Adv. Appl. Prob. **16,** 695–696 (1984)
Printed in N. Ireland

QUASI-INPUT PROCESS IN THE $M/G/1/\infty$ QUEUE

G. I. FALIN,* *Moscow State University*

Abstract

We consider the flow of service starting times in the $M/G/1/\infty$ queue and study some of its equilibrium properties.

RANDOM FLOWS; QUEUEING SYSTEMS

Introduction

We shall consider the $M/G/1/\infty$ queue with FIFO discipline. Let ξ_i be the instant at which the ith service begins, and let η_i be the instant when this service is completed. The sequence of points η_i on the time axis forms the *departure* process. It is studied in detail in [1], for example, whereas the closely connected process of service starting times $\{\xi_i\}_{i\geqq 1}$ has, as far as we know, never been considered. The process $\{\xi_i\}_{i\geqq 1}$ is an input process for the server, so we shall call it the *quasi-input* process. The present note gives some results obtained for such a quasi-input flow. The problem is all the more important in that often we can observe only the moments when one server is turned 'on' or 'off', i.e. the quasi-input and output processes.

Let λ be the rate of input flow, $\beta(s)$ the Laplace–Stieltjes transform of the service-time distribution function $B(x)$, $\beta_k=(-1)^k\beta^k(0)$, $\rho=\lambda\beta_1$. $S_i=\eta_i-\xi_i$ is the service time of the ith call, and N_i is the queue length immediately before the instant η_i.

Distribution of intervals between service beginnings

Let $A_i=\xi_{i+1}-\xi_i$. It is easy to see that, if $N_i\geqq 1$, then $\xi_{i+1}=\eta_i$ and so $A_i=S_i$; but if $N_i=0$, then $A_i=S_i$ plus the interval until the $(i+1)$th call arrives in the system. This simple fact yields that Laplace transform of the random variable A_i is

$$E\exp(-sA_i)=[P\{N_{i-1}=0\}+P\{N_{i-1}=1\}]\cdot\left[\beta(s)-\beta(s+\lambda)+\beta(s+\lambda)\frac{\lambda}{s+\lambda}\right]$$
$$+P\{N_{i+1}\geqq 2\}\cdot\beta(s).$$

One can see that the dependence of $E\exp(-sA_i)$ on i is connected with the dependence of the embedded Markov chain $\{N_i\}$ on i. Thus, if $\rho<1$ and the queue is in a steady state, i.e. $P\{N_i=k\}=\pi_k$ is an invariant probability measure of the chain $\{N_i\}$,

Received 14 May 1984.
* Postal address: Department of Probability, Mechanics and Mathematics Faculty, Moscow State University, Moscow, 119899, USSR.

then all intervals A_i have the same distribution function with Laplace–Stieltjes transform

$$\omega(s) = E \exp(-sA_i) = \beta(s) - \frac{1-\rho}{\beta(\lambda)} \beta(s+\lambda) \frac{s}{s+\lambda}.$$

Here we have used the well-known equalities $\pi_0 = 1-\rho$, $\pi_1 = (1-\rho)(1-\beta(\lambda))/\beta(\lambda)$. In particular, when the queue is in steady state

$$EA_i = 1/\lambda, \qquad \text{Var } A_i = \beta_2 - 2(1-\rho)\beta'(\lambda)/\beta(\lambda) + (1-2\rho)/\lambda^2.$$

These formulae lead us to some interesting qualitative conclusions about the quasi-input process.

Let $\beta(s) = \nu/(s+\lambda)$, i.e. the service time has negative exponential distribution with parameter ν. Then

$$\text{Var } A_i = \frac{2}{\nu^2} + \frac{2(1-\rho)}{\lambda(\nu+\lambda)} + \frac{1-2\rho}{\lambda^2},$$

$$\text{Var } A_i/(EA_i)^2 = \lambda^2 \cdot \text{Var } A_i = 1 - \frac{1-\rho}{1+\rho} 2\rho^2 < 1,$$

so that A_i is not negatively exponentially distributed. Therefore, the quasi-input process in the $M/M/1/\infty$ queue is not Poisson, while the departure process is Poisson.

In the general case, if $\lambda \to 0$ then $\lambda^2 \text{ Var } A_i = 1 - \lambda^2\beta_2 + o(\lambda^2)$. Thus there is no $B(x)$ such that A_i has a negative exponential distribution for all λ (except for the trivial case of instantaneous service, of course).

In connection with this result it would be interesting to find out when the quasi-input process is a renewal process.

Reference

DISNEY, R. L. (1975) Random flow in queueing networks: a review and critique. *AIIE Trans.* **7**, 268–288.