z(x,0) + iZ_x(x,0)v)\psi(x)\mathrm{d}x.$$

By the edge-of-the-wedge theorem, there is a holomorphic function $v(z)$ defined in a neighborhood of $\{Z(x,0) : A_2 < x < A_3\}$ that extends $G$ and $G_1$. Hence $F(z) = G(z)$ in $\Omega_1$. We have thus shown that $F \equiv G$ on $\Omega_1$.

Now for almost every $p \in (-A_1, A_1)$, $G(z)$ converges nontangentially at $Z(p, 0)$ (in $\Omega_1$) to $bu(p)$. Pick such a point $p$ and let $\tilde{\Gamma}(p)$ be a nontangential approach region for $G(z)$ at $Z(p, 0)$. Define $\Gamma(p) = Z^{-1}(\tilde{\Gamma}(p))$. Then

$$\begin{aligned}\lim_{\Gamma(p) \ni (x,t) \to p} u(x,t) &= \lim_{\Gamma(p) \ni (x,t) \to p} F(Z(x,t)) \\ &= \lim_{\tilde{\Gamma}(p) \ni z} G(z) = bu(p).\end{aligned}$$

We have thus shown that if $p$ is a type II point, then there is an interval around it such that a.e. in the interval, pointwise convergence holds as asserted. Consider now a type I point $(x_0, 0)$. Then $Z(x_0, t) \equiv Z(x_0, 0)$ for $t$ in some interval $[0, \epsilon]$. This implies that $F_k(Z(x_0, t)) \equiv F_k(Z(x_0, 0))$ for $t \in [0, \epsilon]$, and so because of the a.e. convergence stated in (a) and (b), we conclude that for almost every type I point $x$, $u(x, t) \to bu(x)$ as $t \to 0$. □

## VI.3 One-sided local solvability in the plane

In Section VI.4 we will explore the boundary regularity of solutions of the inhomogeneous equation $Lf = g$ where

$$L = A(x,t)\frac{\partial}{\partial t} + B(x,t)\frac{\partial}{\partial x}$$

is a smooth, locally integrable complex vector field defined on a subdomain $\Omega$ of $\mathbb{R}^2$.

If $Lf = g$ in $\Omega$, and $f$ has a trace $bf$ on $\partial\Omega$ with a certain degree of regularity, we will investigate whether the regularity persists near $\partial\Omega$ under some smoothness assumption on $g$. As usual, the motivation comes from what is known in the elliptic case. Suppose $h(z)$ is a holomorphic function of one variable defined on the rectangle $Q = (-A, A) \times (0, T)$ with a weak trace $bh$

at $y = 0$. From the local version of the classical Hardy space ($H^p$) theory for holomorphic functions in the unit disk, we have:

(i) if $bh \in C^\infty(-A, A)$, then $h$ is $C^\infty$ up to $y = 0$;
(ii) if $bh \in L^p(-A, A)$ $(1 \leq p \leq \infty)$, then for any $B < A$, the norms of the traces $h(\cdot, y)$ in $L^p(-B, B)$ are uniformly bounded as $y \to 0^+$.

The main results of Section VI.4 will extend (i) and (ii) above to solutions of complex vector fields that satisfy a one-sided solvability condition. In the elliptic case, property (i) follows easily from part (ii) of Theorem VI.1.1. We will show in Section VI.4 that in general, property (i) follows from property (ii) above and a boundary solvability condition. When a vector field exhibits property (ii), we will say that it has the $H^p$ property. To describe the class of vector fields with the $H^p$ property, consider a curve $\Sigma$ in $\Omega$ such that $\Omega\backslash\Sigma$ has two connected components, $\Omega\backslash\Sigma = \Omega^+ \cup \Omega^-$. It turns out that the local solutions of the equation $Lu = 0$ on $\Omega^+$ possess the ($H^p$) property at $q \in \Sigma$ if and only if there is a neighborhood $U$ of $q$ such that $L$ satisfies the solvability condition $(\mathcal{P})$ of Nirenberg and Treves ([**NT**]) on $U \cap \Omega^+$. This leads to a one-sided version of $(\mathcal{P})$ that we denote by $(\mathcal{P}^+)$ (or $(\mathcal{P}^-)$ if $\Omega^+$ is replaced by $\Omega^-$) to indicate the side where it holds. If $(\mathcal{P})$ holds at $q$, then both $(\mathcal{P}^+)$ and $(\mathcal{P}^-)$ hold at $q$. However, $(\mathcal{P}^+)$ and $(\mathcal{P}^-)$ may hold at $q \in \Sigma$ and yet $(\mathcal{P})$ may not hold in a neighborhood of $q$. The Mizohata vector field provides an example illustrating this. Write $L = X + iY$ with $X$ and $Y$ real. Let $\mathcal{O} \subset U$ be a two-dimensional orbit of $L$ in $U$ and consider $X \wedge Y \in C^\infty(U; \bigwedge^2(T(U)))$. Since $\bigwedge^2(T(U))$ has a global nonvanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $X \wedge Y$ does not change sign on any two-dimensional orbit $\mathcal{O}$ of $\{X, Y\}$ in $U$. Recall from Chapter IV that the vector field $L$ satisfies condition $(\mathcal{P})$ at $p \in \Sigma$ if there is a disk $U \subseteq \Omega$ centered at $p$ such that $X \wedge Y$ does not change sign on any two-dimensional orbit of $L$ in $U$.

Definition VI.3.1. *We say that $L$ satisfies condition $(\mathcal{P}^+)$ at $p \in \Sigma$ if there is a disk $U \subseteq \Omega$ centered at $p$ such that $X \wedge Y$ does not change sign on any two-dimensional orbit of $L$ in $U^+ = U \cap \Omega^+$.*

Definition VI.3.2. *We say that $L$ is one-sided locally solvable in $L^p$, $1 < p < \infty$ (resp. in $C^\infty$) at $q \in \Sigma$ if there is a neighborhood $U \subseteq \Omega$ of $q$ such that—after interchanging $\Omega^+$ and $\Omega^-$ if necessary—for every $f \in L^p(U)$ (resp. $f \in C^\infty(U \cap \Omega^+)$ ) there exists $u \in L^p(U)$ (resp. $u \in C^\infty(U \cap \Omega^+)$) such that $Lu = f$ on $U^+ = U \cap \Omega^+$.*

DEFINITION VI.3.3. *We say that $L$ is one-sided locally integrable at $p \in \Sigma$ if there is a disk $U \subset \Omega$ centered at $p$ such that—after interchanging $\Omega^+$ and $\Omega^-$ if necessary—there exists $Z \in C^\infty(U)$ such that:*

(1) *$LZ$ vanishes identically on $U^+ = U \cap \Omega^+$;*
(2) $\mathrm{d}Z(p) \neq 0$.

Let us assume that $L$ is one-sided locally integrable at $p \in \Sigma$ and let $Z$ satisfy (1) and (2) of Definition VI.3.3. Replacing $Z$ by $iZ$ if necessary and decreasing $U$ we may choose local coordinates $(x, t)$ such that $x(p) = t(p) = 0$,

$$Z(x, t) = x + i\varphi(x, t) \tag{VI.33}$$

with $\varphi$ real, $U$ is the rectangle $U = (-a, a) \times (-T, T)$, $\Sigma \cap U = \{(x, 0) : |x| < a\}$ and $U^+ = (-a, a) \times (0, T)$. Thus, modulo a nonvanishing multiple, we may assume that

$$L = \frac{\partial}{\partial t} - i\frac{\varphi_t(x, t)}{1 + i\varphi_x(x, t)}\frac{\partial}{\partial x}, \tag{VI.34}$$

$$X = \frac{\partial}{\partial t} + \frac{\varphi_t\varphi_x}{1+\varphi_x^2}\frac{\partial}{\partial x}, \qquad Y = -\frac{\varphi_t}{1+\varphi_x^2}\frac{\partial}{\partial x},$$

and so

$$X \wedge Y = \frac{\varphi_t(x, y)}{1+\varphi_x^2}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t}.$$

The proof of the following lemma is essentially the same as the one for Lemma IV.2.2.

LEMMA VI.3.4. *Let $Z(x, t)$ and $L$ be given by (VI.33) and (VI.34) respectively. Then, $L$ satisfies $(\mathcal{P}^+)$ at the origin if and only there exist $T, a > 0$ such that $(0, T) \ni t \mapsto \varphi(x, t)$ is monotone for every $x \in (-a, a)$.*

We now recall from [**BH6**] the local equivalence between $(\mathcal{P}^+)$ and one-sided solvability. More precisely,

THEOREM VI.3.5. *Let $Z(x, t)$ and $L$ be given by (VI.33) and (VI.34) respectively. The following properties are equivalent:*

(1) *$L$ satisfies $(\mathcal{P}^+)$ ( or $(\mathcal{P}^-)$) at the origin;*
(2) *$L$ is one-sided locally solvable in $L^p$, $1 < p < \infty$, at the origin;*
(3) *$L$ is one-sided locally solvable in $C^\infty$ at the origin.*

The following proposition is concerned with continuous solvability up to the boundary and will be useful in the applications to boundary regularity in Section VI.4.

PROPOSITION VI.3.6. *Let $Z(x,t)$ and $L$ be given by (VI.33) and (VI.34) respectively and assume that $L$ satisfies $(\mathcal{P}^+)$ at the origin, i.e., for some $U^+ = (-r,r)\times(0,T)$, the function $(0,T)\ni t\mapsto \varphi(x,t)$ is monotone for $|x|<r$. If $f(x,t)\in \mathrm{Lip}(U)$ there exists $u\in\bigcap_{0<\alpha<1} C^\alpha((-r,r)\times[0,T))$ such that $Lu=f$ in $U^+$.*

The proof of the proposition is based on the following lemma.

LEMMA VI.3.7. *Let $F(\zeta)\in L^\infty_c(\mathbb{C})$ and let $f(x,t)=F\circ Z(x,t)$. There exists $v\in\bigcap_{0<\alpha<1} C^\alpha((-r,r)\times(-T,T))$ such that $Lv=2i\varphi_t Z_x^{-1} f$ on $Q=(-r,r)\times(-T,T)$.*

PROOF. Let $E=1/(\pi\zeta)$ be the fundamental solution of $\partial/\partial\overline{\zeta}$ and set $V=E*F$. Then $V\in\bigcap_{0<\alpha<1} C^\alpha$ locally and $\overline{\partial_\zeta} V=F$ in the sense of distributions. If we set $v=V\circ Z$ it follows that $v$ is in $\bigcap_{0<\alpha<1} C^\alpha((-r,r)\times[0,T))$ and the chain rule gives $Lv=-2i\varphi_t Z_x^{-1}\,(\overline{\partial_\zeta} V)\circ Z=-2i\varphi_t Z_x^{-1} f$. □

PROOF OF PROPOSITION VI.3.6. Let $f\in\mathrm{Lip}(U)$. Set $u_0(x,t)=\int_0^t f(x,s)\,\mathrm{d}s$. Then, $u_0\in\mathrm{Lip}(U)$ and $Lu_0-f=-i\varphi_t Z_x^{-1}\int_0^t\partial_x f\,\mathrm{d}s=2i\varphi_t Z_x^{-1} f_1$ where $f_1$ is bounded. It is clear that we will be able to solve $Lu=f$ on $Q^+$ if we can solve

$$Lu_1 = 2i\varphi_t Z_x^{-1} f_1 \quad \text{on } Q^+ \tag{VI.35}$$

by setting $u=u_0-u_1$. In view of Lemma VI.3.7 we wish to write $f_1=F_1\circ Z(x,t)$ and the obstruction to doing so is the fact that $f_1$ may not be constant on the fibers $Z^{-1}(\zeta)$, $\zeta\in Z(Q^+)$. However, we are free to modify arbitrarily $f_1$ on the set $\{\varphi_t=0\}\cup\{t\le 0\}$ without modifying the right-hand side of (VI.35). Hence, we declare that $f_1$ vanishes on $\{\varphi_t=0\}$ as well as on $t\le 0$. Since $Z$ is a diffeomorphism on $Q^+\setminus\{\varphi_t=0\}$, we may write $f_1=F_1\circ Z(x,t)$ with $F_1$ bounded on $Z(Q^+)$ and extend $F_1$ as zero outside $Z(Q^+)$, so $F_1\in L^\infty_c(\mathbb{C})$. An application of Lemma VI.3.7 shows that there exists a function $u_1$ of class $C^\alpha(U)$ for any $0<\alpha<1$ whose restriction to $U^+$ satisfies (VI.35). Then $u=u_0-u_1\in C^\alpha(U^+)=C^\alpha\left(\overline{U^+}\right)$. □

## VI.4 The $H^p$ property for vector fields

Consider a one-sided locally integrable smooth vector field

$$L=\frac{\partial}{\partial t}+a(x,t)\frac{\partial}{\partial x}$$

defined on a neighborhood $Q=(-A,A)\times(-B,B)$ of the origin with a one-sided first integral $Z(x,t)=x+i\varphi(x,t)$ defined on $Q$ satisfying $LZ=0$

for $t \geq 0$. In this section we will assume that $L$ satisfies condition $(\mathcal{P}^+)$ at the origin in $\Sigma = (-A, A) \times \{0\}$. We may clearly assume that $\varphi(0,0) = \varphi_x(0,0) = 0$ and

$$|\varphi_x(x,t)| < \frac{1}{2} \quad \text{on a neighborhood of } Q.$$

After a further contraction of $Q$ about the origin, Lemma VI.3.4 shows that

$$\text{for every } x \in (-A, A), \quad \text{the map } (0, B) \ni t \mapsto \varphi(x,t) \text{ is monotone.}$$

The main result of this section is as follows:

THEOREM VI.4.1. *Suppose $f$ is a distribution solution of $Lf = 0$ in the rectangle $Q = (-A, A) \times (0, B)$. Assume $f$ has a weak boundary value $bf = f(x, 0)$ at $y = 0$. Then there exist $A_0 > 0$ and $T_0 > 0$ such that for any $0 < T \leq T_0$ and $0 < a < A_0$, if $f(.,0)$ and $f(.,T) \in L^p(-A_0, A_0)$, $f(.,t) \in L^p(-a,a)$ for any $0 < t < T$ and for almost all $0 < a < A_0$, there exists $C = C(a,T)$ such that*

(i) *if $1 \leq p < \infty$, then*

$$\begin{aligned}\int_{-a}^{a} |f(x,t)|^p \,\mathrm{d}x \leq & C\left(\int_{-a}^{a} |f(x,0)|^p \,\mathrm{d}x + \int_{-a}^{a} |f(x,T)|^p \,\mathrm{d}x\right. \\ & + \int_0^T |f(a,s)|^p |\varphi_s(a,s)| \,\mathrm{d}s \\ & \left. + \int_0^T |f(-a,s)|^p |\varphi_s(-a,s)| \,\mathrm{d}s\right);\end{aligned}$$

(ii) *if $p = \infty$, then $f \in L^\infty((-a,a) \times (0,T))$.*

Before proving Theorem VI.4.1, we will need to recall some concepts and results from the classical theory of Hardy spaces for bounded, simply connected domains in the complex plane. Let $D$ be a such a domain with rectifiable boundary. There are several definitions of a Hardy space for such a domain (see [**L**] and [**Du**]). For our purpose here, we need to recall two of the definitions:

DEFINITION VI.4.2. [**Du**] *For $1 \leq p < \infty$, a holomorphic function $g$ on a bounded domain $D$ with rectifiable boundary is said to be in $E^p(D)$ if there exists a sequence of rectifiable curves $C_j$ in $D$ tending to $bD$ in the sense that the $C_j$ eventually surround each compact subdomain of $D$, such that*

$$\int_{C_n} |g(z)|^p |\mathrm{d}z| \leq M < \infty.$$

*The norm of $g \in E^p(D)$ is defined as*

$$||g||^p_{E^p(D)} = \inf \sup_j \int_{C_j} |g(z)|^p |\mathrm{d}z|$$

*where the inf is taken over all sequences of rectifiable curves $C_j$ in $D$ tending to $\partial D$.*

DEFINITION VI.4.3. *Suppose for a bounded region $\Omega \subseteq \mathbb{C}$ there is $\alpha = \alpha(\Omega) > 0$ with the property that almost every point $p$ in the boundary admits a nonempty nontangential approach subregion*

$$\Gamma_\alpha(p) = \{z \in \Omega : |z - p| \leq (1+\alpha)\mathrm{dist}\,(z, \partial\Omega)\}$$

*that is, for a.e. $p \in \partial\Omega$, $\Gamma_\alpha(p)$ is open and $p$ is in the closure of $\Gamma_\alpha(p)$. Let $u$ be a function defined on $\Omega$. The nontangential maximal function of $u$, $u^*$, and the nontangential limit of $u$, $u^+$, are defined as follows:*

$$u^*(p) = \sup_{\zeta \in \Gamma_\alpha(p)} |u(\zeta)|, \quad a.e. \quad p \in \partial\Omega,$$

$$u^+(p) = \lim_{\zeta \in \Gamma_\alpha(p)} u(\zeta), \quad a.e. \quad p \in \partial\Omega.$$

DEFINITION VI.4.4. *For $1 \leq p < \infty$ the Hardy space $H^p(\Omega)$ is defined by*

$$H^p(\Omega) = \{G \in \mathbb{O}(\Omega) : G^* \in L^p(\partial\Omega)\}$$

*where $\mathbb{O}(\Omega)$ denotes the holomorphic functions on $\Omega$ and $G^*$ denotes the nontangential maximal function defined using the $\Gamma_\alpha(p)$ as in the definition above.*

When $\Omega$ is the unit disk, it is a classical fact that both definitions of Hardy spaces agree ([**Du**]). By the Riemann mapping theorem, this is also true for any bounded, simply connected domain with a smooth boundary. In the work [**L**], it is shown that when $1 < p < \infty$, these spaces agree if $\Omega$ is bounded, simply connected with a Lipschitz boundary.

DEFINITION VI.4.5. *For $1 < q < \infty$, the maximal operator $T_*$ on $L^q(\partial\Omega)$ is defined by*

$$T_* u(p) = \sup_{\epsilon > 0} \left| \int_{|\zeta - p| > \epsilon} \frac{1}{\zeta - p} u(\zeta)\, \mathrm{d}\zeta \right|, \quad a.e. \quad p \in \partial\Omega.$$

Let us denote the Cauchy integral of a function $u$ by $Cu$. We will be interested in the $L^p$ boundedness of the nontangential maximal operator $(Cu)^*$ on certain kinds of domains which we now describe:

Definition VI.4.6. *A bounded, simply connected domain* $\Omega$ *is called Ahlfors-regular if there is a constant* $c > 0$ *such that for every* $q \in \partial\Omega$, *and for every* $r > 0$, *the arclength measure of the portion of the boundary contained in the disk of radius* $r$ *centered at* $q$ *is less than* $cr$.

We note that examples of Ahlfors-regular domains include simply connected domains with Lipschitz boundary. Ahlfors-regular domains admit nontangential approach regions $\Gamma_\alpha(p)$ as in Definition VI.4.3. The study of the boundedness of the operator $T_*$ on domains with Lipschitz boundary was initiated by A. Calderón in the 1970s. He proved that $T_*$ is well-defined and bounded on $L^q(\partial\Omega)$ $(1 < q < \infty)$ provided the Lipschitz character of $\Omega$ is smaller than an absolute constant. Later, R. Coifman, A. McIntosh and Y. Meyer extended this result to the entire Lipschitz class. G. David has shown that the Ahlfors-regular domains are the largest rectifiable domains on which $T_*$ is bounded. More precisely, he proved:

Theorem VI.4.7. [**D**] *Let* $\Omega \subseteq \mathbb{C}$ *be a bounded, simply connected domain with rectifiable boundary. Then* $T_*$ *is bounded on* $L^q(\partial\Omega)$, $1 < q < \infty$, *if and only if* $\Omega$ *is an Ahlfors-regular domain.*

The Hardy–Littlewood maximal function $Mu$ on $\partial\Omega$ is defined by

$$Mu(z) = \sup \frac{1}{|I|} \int_I |u(\zeta)|\,|\mathrm{d}\zeta|$$

where the sup is taken over all subarcs $I \subseteq \partial\Omega$ that contain $z$ and $|I|$ denotes the arclength of $I$. It is well known that the Hardy–Littlewood maximal function of $\partial\Omega$ is $L^p$ bounded $(1 < p < \infty)$ for a class of domains that includes the Ahlfors-regular domains ([**D**]). The following lemma therefore reduces the boundedness of $(Cu)^*$ to that of $T_*$.

Lemma VI.4.8. *Let* $\Omega \subseteq \mathbb{C}$ *be an Ahlfors-regular domain. The following inequality holds for every* $u \in L^q(\partial\Omega)$, $1 < q < \infty$, *and every* $p \in \partial\Omega$:

$$(Cu)^*(p) \leq T_*u(p) + c(\alpha)Mu(p), \tag{VI.36}$$

*where* $(Cu)^*$ *denotes the nontangential maximal function of the Cauchy integral of* $u$ *and* $c(\alpha)$ *is a positive constant depending exclusively on the aperture of the cone* $\Gamma_\alpha(p)$.

Proof. For $p \in \partial\Omega$ arbitrary, it suffices to show that

$$|Cu(x)| \leq T_*u(p) + c(\alpha)Mu(p) \quad \text{for every } x \in \Gamma_\alpha(p).$$


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Let $r := |x-p|$. We have

$$2\pi i C u(x) = \int_{|\zeta-p|>2r} \frac{u(\zeta)}{\zeta-p}\,\mathrm{d}\zeta + \int_{|\zeta-p|>2r} \left(\frac{u(\zeta)}{\zeta-x} - \frac{u(\zeta)}{\zeta-p}\right)\mathrm{d}\zeta + \int_{|\zeta-p|<2r} \frac{u(\zeta)}{\zeta-x}\,\mathrm{d}\zeta$$

$= I_1 + I_2 + I_3$. We will now proceed to estimate $|I_i|$, $i = 1, 2, 3$. Clearly, $|I_1| \le T_* u(p)$.

To estimate $I_2$ observe that

$$\left|\frac{1}{\zeta-x} - \frac{1}{\zeta-p}\right| = \frac{r}{|\zeta-x|\,|\zeta-p|}. \tag{VI.37}$$

But $|\zeta-p| \le |\zeta-x| + |x-p|$ and since $x \in \Gamma_\alpha(p)$, we have: $|\zeta-p| \le (2+\alpha)|x-\zeta|$. Hence (VI.37) becomes

$$\left|\frac{1}{\zeta-x} - \frac{1}{\zeta-p}\right| \le \frac{(2+\alpha)r}{|\zeta-p|^2}.$$

$I_2$ can thus be estimated as follows:

$$\begin{aligned}
|I_2| &\le (2+\alpha)\int_{|\zeta-p|>2r} \frac{r}{|p-\zeta|^2}|u(\zeta)|\,\mathrm{d}\sigma(\zeta)\\
&\le (2+\alpha)\sum_{j=1}^{\infty}\int_{2^j r<|p-\zeta|<2^{j+1}r} \frac{r}{(2^j r)^2}|u(\zeta)|\,\mathrm{d}\sigma(\zeta)\\
&\le 2(2+\alpha)\sum_{j=1}^{\infty}\frac{1}{2^j}\left(\frac{1}{2^{j+1}r}\int_{|p-\zeta|<2^{j+1}r}|u(\zeta)|\,\mathrm{d}\sigma(\zeta)\right)\\
&\le c(\alpha)Mu(p).
\end{aligned}$$

Finally, in order to estimate $I_3$ we observe that $x \in \Gamma_\alpha(p)$ and $\zeta \in \partial\Omega$ imply

$$\frac{1}{|\zeta-x|} \le \frac{1+\alpha}{r}.$$

Using the latter estimate we obtain:

$$|I_3| \le \frac{(1+\alpha)}{2\pi r}\int_{|p-\zeta|<2r}|u(\zeta)|\,\mathrm{d}\sigma(\zeta) \le c(\alpha)Mu(p).$$

□

Our next aim is to prove that $E^p(\Omega) = H^p(\Omega)$ for a particular class of domains $\Omega$ that includes the domains $U_k$ that will appear in the proof of Theorem VI.4.1. We consider smooth regions $U$ that are bounded by two smooth curves $C_1$ and $C_2$ that cross each other at two points $A$ and $B$ where they meet at angles $0 \le \theta(A),\ \theta(B) < \pi$. If $\theta(A), \theta(B) > 0$ then $U$ has a Lipschitz boundary and by the result in [**L**] we know that $E^p(U) = H^p(U)$ for $p > 1$. Our methods will show that this equivalence still holds when the values $\theta(A) = 0$, $\theta(B) = 0$, and $p = 1$ are allowed. By a conformal map argument we may assume that

(1) $A = 0$ and $B = 1$;
(2) the part $C_1$ in the boundary of $U$ is given by $[0, 1] \ni t \mapsto t$;
(3) the part $C_2$ in the boundary of $U$ is given by $[0, 1] \ni t \mapsto x(t) + iy(t)$ where $x(t), y(t)$ are smooth real functions such that $x(0) = y(0) = y(1) = 0$, $x(1) = 1$.

We first prove that $H^p(U) \subseteq E^p(U)$. We construct for a large integer $j$ a curve $C_j$ as follows. To every point $z \in C_2 \cap \partial U$ we assign the point $\gamma_{j,2}(z) = z + j^{-1}\mathbf{n}(z)$ where $\mathbf{n}(z)$ is the inward unit normal to $C_2$ at $z$. For large $j$, $C_2 \ni z \mapsto \gamma_{j,2}(z)$ is a diffeomorphism and

$$\operatorname{dist}(\gamma_{j,2}(z), C_2) = |\gamma_{j,2}(z) - z| = \frac{1}{j}. \tag{VI.38}$$

Observe that the set

$$D_j = \left\{ z : \ \operatorname{dist}(z, [0, 1] \times \{0\}) \le \frac{1}{j} \right\}$$

has a $C^1$ boundary $\partial D_j$ formed by two straight segments and two circular arcs. Fix a point $z_0 \in C_2$, choose $j$ such that $z_0 \notin D_j$ and consider the connected component of

$$\left\{ z : \ \operatorname{dist}(\gamma_{j,2}(z), D_j) \ge \frac{1}{j} \right\}$$

that contains $z_0$. Thus, we obtain a curve $C_{j,2}$ given by $[0, 1] \supseteq [a_j, b_j] \ni t \mapsto \gamma_{j,2}(x(t) + iy(t)) \subset U$ that meets $\partial D_j$ at its endpoints $A_j$, $B_j$ and remains off $D_j$ for $a_j < t < b_j$. Hence, we obtain a closed curve $C_j$ completing the curve $C_{j,2}$ with the portion $C_{j,1}$ of $\partial D_j$ contained in $U$ that joins $A_j$ to $B_j$. Because we are assuming that $\theta(A), \theta(B) < \pi$ we see that, for large $j$, $C_{j,1}$ is a horizontal segment at height $1/j$. It is clear that all points in $C_j$ have distance $1/j$ to the boundary. Furthermore, if $q \in C_{j,2}$, $q \ne A_j$, and $q \ne B_j$ then $\operatorname{dist}(q, \partial U) = \operatorname{dist}(q, C_2) = 1/j$ because of (VI.38) and the fact that $\operatorname{dist}(q, [0, 1] \times \{0\}) > 1/j$. Similarly, if $q \in C_{j,1}$, $q \ne A_j$, and $q \ne B_j$ then $\operatorname{dist}(q, \partial U) = \operatorname{dist}(q, C_1) = 1/j$. Thus, every point $q \in C_j$ is at a distance $1/j$ of $\partial U$, we can always find $z \in \partial U$ such that $|q - z| = \operatorname{dist}(q, \partial U)$, and $z$ is uniquely determined by $q$ except when $q = A_j$ or $q = B_j$ (in which case the distance may be attained at two distinct boundary points). In particular, whatever the value of $\alpha > 0$, $q \in \Gamma_\alpha(z)$ for all $q \in C_j$ and $|g(q)| \le g^*(z)$ for any function $g$ defined on $U$. Given $g \in H^p(U)$ we must show that

$$\sup_j \int_{C_j} |g(z)|^p |\mathrm{d}z| \le M < \infty. \tag{VI.39}$$

We have

$$\begin{aligned}\int_{C_{j,2}} |g(q)|^p |\mathrm{d}q| &= \int_{\gamma_{j,2}^{-1}(C_j)} |g(\gamma_{j,2}(z))|^p \, |\gamma'_{j,2}(z)| \, |\mathrm{d}z| \\ &\le \int_{\gamma_{j,2}^{-1}(C_j)} |g^*(z)|^p \, |\gamma'_{j,2}(z)| \, |\mathrm{d}z| \\ &\le C \int_{C_2} |g^*(z)|^p \, |\mathrm{d}z|. \end{aligned} \tag{VI.40}$$

Similarly, using the map $\gamma_{j,1}(x) = x + i(1/j) \in C_{j,1}$, we get

$$\int_{C_{j,1}} |g(q)|^p |\mathrm{d}q| \le C \int_{C_1} |g^*(z)|^p \, |\mathrm{d}z|, \tag{VI.41}$$

so adding (VI.40) and (VI.41) we obtain

$$\int_{C_j} |g(q)|^p |\mathrm{d}q| \le C \int_{\partial U} |g^*(z)|^p \, |\mathrm{d}z|$$

which implies (VI.39) with $M = C\,||g||^p_{H^p}$.

To prove the other inclusion we first assume that $p = 2$. Given $f \in E^2(U) \subseteq E^1(U)$ it has an a.e defined boundary value $f^+ = bf \in L^2(\partial U)$ and the Cauchy integral representation

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{bf(\zeta)}{\zeta - z} \, \mathrm{d}\zeta, \quad z \in U$$

is valid ([**Du**], theorem 10.4). Furthermore, $||f||_{E^p(U)} \simeq \|f^+\|_{L^p(\partial U)}$. Next we recall Lemma VI.4.8 that gives the estimate

$$f^*(z) \le T_* f^+(z) + CMf^+(z), \quad z \in \partial U \backslash \{A, B\}. \tag{VI.42}$$

It is well known that $M$ is bounded in $L^2(\partial U)$. Furthermore, $T_*$ is also bounded in $L^2(\partial U)$ by Theorem VI.4.7. Therefore (VI.42) implies that

$$\|f\|_{H^2(U)} = \|f^*\|_{L^2(\partial U)} \le C \|f^+\|_{L^2(\partial U)} \le C' \|f\|_{E^2(U)}.$$

The same technique leads to the inclusion $E^p(U) \subset H^p(U)$ for $p > 1$ because $T_*$ and $M$ are bounded as well in $L^p(\partial U)$ for $1 < p < \infty$ but the method breaks down for $p = 1$. This case will be handled in the proof of Theorem VI.4.1 using the fact that if $f \in E^p(U)$, $1 \le p < \infty$, $f$ has a canonical factorization $f = FB$ where $F$ has no zeros, and $|B| \le 1$. This is classical for the unit disk

$\Delta$, where $B$ is obtained as a Blaschke product and the general case is obtained from the classical result.

We are now ready to present the proof of Theorem VI.4.1. We begin by defining

$$m(x) = \min_{0\le y\le B} \varphi(x, y), \qquad M(x) = \max_{0\le y\le B} \varphi(x, y), \quad -A \le x \le A.$$

The function $Z(x, y)$ takes the rectangle $Q = [-A, A] \times [0, B]$ onto

$$Z(Q) = \{\xi + i\eta : \quad -A \le \xi \le A, \quad m(\xi) \le \eta \le M(\xi)\}.$$

The interior of $Z(Q)$ is

$$\{\xi + i\eta : \quad -A < \xi < A, \quad m(\xi) < \eta < M(\xi)\}.$$

We will consider three essential cases, in each of which we will show that the assertions of the theorem are valid on a half-interval $[0, a]$. Since the same arguments also apply to the half-intervals $[-a, 0]$, the theorem will follow.

**Case 1**: Assume that $M(0) = m(0)$ and $M(a) = m(a)$ for some $a > 0$. In this case we will first assume that the solution $f$ is smooth on $\overline{Q}$. If $M(x) = m(x)$ for every $x \in [0, a]$, then $L$ would be $\frac{\partial}{\partial t}$ in $[0, a]$ and $f(x, t) = f(x, 0)$ for all $t \in [0, B]$, which trivially leads to the inequality we seek on the half-interval $[0, a]$. Hence we may assume that there is $x \in (0, a)$ for which $m(x) < M(x)$. Then the set $Z((0, a) \times (0, B))$ has nonempty interior. Every component of the interior of this set has the form

$$\{\xi + i\eta : \alpha < \xi < \beta, m(\xi) < \eta < M(\xi)\}$$

where $(\alpha, \beta)$ is a component of the open set $\{x \in (0, a) : M(x) > m(x)\}$. Let

$$\{x \in (0, a) : M(x) > m(x)\} = \bigcup_k (\alpha_k, \beta_k)$$

be a decomposition into components. Fix $k$ and consider one of these components $(\alpha_k, \beta_k)$. Note that $m(\alpha_k) = M(\alpha_k)$ and $m(\beta_k) = M(\beta_k)$. Since for each $x$, the function

$$t \longmapsto \varphi(x, t) \quad \text{is monotonic,}$$

either $m(x) = \varphi(x, 0)$ and $M(x) = \varphi(x, B)$ or $m(x) = \varphi(x, B)$ and $M(x) = \varphi(x, 0)$ on $(\alpha_k, \beta_k)$. Without loss of generality, we may assume that $m(x) = \varphi(x, 0)$ and $M(x) = \varphi(x, B)$ for every $x \in (\alpha_k, \beta_k)$. Let $U_k =$ the interior of $Z((\alpha_k, \beta_k) \times (0, B))$. Thus

$$U_k = \{x + iy : \alpha_k < x < \beta_k, \ \varphi(x, 0) < y < \varphi(x, B)\}.$$


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Since the solution $f$ is assumed smooth on $\overline{Q}$ in the case under consideration, by the Baouendi–Treves approximation theorem, there exists $F_k \in C^\infty(\overline{U_k})$, holomorphic in $U_k$ such that

$$f(x, y) = F_k(Z(x, y)) \quad \forall (x, y) \in [\alpha_k, \beta_k] \times [0, B].$$

Note that $U_k$ is a bounded, simply connected region lying between two smooth graphs and its boundary $\partial U_k$ is smooth except at the two end points $(\alpha_k, M(\alpha_k))$ and $(\beta_k, M(\beta_k))$. Note also that $U_k$ has a rectifiable boundary of length bounded by

$$\begin{aligned} |\partial U_k| &\le \int_{\alpha_k}^{\beta_k} \sqrt{1+\varphi_x^2(x, B)}\,\mathrm{d}x + \int_{\alpha_k}^{\beta_k} \sqrt{1+\varphi_x^2(x, 0)}\,\mathrm{d}x \\ &\le 2(\beta_k - \alpha_k)\sqrt{1+\sup_Q |\nabla\varphi|^2} = K(\beta_k - \alpha_k) \end{aligned}$$

where the constant $K$ is independent of $k$. For each $p \in \partial U_k$, and $p \notin \{(\alpha_k, M(\alpha_k)), (\beta_k, M(\beta_k))\}$, define the approach region

$$\Gamma_p = \{z \in U_k : |z - p| \le 2\,\mathrm{dist}(z, \partial U_k)\}.$$

Define the maximal functions $F_k^*$ and $T_* F_k$ on $\partial U_k$ (except at the two cusps) by

$$F_k^*(p) = \sup_{\zeta \in \Gamma_p} |F_k(\zeta)|$$

and

$$T_* F_k(z) = \sup_{\epsilon > 0} \left| \int_{\{\zeta \in \partial U_k : |\zeta - z| > \epsilon\}} \frac{1}{\zeta - z} F_k(\zeta)\,\mathrm{d}\zeta \right|, \quad z \in \partial U_k.$$

Recall the Hardy–Littlewood maximal function

$$MF_k(z) = \sup \frac{1}{|I|} \int_I |f^+(\zeta)|\,|\mathrm{d}\zeta|, \quad z \neq \alpha_k + iM(\alpha_k), \beta_k + iM(\beta_k)$$

where the sup is taken over all subarcs $I \subseteq \partial U_k$ that contain $z$ and $|I|$ denotes the arclength of $I$. Next, since each $U_k$ is Ahlfors-regular, Lemma VI.4.8 gives the estimate

$$F_k^*(z) \le T_* F_k(z) + CMF_k(z), \quad z \in \partial U_k \backslash \{\alpha_k + iM(\alpha_k), \beta_k + iM(\beta_k)\}. \tag{VI.43}$$

The constant $C$ in (VI.43) is independent of $k$ because the aperture of the $\Gamma_p$ is independent of $k$. Next we will show that any $z \in U_k$ lies in $\Gamma_p$ for some $p \in \partial U_k$. Let $z \in U_k$. Then for some $(x, t) \in (\alpha_k, \beta_k) \times (0, B)$, $z = x + i\varphi(x, t)$

and $\varphi(x, 0) < \varphi(x, t) < \varphi(x, B)$. Let $p = x + i\varphi(x, B)$ and $q = x + i\varphi(x, 0)$. We claim that $z \in \Gamma_p \cup \Gamma_q$. Indeed suppose first

$$|\varphi(x, B) - \varphi(x, t)| \leq |\varphi(x, t) - \varphi(x, 0)|. \tag{VI.44}$$

Then for any $y$:

$$\begin{aligned} |x + i\varphi(x, t) - y - i\varphi(y, B)| &\geq \frac{1}{2}(|x - y| + |\varphi(x, t) - \varphi(y, B)|) \\ &\geq \frac{1}{2}(|x - y| + |\varphi(x, t) - \varphi(x, B)| - |\varphi(x, B) - \varphi(y, B)|) \\ &\geq \frac{1}{2}(|\varphi(x, t) - \varphi(x, B)| \quad \text{since} |\varphi_x| \leq \frac{1}{2} \\ &= \frac{1}{2}|z - p|. \end{aligned} \tag{VI.45}$$

We also have:

$$\begin{aligned} |x + i\varphi(x, t) - y - i\varphi(y, 0)| &\geq \frac{1}{2}(|x - y| + |\varphi(x, t) - \varphi(y, 0)|) \\ &\geq \frac{1}{2}(|\varphi(x, t) - \varphi(x, 0)| \\ &\geq \frac{1}{2}(|\varphi(x, B) - \varphi(x, t)| \quad \text{by} \quad \text{(VI.44)} \\ &= \frac{1}{2}|z - p|. \end{aligned} \tag{VI.46}$$

From (VI.45) and (VI.46) we see that if (VI.44) holds, then $z \in \Gamma_p$. By a similar reasoning, if (VI.44) does not hold, then $z \in \Gamma_q$. We have thus shown that

$$U_k \subseteq \bigcup_{p \in \partial U_k} \Gamma_p. \tag{VI.47}$$

Next fix $(x, t) \in (\alpha_k, \beta_k) \times (0, B)$. If $x + i\varphi(x, t) \in U_k$, i.e., if $\varphi(x, 0) < \varphi(x, t) < \varphi(x, B)$, then by (VI.47),

$$|F_k(x + i\varphi(x, t)| \leq F_k^*(x + i\varphi(x, 0)) + F_k^*(x + i\varphi(x, B)). \tag{VI.48}$$

On the other hand, if $\varphi(x, t) = \varphi(x, 0)$, then since $\varphi(x, 0) < \varphi(x, B)$, there exists $t \leq y < B$ such that $\varphi(x, y) = \varphi(x, 0) = \varphi(x, t)$ and $y$ is the maximum such. Let $y_m \to y, \quad y_m > y$. Then by (VI.48),

$$|F_k(x + i\varphi(x, y_m)| \leq F_k^*(x + i\varphi(x, 0)) + F_k^*(x + i\varphi(x, B)).$$

Letting $m \to \infty$, we get

$$\begin{aligned} |F_k(x + i\varphi(x, t)| &= |F_k(x + i\varphi(x, y))| \\ &\leq F_k^*(x + i\varphi(x, 0)) + F_k^*(x + i\varphi(x, B)). \end{aligned}$$

Thus for any $(x,t) \in (\alpha_k, \beta_k) \times (0, B)$, we have:

$$\begin{aligned} |f(x,t)| = &|F_k(x+i\varphi(x,t))| \le F_k^*(x+i\varphi(x,0)) \\ &+ F_k^*(x+i\varphi(x,B)). \end{aligned} \tag{VI.49}$$

From (VI.43) and (VI.49), for any $(x,t) \in (\alpha_k, \beta_k) \times (0, B)$, we have:

$$\begin{aligned} |f(x,t)| \le\, &T_* F_k(x+i\varphi(x,0)) + T_* F_k(x+i\varphi(x,B)) \\ &+ C(MF_k(x+i\varphi(x,0)) + MF_k(x+i\varphi(x,B))), \end{aligned} \tag{VI.50}$$

where we recall that the constant $C$ is independent of $k$. Let $1 < p < \infty$. The cases $p = 1, \infty$ will be treated separately at the end. Since $U_k$ is an Ahlfors-regular domain, both $T_*$ and $M$ are bounded in $L^p(\partial U_k)$ ([**D**]) and so (VI.50) leads to

$$\int_{\alpha_k}^{\beta_k} |f(x,t)|^p \, \mathrm{d}x \le C \int_{\partial U_k} |F_k(z)|^p \, |\mathrm{d}z| \quad \text{for any } 0 < t < B. \tag{VI.51}$$

Since $f(x,t) = F_k(Z(x,t))$ on $[\alpha_k, \beta_k] \times [0, B]$, we conclude that for any $0 < t < B$:

$$\int_{\alpha_k}^{\beta_k} |f(x,t)|^p \, \mathrm{d}x \le C \left( \int_{\alpha_k}^{\beta_k} |f(x,0)|^p \, \mathrm{d}x + \int_{\alpha_k}^{\beta_k} |f(x,B)|^p \, \mathrm{d}x \right) \tag{VI.52}$$

where $C$ is independent of $k$. We can write

$$(0, a) = \left( \bigcup_k (\alpha_k, \beta_k) \right) \bigcup S$$

where $S = \{x \in (0,a) : \varphi(x,0) = \varphi(x,B)\}$. Observe that for $x \in S$, the function $t \longmapsto f(x,t)$ is constant since $L = \frac{\partial}{\partial t}$ on $\{x\} \times (0, B)$. Hence for any $0 \le t \le B$,

$$\int_S |f(x,t)|^p \, \mathrm{d}x = \int_S |f(x,B)|^p \, \mathrm{d}x. \tag{VI.53}$$

Using (VI.53) and summing up over $k$ in (VI.52), we conclude:

$$\int_0^a |f(x,t)|^p \, \mathrm{d}x \le C \left( \int_0^a |f(x,0)|^p \, \mathrm{d}x + \int_0^a |f(x,B)|^p \, \mathrm{d}x \right) \tag{VI.54}$$

for any $0 < t < B$. Finally, we use a refinement of the approximation theorem as in Theorem II.4.12 to remove the smoothness of $f$.

**Case 2**: Assume that $M(0) = m(0)$ and $M(x) > m(x)$ for every $0 < x \le A$. We will need to use the boundary version of the Baouendi–Treves approximation formula. Let $h(x) \in C_0^\infty(-A, A)$, $h(x) \equiv 1$ in a neighborhood of 0. For $\tau > 0$, define

$$E_\tau f(x,t) = (\tau/\pi)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{-\tau[Z(x,t) - Z(x',0)]^2} f(x',0) h(x') Z_x(x',0) \, \mathrm{d}x'$$

and

$$G_\tau f(x,t) = (\tau/\pi)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2} f(x',t)h(x')Z_x(x',t)\,\mathrm{d}x'$$

where $f(x',t)$ is the distribution trace of $f$ at $t \geq 0$. Let

$$R_\tau f(x,t) = E_\tau f(x,t) - G_\tau f(x,t).$$

The Baouendi–Treves approximation theorem asserts that after decreasing $A$ and $B$, $E_\tau f(x,t)$ converges to $f(x,t)$ in the sense of distributions in the open set $(-A,A)\times(0,B)$. However, here we need the refined boundary result in Chapter II (Theorem II.4.12) which guarantees convergence up to $t=0$ in appropriate function spaces. More precisely, according to the result, there exist $a, b > 0$ such that

$$R_\tau f(x,t) \to 0 \quad \text{in } C^\infty([-a,a]\times[0,b]).$$

Since it is clear that $G_\tau f(x,t) \to f(x,t)$ in $L^p(-a,a)$ whenever $f(.,t) \in L^p(-a,a)$, it follows that

$$E_\tau f(x,t) \to f(x,t) \quad \text{in } L^p([-a,a]), \quad \text{if } f(.,t) \in L^p(-a,a). \tag{VI.55}$$

Let $F_\tau(z)$ be the entire function satisfying $F_\tau(Z(x,t)) = E_\tau f(x,t)$. Let $U_a =$ the interior of $Z((0,a)\times(0,b))$. Recall that $m(0) = M(0)$ but $m(x) < M(x)$ for any $0 < x \leq A$. The domain $U_a$ is also an Ahlfors-regular domain. Therefore, we can apply the arguments in Case 1 to the smooth functions $E_\tau f$ to arrive at:

$$\int_0^a |E_\tau f(x,t)|^p\,\mathrm{d}x \leq C \int_{\partial U_a} |F_\tau f(z)|^p\,|\mathrm{d}z|. \tag{VI.56}$$

Note that this time $\partial U_a$ has three pieces and so (VI.56) leads to:

$$\int_0^a |E_\tau f(x,t)|^p\,\mathrm{d}x \leq C\left(\int_0^a |E_\tau f(x,0)|^p\,\mathrm{d}x + \int_0^a |E_\tau f(x,b)|^p\,\mathrm{d}x \right.$$
$$\left. + \int_0^b |E_\tau f(a,s)|^p |\varphi_s(a,s)|\,\mathrm{d}s\right), \quad 0 < t < b. \tag{VI.57}$$

We now wish to let $\tau \to \infty$ in (VI.57). From (VI.55) we know that if $f(.,0)$ and $f(.,b)$ are in $L^p(-a,a)$, then

$$\int_0^a |E_\tau f(x,0)|^p\,\mathrm{d}x \to \int_0^a |f(x,0)|^p\,\mathrm{d}x \quad \text{and}$$
$$\int_0^a |E_\tau f(x,b)|^p\,\mathrm{d}x \to \int_0^a |f(x,b)|^p\,\mathrm{d}x.$$

We thus need only compute the limit of the $s$ integral in (VI.57). We will show that for almost all $a'$,

$$\int_0^b |E_\tau f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s \to \int_0^b |f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s. \qquad \text{(VI.58)}$$

We know that $M(x) > m(x)$ for every $0 < x \le A$. We may also assume that $\varphi(x,t) > \varphi(x,0)$ for every $x \in (0,A]$, $t \in (0,b]$. Indeed, otherwise, we will be placed in the context of Case 1. The approximation theorem then implies that for each $x > 0$, $f$ is continuous at $(x,t)$ for $t > 0$ small. Since $R_\tau f(x,t) \to 0$ uniformly in $[0,a] \times [0,b]$, (VI.58) will follow if we show that for almost all $a'$,

$$\int_0^b |G_\tau f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s \to \int_0^b |f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s. \qquad \text{(VI.59)}$$

Choose two numbers $a_1, a_2$ such that $0 < a_1 < a < a_2 \le A$. By the approximation theorem, after decreasing $b$, since $f$ is continuous at $(x,t)$ for $t = t(x) > 0$ small, there exists $F$ continuous in $Z((a_1,a_2) \times (0,b))$, holomorphic in $W =$ the interior of $Z((a_1,a_2) \times (0,b))$ such that $F(Z(x,t)) = f(x,t)$. Observe that

$$W = \{x + iy : x \in (a_1,a_2),\ \varphi(x,0) < y < \varphi(x,b)\}$$

and $F$ has a distributional boundary value $= f(x,0)$ on the curve $\{x + i\varphi(x,0) : a_1 < x < a_2\}$. For $x \in (a_1,a_2)$, define

$$F^*(x) = \sup_{0<t<b} |F(x + i\varphi(x,t))|.$$

Since $F$ has an $L^p$ boundary value, it is well known (see, for example, [**Ro**]) that $F^* \in L^p_{\mathrm{loc}}(a_1,a_2)$. Let $\psi \in C_0^\infty(a_1,a_2)$, $\psi \ge 0$, $\psi(x) \equiv 1$ near $a$. Write $G_\tau f(x,t) = G_\tau^1 f(x,t) + G_\tau^2 f(x,t)$, where

$$G_\tau^1 f(x,t) = (\tau/\pi)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{-\tau[Z(x,t)-Z(x',t)]^2} \psi(x') f(x',t) h(x') Z_x(x',t)\,\mathrm{d}x'$$

and $G_\tau^2 f(x,t) = G_\tau f(x,t) - G_\tau^1 f(x,t)$. Consider first $G_\tau^2 f(x,t)$ for $x$ near $a$. Observe that the integrand is zero for $x'$ near $a$ and hence for $x$ near $a$ and $t \in [0,b]$,

$$G_\tau^2 f(x,t) \to 0 \quad \text{uniformly}. \qquad \text{(VI.60)}$$

In the integrand of $G_\tau^1 f(x,t)$, $f(x',t)$ can be replaced by $F(Z(x',t)) = F(x' + i\varphi(x',t))$ and hence we have:

$$|G_\tau^1 f(x,t)| \le C(\tau/\pi)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\tau|x-x'|^2} \psi(x') F^*(x')\,\mathrm{d}x' \qquad \text{(VI.61)}$$

where $C$ is independent of $\tau$. Thus if we define

$\eta(x) = \pi^{-1/2}e^{-\frac{x^2}{2}}$, and $\eta_\tau(x) = \tau^{1/2}\eta(\tau^{1/2}x)$, then (VI.61) says that

$$|G^1_\tau f(x,t)| \le C(\eta_\tau * \psi F^*)(x) \quad \forall t \in [0,b]. \tag{VI.62}$$

Since $\psi F^* \in L^p(-\infty,\infty)$ and $\eta$ is a radial decreasing function in $|x|$, by a proposition in [**S2**, page 57],

$$\sup_{\tau>0} \eta_\tau * \psi F^*(x) \quad \text{is finite a.e.}$$

Pick a point $x_0$ where this supremum is finite and where $F^*(x_0) < \infty$. Then at such a point, the functions $|G^1_\tau f(x_0,t)|$ are bounded on $[0,b]$. Since pointwise,

$$G^1_\tau f(x_0,t) \to f(x_0,t) \quad \forall t \in [0,b],$$

it follows that

$$\int_0^b |G_\tau f(x_0,s)|^p |\varphi_s(x_0,s)|\,\mathrm{d}s \to \int_0^b |f(x_0,s)|^p |\varphi_s(x_0,s)|\,\mathrm{d}s. \tag{VI.63}$$

From (VI.59) and (VI.63), we conclude that

$$\int_0^b |E_\tau f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s \to \int_0^b |f(a',s)|^p |\varphi_s(a',s)|\,\mathrm{d}s \tag{VI.64}$$

for almost all $a'$. We can therefore let $\tau \to \infty$ in (VI.57) and conclude that for almost all $a$:

$$\int_0^a |f(x,t)|^p\,\mathrm{d}x \le C\left(\int_0^a |f(x,0)|^p\,\mathrm{d}x + \int_0^a |f(x,b)|^p\,\mathrm{d}x \right.$$
$$\left. + \int_0^b |f(a,s)|^p |\varphi_s(a,s)|\,\mathrm{d}s\right), \quad 0 < t < b. \tag{VI.65}$$

**Case 3**: Assume $M(0) > m(0)$. Let $a > 0$ such that $M(x) > m(x)$ for every $x \in (-a,a)$. If $W_a = Z((-a,a)\times(0,B))$, there is a function $F$ holomorphic on the interior of $W_a$ such that $f(x,y) = F(Z(x,y))$. This time the boundary of $W_a$ has four pieces. One can then reason as in the previous case to get the required estimate on the interval $(-a,a)$. Finally, observe that estimates on the interval of the form $[-a,0]$ are also valid under Cases 1 and 2. The theorem for $1 < p < \infty$ follows from these three cases.

We consider next the case when $p = 1$.

Assume we are in the situation of Case 1 where $M(0) = m(0)$ and $M(a) = m(a)$ for some $a > 0$. As before we assume first that $f(x,t)$ is smooth on $\overline{Q^+}$, $F_k \in C^\infty(\overline{U_k})$, holomorphic in $U_k$ and $f(x,y) = F_k(Z(x,y))$ on $[\alpha_k,\beta_k]\times[0,B]$. Since $U_k$ is simply connected, by a classical result (see the corollary of theorem 10.1 in [**Du**]), $F_k$ has a factorization $F_k = G_k B_k$ where each factor is holomorphic in $U_k$, $G_k$ has no zeros, $G_k \in E^1(U_k)$, $|B_k(z)| \le 1$, and $|B_k(z)| = 1$ on $\partial U_k$. The fact that $G_k \in E^1(U_k)$ implies (see theorem 10.4 in [**Du**]) that it has a nontangential limit $bG_k$ a.e. on $\partial U_k$, and $G_k$ equals the Cauchy transform of $bG_k$. Observe that since $|B_k(z)| = 1$ on $\partial U_k$, $|bG_k(z)| = |F_k(z)|$ on $\partial U_k$. Since $G_k$ has no zeros on the simply connected region $U_k$, it has a holomorphic square root $H_k$. Note that $H_k \in E^2(U_k) = H^2(U_k)$ (by the discussion preceding this proof). We have

$$H_k^*(z) \le T_*(bH_k)(z) + CM(bH_k)(z). \tag{VI.66}$$

Using (VI.66) and the equality $|G_k| = |F_k|$ on $\partial U_k$ we get:

$$\begin{aligned}
\int_{\alpha_k}^{\beta_k} |f(x,t)|\,\mathrm{d}x &= \int_{\alpha_k}^{\beta_k} |F_k(x+i\varphi(x,t))|\,\mathrm{d}x \\
&\le \int_{\alpha_k}^{\beta_k} |G_k(x+i\varphi(x,t))|\,\mathrm{d}x = \int_{\alpha_k}^{\beta_k} |H_k(x+i\varphi(x,t))|^2\,\mathrm{d}x \\
&\le \int_{\partial U_k} |H_k^*(z)|^2\,|\mathrm{d}z| \\
&\le C\int_{\partial U_k} |bH_k(z)|^2\,|\mathrm{d}z| \quad \text{by the } L^2 \text{ boundedness of } T_* \text{ and } M \\
&= C\left(\int_{\alpha_k}^{\beta_k} |f(x,0)|\,\mathrm{d}x + \int_{\alpha_k}^{\beta_k} |f(x,B)|\,\mathrm{d}x\right) \text{ for any } 0 < t < B.
\end{aligned} \tag{VI.67}$$

Summing up over $k$ and adding the contributions from the set $S = (0,a)\setminus \cup_k (\alpha_k, \beta_k)$, we get:

$$\int_0^a |f(x,t)|\,\mathrm{d}x \le C\left(\int_0^a |f(x,0)|\,\mathrm{d}x + \int_0^a |f(x,B)|\,\mathrm{d}x\right) \quad \text{for } 0 < t < B, \tag{VI.68}$$

whenever $f$ is a solution and $f \in C^\infty(\overline{Q}^+)$. In general, for $f \in \mathcal{D}'(Q^+)$ satisfying the hypotheses of Theorem VI.4.1, let $\{f_m(x,t)\}$ be a sequence of $C^\infty$ solutions on $\overline{Q^+}$ satisfying:

(i) for each $0 \le t \le B$, $f_m(.,t) \to f(.,t)$ in $\mathcal{D}'(-a,a)$;
(ii) $f_m(x,0) \to f(x,0)$ and $f_m(x,B) \to f(x,B)$ in $L^1(-a,a)$.

We now apply inequality (VI.68) to $f_m - f_n$, let $m$ and $n$ tend to $\infty$, and use (i) and (ii) above to conclude that (VI.68) also holds for $f$. Cases 2 and 3 are also treated in a similar fashion. Finally we consider the case where $p = \infty$. Suppose we are in the situation of Case 1 where $M(0) = m(0)$ and

$M(a) = m(a)$ for some $a > 0$. Assume first that $f(x,t) \in C^\infty(\overline{Q})$ and for $k$ fixed as before, let

$$U_k = \{x + iy : \alpha_k < x < \beta_k,\ \varphi(x,0) < y < \varphi(x,B)\}$$

and $f(x,y) = F_k(Z(x,y))$ on $[\alpha_k, \beta_k] \times [0,B]$, $F_k$ holomorphic on $U_k$ and continuous on the closure. We apply the maximum modulus principle to $F_k$ and use the constancy of $f$ on the vertical segments $x = \alpha_k$ and $x = \beta_k$ to conclude that

$$|f(x,y)| \le ||f(.,0)||_{L^\infty(0,a)} + ||f(.,B)||_{L^\infty(0,a)} \quad \forall (x,y) \in [\alpha_k, \beta_k] \times [0,B].$$

If $S$ is the set as before with

$$(0,a) = \left( \bigcup_k (\alpha_k, \beta_k) \right) \bigcup S,$$

then $f(x,y) = f(x,B) \quad \forall (x,y) \in S \times (0,B)$, and so we conclude that

$$\begin{aligned} |f(x,y)| \le & ||f(.,0)||_{L^\infty(0,a)} + ||f(.,B)||_{L^\infty(0,a)} \\ & \forall (x,y) \in (0,a) \times (0,B). \end{aligned} \tag{VI.69}$$

For a solution $f \in \mathcal{D}'(Q^+)$ satisfying $f(.,0)$ and $f(.,B) \in L^\infty(-A,A)$, we use the refinement of the approximation theorem in Chapter II according to which

$$f(x,y) = \lim_{\tau \to \infty} E_\tau f(x,y) \quad \text{a.e. in} \quad (0,a) \times (0,B), \tag{VI.70}$$

provided that $A$ and $B$ are small enough. Moreover,

$$\begin{aligned} |G_\tau f(x,B)| &\le c_1 \tau^{\frac{1}{2}} \int e^{-c_2 \tau |x - x'|^2} |f(x',B)| |h(x')| \, dx' \\ &\le c_3 ||f(.,B)||_{L^\infty} \quad \forall \tau > 0 \end{aligned} \tag{VI.71}$$

and likewise,

$$|G_\tau f(x,0)| \le c||f(.,0)||_{L^\infty}. \tag{VI.72}$$

Letting $\tau \to \infty$, and recalling that $R_\tau f \to 0$ uniformly, we get

$$\begin{aligned} &\overline{\lim_{\tau \to \infty}} |E_\tau f(x,0)| \le C||f(.,0)||_{L^\infty} \quad \text{and} \\ &\overline{\lim_{\tau \to \infty}} |E_\tau f(x,B)| \le C||f(.,B)||_{L^\infty} \end{aligned} \tag{VI.73}$$

for some $C > 0$. From (VI.69) (applied to $E_\tau f$), (VI.70) and (VI.73), we conclude that for every $(x,y) \in (0,a) \times (0,B)$,

$$|f(x,y)| \le C \left( ||f(.,0)||_{L^\infty(0,a)} + ||f(.,B)||_{L^\infty(0,a)} \right). \tag{VI.74}$$

Next we consider Case 2 where $M(0) = m(0)$ and $M(x) > m(x)$ for every $0 < x \le A$. As before, let $a, b > 0$ such that

$$E_\tau f(x, t) \to f(x, t) \quad \text{a.e. in } [-a, a] \times [0, b]. \tag{VI.75}$$

Let $U_a = Z((0, a) \times (0, b))$ and consider the holomorphic function $F_\tau$ such that $F_\tau(Z(x, t)) = E_\tau f(x, t)$. The maximum principle applied to $F_\tau$ on $U_a$ leads to

$$\begin{aligned}|E_\tau f(x, y)| \le ||E_\tau f(., 0)||_{L^\infty(0,a)} + ||E_\tau f(., b)||_{L^\infty(0,a)} \\ + ||E_\tau f(a, .)||_{L^\infty(0,b)} \quad \forall (x, y) \in [0, a] \times [0, b].\end{aligned} \tag{VI.76}$$

As observed already, the terms $||E_\tau f(., 0)||_{L^\infty(0,a)}$ and $||E_\tau f(., b)||_{L^\infty(0,a)}$ are dominated by a constant multiple of

$$||f(., 0)||_{L^\infty(0,a)} + ||f(., b)||_{L^\infty(0,a)}.$$

We therefore only need to estimate the term $||E_\tau f(a, .)||_{L^\infty(0,b)}$ for which it suffices to estimate $||G_\tau f(a, .)||_{L^\infty(0,b)}$. Let $0 < a_1 < a < a_2 < A$ be as before, $F$ holomorphic such that

$$f(x, y) = F(x + i\varphi(x, y)) \quad \text{on} \quad [a_1, a_2] \times (0, b].$$

Since $bF = bf \in L^\infty(a_1, a_2)$, by the generalized maximum principle applied to $F$ there exists $M > 0$ such that

$$|F(x + i\varphi(x, y))| = |f(x, y)| \le M \quad \text{on} \quad [a_1', a_2'] \times (0, b],$$

for some $a_1 < a_1' < a < a_2' < a_2$. We write $G_\tau f = G_\tau^1 f + G_\tau^2 f$ as before, except that this time $\psi$ is supported in $(a_1', a_2')$. Recall that $G_\tau^2 f \to 0$ uniformly while

$$|G_\tau^1 f(x, t)| \le C \sup |\psi(x') f(x', t)| \le CM.$$

Hence for some $C > 0$,

$$||E_\tau f(a, .)||_{L^\infty(0,b)} \le C \quad \forall \tau > 0.$$

We have shown that $f \in L^\infty((0, a) \times (0, b))$ in this case. Case 3 is treated likewise. We conclude that $f$ is bounded. Theorem VI.4.1 has now been proved.

Corollary VI.4.9. *Suppose $f$ is a distribution solution of $Lf = g$ in the rectangle $Q = (-A, A) \times (0, B)$. Suppose $f$ has a weak boundary value $bf = f(x, 0)$ at $y = 0$ and that $g$ is a Lipschitz function. Then there exist $A_0 > 0$ and $T_0 > 0$ such that for any $0 < T \le T_0$ and $0 < a < A_0$, if $f(., 0)$ and $f(., T) \in L^p(-A_0, A_0)$, $f(., t) \in L^p(-a, a)$ for any $0 < t < T$.*


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PROOF. Using Proposition VI.3.6 we may find a function $f_0$, uniformly continuous on $Q$, such that $Lf_0 = g$. Then, $f_1 = f - f_0$ satisfies the hypothesis of Theorem VI.4.1. It follows that (i) holds for $f_1$ if $1 \le p < \infty$ or (ii) if $p = \infty$ and the same conclusion applies to $f = f_0 + f_1$ because $f_0$ is continuous up to the boundary. □

COROLLARY VI.4.10. *Let $L$ be as above, $f \in \mathcal{D}'(Q^+)$, $Lf = g$ in $Q^+$ where $g \in C^\infty(\overline{Q^+})$. Let $A_0$ and $T_0$ be as in Theorem VI.4.1. If $f$ has a weak trace $f(x, 0) \in C^\infty(-A_0, A_0)$ and $f(., T_0)$ is in $C^\infty(-A_0, A_0)$, then for all $0 < a < A_0$ and $0 < T < T_0$, $f \in C^\infty([-a, a] \times [0, T])$. In particular, $f$ is smooth up to the boundary $t = 0$.*

PROOF. By Proposition VI.3.6, we can get $u \in C^0((-A, A) \times [0, B))$ that solves $Lu = g$ in $Q^+$. Hence $L(u - f) = 0$ in $Q^+$ and so by Theorem VI.4.1 and the continuity of $u$ up to the boundary, for any $0 < a < A_0$ and $0 < t \le T_0$ there is a constant $C > 0$ such that

$$\int_{-a}^{a} |f(x, t)|^2 \, dx \le C \quad \forall t \in [0, T_0]. \tag{VI.77}$$

Define the vector field $M = \frac{1}{Z_x(x,t)} \frac{\partial}{\partial x}$. Since the bracket $[L, M] = 0$ and $Lf = g$, the distribution $Mf$ is also a solution of $L(Mf) = Mg$ in $Q^+$. Moreover, since the traces $Mf(., T_0)$ and $Mf(., 0)$ are smooth, by repeating the same arguments, for any $0 < a < A_0$ and $0 < T < T_0$ there is a constant $C > 0$ such that

$$\int_{-a}^{a} |Mf(x, t)|^2 \, dx \le C \quad \forall t \in [0, T]. \tag{VI.78}$$

Since $\frac{\partial f}{\partial t} = -a(x, t)\frac{\partial f}{\partial x} + g(x, t)$, (VI.78) implies that for some constant $C'$,

$$\int_{-a}^{a} \left| \frac{\partial f}{\partial t}(x, t) \right|^2 \, dx \le C' \quad \forall t \in [0, T].$$

By iterating this argument, we derive that for every $m, n = 1, 2, \ldots,$ there exists $C = C(m, n) > 0$ such that

$$\int_{-a}^{a} |D_x^m D_t^n f(x, t)|^2 \, dx \le C \quad \forall t \in [0, T]. \tag{VI.79}$$

From (VI.79) we conclude that $f \in C^\infty([-a, a] \times (0, T])$. Smoothness up to the boundary now follows from the case $p = \infty$ in Theorem VI.4.1. □

REMARK VI.4.11. Conversely, if a locally integrable vector field $L$ shares the $H^p$ property as in Theorem VI.4.1, then $L$ has to satisfy condition $(\mathcal{P}^+)$ at the origin in $\Sigma = (-A, A) \times \{0\}$. See [**BH6**] for the proof.

COROLLARY VI.4.12. *Let $L$ satisfy $(\mathcal{P}^+)$ at the origin as above. Suppose $Lf = g$ in $Q^+$, $g \in C^\infty(\overline{Q^+})$, and $f \in C^\infty(Q^+)$. If the trace $bf = f(x,0)$ exists and $f(x,0) \in C^\infty(-A, A)$, then $f$ is $C^\infty$ up to the boundary $t = 0$.*

Example 4.3 in [**BH6**] provides a real-analytic vector field $L$ for which Corollary VI.4.12 is not valid even for a solution of the homogeneous equation $Lf = 0$. Example 4.4 in the same paper shows that in Theorem VI.4.1, one needs to assume the integrability of two traces. That is, if we only assume that $bf = f(x,0) \in L^1$, the traces $f(., t)$ may not be in $L^1$.

## Notes

The results of this chapter in the holomorphic case are classical. For a discussion of the conditions that guarantee the existence of a boundary value we refer to the books [**BER**] and [**H2**]. The basic theory of Hardy spaces for bounded, simply connected domains in the complex plane is exposed in [**Du**] (see also [**Po**]). The paper [**L**] and the references in it contain more recent developments on the subject. The planar case of Theorem VI.1.3 as well as the necessity in the real-analytic, planar situation was proved in [**BH5**]. Lemma VI.4.8 is taken from [**L**]. Theorem VI.4.1 and its corollaries appeared in [**BH6**]. The work [**HH**] extends Theorem VI.4.1 to the case $0 < p < 1$ for vector fields with real-analytic coefficients.