A NOTE ON NORMAL MATRICES

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1. Introduction. Let U_n be an n-dimensional unitary space with inner product (u, v). For vectors $u_1, \ldots, u_r \in U_n$, $r \leqslant n$, let $u_1 \wedge \ldots \wedge u_r$ denote the Grassmann exterior product **(4)** of the u_i; it is a vector in U_m where $m = {}_nC_r$. If also $v_1, \ldots, v_r \in U_n$, then $(u_1 \wedge \ldots \wedge u_r, v_1 \wedge \ldots \wedge v_r)$ is the determinant of the $r \times r$ matrix $((u_i, v_j))$, $1 \leqslant i, j \leqslant r$. If A is a linear transformation of U_n to itself, the rth compound of A is defined by

$$C_r(A)u_1 \wedge \ldots \wedge u_r = (Au_1) \wedge \ldots \wedge (Au_r).$$

For $1 \leqslant r \leqslant k \leqslant n$, denote by $Q_{k,r}$ the set of all ${}_kC_r$ sequences $\omega = \{i_1, \ldots, i_r\}$ such that $1 \leqslant i_1 < \ldots < i_r \leqslant k$. For a set of vectors $x_1, \ldots, x_k \in U_n$ set

$$x_\omega = x_{i_1} \wedge \ldots \wedge x_{i_r},$$

$$g_r = g_r(x_1, \ldots, x_k) = \sum_{\omega \in Q_{k,r}} (C_r(A)x_\omega, x_\omega).$$

Let $E_r(a_1, \ldots, a_k)$ denote the elementary symmetric function of $a_1, \ldots, a_k$ of degree r and let $\lambda_1, \ldots, \lambda_n$ denote the characteristic values of the linear transformation A. In **(2)** it was shown that if A is Hermitian, then

$$\max g_r = E_r(\xi_1, \ldots, \xi_k),$$

$$\min g_r = E_r(\eta_1, \ldots, \eta_k),$$

where $\{\xi_1, \ldots, \xi_k\}$ and $\{\eta_1, \ldots, \eta_k\}$ are certain subsets of $\{\lambda_1, \ldots, \lambda_n\}$ and where the max and min are taken over all sets of k orthonormal vectors $x_1, \ldots, x_k$ in U_n. In this note we offer the following generalization of this fact.

THEOREM 1. *If A is normal and if $x_1, \ldots, x_k$ are orthonormal vectors in U_n, then $g_r(x_1, \ldots, x_k)$ lies in the convex hull H_r of all the complex numbers*

$$E_r(\lambda_{j_1}, \ldots, \lambda_{j_k}), \qquad \{j_1, \ldots, j_k\} \in Q_{n,k}.$$

In Theorem 2 we identify the complex numbers which are of the form $g_1(x_1, \ldots, x_k)$ for orthonormal vectors $x_1, \ldots, x_k \in U_n$.

THEOREM 2. *If A is normal, the set of all sums $(Ax_1, x_1) + \ldots + (Ax_k, x_k)$ for orthonormal vectors $x_1, \ldots, x_k \in U_n$ is H_1.*

Received January 19, 1962. This paper presents research completed while the author held a fellowship with the Summer Research Institute of the Canadian Mathematical Congress, 1961.

We shall give an example to show that the analogue of Theorem 2 for $r > 1$ is, in general, false.

2. Proofs.

Let $|X|$ and $||X||$ denote, respectively, the determinant and the absolute value of the determinant of the matrix X. If $\omega = \{i_1, \ldots, i_r\}$ and $\tau = \{j_1, \ldots, j_r\}$ are in $Q_{n,r}$, then $X[\omega|\tau]$ denotes the submatrix of X which lies in the intersection of rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ of X. Set $\sigma(\tau) = j_1 + \ldots + j_r$ and let $Q_{n,s} - \tau$ denote the set of sequences $\{m_1, \ldots, m_s\} \in Q_{n,s}$ for which $\{m_1, \ldots, m_s\} \cap \{j_1, \ldots, j_r\}$ is empty. If $\omega \in Q_{n,s}$, then ω' will denote the only element of $Q_{n,n-s} - \omega$; ω' contains the integers $1, \ldots, n$ which are not in ω.

LEMMA. *Let $B = (b_{i,j})$, $1 \leqslant i, j \leqslant n$, be a unitary matrix. Let $\tau = \{j_1, \ldots, j_r\}$ be a fixed member of $Q_{n,r}$, and $\mu = \{k + 1, \ldots, n\}$ be a fixed member of $Q_{n,n-k}$. Then, for $1 \leqslant r \leqslant k < n$,*

$$\sum_{\omega \in Q_{k,r}} ||B[\omega|\tau]||^2 = \sum_{\rho \in Q_{n,n-k}-\tau} ||B[\mu|\rho]||^2. \tag{1}$$

Proof. For $1 \leqslant s < n$ let γ, δ be two elements of $Q_{n,s}$. Let $B_{i,j}$ denote the cofactor of $b_{i,j}$ in B and let $(B_{i,j})$ denote the $n \times n$ matrix with $B_{i,j}$ in row i and column j, $1 \leqslant i, j \leqslant n$. For any matrix B (not necessarily unitary) the following identity is known (**3**, Eq. 8.6):

$$|(B_{i,j})[\gamma|\delta]| = (-1)^{\sigma(\gamma)+\sigma(\delta)}|B[\gamma'|\delta']|\,|B|^{s-1}. \tag{2}$$

If B is unitary, then $B_{i,j}/|B| = \bar{b}_{i,j}$, the complex conjugate of $b_{i,j}$. Hence (2) becomes

$$|B|\,|\bar{B}[\gamma|\delta]| = (-1)^{\sigma(\gamma)+\sigma(\delta)}\,|B[\gamma'|\delta']|. \tag{3}$$

Let $C = (c_{i,j})$, $1 \leqslant i, j \leqslant n$, where

$$c_{i,j} = \begin{cases} 0 & \text{if } i \in \mu \text{ and } j \in \tau, \\ b_{i,j} & \text{otherwise.} \end{cases} \tag{4}$$

Then, using (3),

$$\sum_{\omega \in Q_{k,r}} ||B[\omega|\tau]||^2 = \sum_{\omega \in Q_{k,r}} |B[\omega|\tau]|\,|\bar{B}[\omega|\tau]|$$

$$= |B|^{-1} \sum_{\omega \in Q_{k,r}} (-1)^{\sigma(\omega)+\sigma(\tau)}|B[\omega|\tau]|\,|B[\omega'|\tau']|.$$

But this expression, apart from the factor $|B|^{-1}$, is just the Laplace expansion of $|C|$ down columns $j_1, \ldots, j_r$; the other terms that would normally appear in this Laplace expansion are all zero because of (4). Hence the left member of (1) is just $|B|^{-1}|C|$.

On the other hand, if we expand $|C|$ across rows $k + 1, \ldots, n$ and use (4) and (3),

https://doi.org/10.4153/CJM-1963-026-8 Published online by Cambridge University Press

$$|B|^{-1}|C| = |B|^{-1} \sum_{\rho \in Q_{n,n-k}-\tau} (-1)^{\sigma(\mu)+\sigma(\rho)} |B[\mu|\rho]|\, |B[\mu'|\rho']|$$
$$= \sum_{\rho \in Q_{n,n-k}-\tau} |B[\mu|\rho]|\, |\bar{B}[\mu|\rho]|$$
$$= \sum_{\rho \in Q_{n,n-k}-\tau} ||B[\mu|\rho]||^2.$$

Proof of Theorem 1. Throughout the rest of this paper $e_1, \ldots, e_n$ denotes an orthonormal set of characteristic vectors of A belonging to the characteristic values $\lambda_1, \ldots, \lambda_n$, respectively. We are given orthonormal vectors $x_1, \ldots, x_k \in U_n$. If $k = n$, the result is clear since the vectors x_ω for $\omega \in Q_{n,r}$ form an orthonormal basis in U_m so that $g_r = \text{trace } C_r(A) = E_r(\lambda_1, \ldots, \lambda_n)$. Suppose $k < n$ and choose $x_{k+1}, \ldots, x_n$ so that $x_1, \ldots, x_n$ is an orthonormal basis for U_n. Let $B = ((x_i, e_j))$, $1 \leqslant i, j \leqslant n$. Then B is a unitary matrix. Now

$$x_i = \sum_{j=1}^{n} (x_i, e_j)e_j, \qquad 1 \leqslant i \leqslant k,$$

$$Ax_i = \sum_{j=1}^{n} \lambda_j(x_i, e_j)e_j, \qquad 1 \leqslant i \leqslant k.$$

Hence, using the multilinear and alternating properties of the Grassmann product, it follows that if $\tau = \{j_1, \ldots, j_r\}$,

$$x_\omega = \sum_{\tau \in Q_{n,r}} |B[\omega|\tau]| e_\tau,$$

$$C_r(A)x_\omega = \sum_{\tau \in Q_{n,r}} \lambda_{j_1} \ldots \lambda_{j_r} |B[\omega|\tau]| e_\tau,$$

so that

$$g_r = \sum_{\omega \in Q_{k,r}} \sum_{\tau \in Q_{n,r}} \lambda_{j_1} \ldots \lambda_{j_r} ||B[\omega|\tau]||^2$$

$$(5) \qquad = \sum_{\tau \in Q_{n,r}} \lambda_{j_1} \ldots \lambda_{j_r} \sum_{\omega \in Q_{k,r}} ||B[\omega|\tau]||^2.$$

For $\rho = \{m_1, \ldots, m_k\} \in Q_{n,k}$, let $h_\rho = |B[\mu|\rho']|$, where $\mu = \{k+1, \ldots, n\}$. Then we claim that

$$(6) \qquad g_r = \sum_{\rho \in Q_{n,k}} |h_\rho|^2 E_r(\lambda_{m_1}, \ldots, \lambda_{m_k}).$$

To see this, note that the coefficient of

$$(7) \qquad \lambda_{j_1} \ldots \lambda_{j_r}$$

in (6) is

$$\sum_{\rho' \in Q_{n,n-k}-\tau} ||B[\mu|\rho']||^2.$$

By the Lemma, this is the same as the coefficient of (7) in (5).

The proof of Theorem 1 will now be complete if we can show that

$$\sum_{\rho \in Q_{n,k}} |h_\rho|^2 = 1.$$

This is immediate since the $|B[\mu|\rho']|$ for $\rho' \in Q_{n,n-k}$ are the co-ordinates of the unit vector $x_{k+1} \wedge \ldots \wedge x_n$ relative to the orthonormal basis $e_{\rho'}$ in the space U_t with $t = {}_nC_{n-k}$.

Proof of Theorem 2. Since any point $P \in H_1$ may be written as a convex combination of three of the vertices of H_1 and since the vertices of H_1 lie among the numbers

$$\lambda_{j_1} + \ldots + \lambda_{j_k}, \qquad \{j_1, \ldots, j_k\} \in Q_{n,k}, \tag{8}$$

it is enough to show that any point P in the convex hull of three of the numbers (8) is of the form $P = g_1(x_1, \ldots, x_k)$ for orthonormal vectors

$$x_1, \ldots, x_k \in U_n.$$

Suppose we are given three sums (8), say S_1, S_2, S_3. With a proper choice of the notation we may assume that

$$\begin{aligned} S_1 = {} & (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+1} + \ldots + \lambda_{p+q}) \\ & + (\lambda_{p+q+1} + \ldots + \lambda_{p+q+r}) + (\lambda_{w+1} + \ldots + \lambda_{w+t}), \\ S_2 = {} & (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+1} + \ldots + \lambda_{p+q}) \\ & + (\lambda_{p+q+r+1} + \ldots + \lambda_{p+q+r+s}) + (\lambda_{w+t+1} + \ldots + \lambda_{w+t+u}), \\ S_3 = {} & (\lambda_1 + \ldots + \lambda_p) + (\lambda_{p+q+1} + \ldots + \lambda_{p+q+r}) \\ & + (\lambda_{p+q+r+1} + \ldots + \lambda_{p+q+r+s}) + (\lambda_{w+t+u+1} + \ldots + \lambda_{w+t+u+v}), \end{aligned}$$

where, for brevity, we have let $w = p + q + r + s$. Here some of p, q, r, s, t, u, v may be zero, in which case not all of the types of terms indicated need actually appear. We have

$$p + q + r + t = k, \tag{9}$$

$$p + q + s + u = k, \tag{10}$$

$$p + r + s + v = k. \tag{11}$$

We may suppose that $t \geqslant u \geqslant v$. Let α, β, θ be three real numbers with $\alpha^2 + \beta^2 + \theta^2 = 1$. We have to find orthonormal vectors $x_1, \ldots, x_k \in U_n$ such that

$$(Ax_1, x_1) + \ldots + (Ax_k, x_k) = \alpha^2 S_1 + \beta^2 S_2 + \theta^2 S_3.$$

If $p > 0$, set $x_i = e_i$ for $1 \leqslant i \leqslant p$. Then

$$(Ax_1, x_1) + \ldots + (Ax_p, x_p) = \lambda_1 + \ldots + \lambda_p.$$

If $v > 0$, set $x_{p+i} = \alpha e_{w+i} + \beta e_{w+t+i} + \theta e_{w+t+u+i}$ for $1 \leqslant i \leqslant v$. Then $(x_{p+i}, x_{p+i}) = 1$ and

$$(Ax_{p+1}, x_{p+1}) + \ldots + (Ax_{p+v}, x_{p+v}) = \alpha^2(\lambda_{w+1} + \ldots + \lambda_{w+v}) + \beta^2(\lambda_{w+t+1} + \ldots + \lambda_{w+t+v}) + \theta^2(\lambda_{w+t+u+1} + \ldots + \lambda_{w+t+u+v}).$$

From (10) and (11) it follows that $r = q + (u - v)$; hence $r \geqslant u - v$. If $u > v$, let

$$x_{p+v+i} = \beta e_{w+t+v+i} + (\alpha^2 + \theta^2)^{\frac{1}{2}} e_{p+q+i} \qquad \text{for } 1 \leqslant i \leqslant u - v.$$

Then $(x_{p+v+i}, x_{p+v+i}) = 1$ and

$$(Ax_{p+v+1}, x_{p+v+1}) + \ldots + (Ax_{p+u}, x_{p+u}) = \beta^2(\lambda_{w+t+v+1} + \ldots + \lambda_{w+t+u}) + (\alpha^2 + \theta^2)(\lambda_{p+q+1} + \ldots + \lambda_{p+q+u-v}).$$

It follows from (9) and (11) that $s = q + (t - v)$. If $t > v$, define

$$x_{p+u+i} = \alpha e_{w+v+i} + (\beta^2 + \theta^2)^{\frac{1}{2}} e_{p+q+r+i} \qquad \text{for } 1 \leqslant i \leqslant t - v.$$

Then $(x_{p+u+i}, x_{p+u+i}) = 1$ and

$$(Ax_{p+u+1}, x_{p+u+1}) + \ldots + (Ax_{p+u+t-v}, x_{p+u+t-v}) = \alpha^2(\lambda_{w+v+1} + \ldots + \lambda_{w+t}) + (\beta^2 + \theta^2)(\lambda_{p+q+r+1} + \ldots + \lambda_{p+q+r+t-v}).$$

Up to this point $p + u + t - v$ vectors x_i have been constructed; these vectors are automatically orthogonal because, when expressed in terms of the e_i, no two x_i involve the same e_i. There remain $k - (p + u + t - v) = 2q$ vectors x_i to be constructed. Let G be the subspace of U_n spanned by

$$f_1 = e_{p+1}, \ldots, f_q = e_{p+q}, \quad f_{q+1} = e_{p+q+u-v+1}, \ldots, f_{2q} = e_{p+q+r},$$
$$f_{2q+1} = e_{p+q+r+t-v+1}, \ldots, f_{3q} = e_{p+q+r+s};$$

and let $\zeta_1, \ldots, \zeta_{3q}$ be the λ_i belonging to $f_1, \ldots, f_{3q}$. Let $y_i = \theta f_i + \beta f_{q+i} + \alpha f_{2q+i}$ for $1 \leqslant i \leqslant q$. Choose $x_{k-2q+1}, \ldots, x_k$ such that $y_1, \ldots, y_q, x_{k-2q+1}, \ldots, x_k$ is an orthonormal basis of G. Then if we compute the trace of the restriction A_G of A to G we get:

$$\begin{aligned}\text{trace } A_G &= \zeta_1 + \ldots + \zeta_{3q} \\ &= (Ay_1, y_1) + \ldots + (Ay_q, y_q) + (Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k) \\ &= \theta^2(\zeta_1 + \ldots + \zeta_q) + \beta^2(\zeta_{q+1} + \ldots + \zeta_{2q}) \\ &\quad + \alpha^2(\zeta_{2q+1} + \ldots + \zeta_{3q}) + (Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k).\end{aligned}$$

Hence we find that

$$(Ax_{k-2q+1}, x_{k-2q+1}) + \ldots + (Ax_k, x_k) = (\alpha^2 + \beta^2)(\lambda_{p+1} + \ldots + \lambda_{p+q}) + (\alpha^2 + \theta^2)(\lambda_{p+q+u-v+1} + \ldots + \lambda_{p+q+r}) + (\beta^2 + \theta^2)(\lambda_{p+q+r+t-v+1} + \ldots + \lambda_{p+q+r+s}).$$

Then $x_1, \ldots, x_k$ are orthonormal vectors in U_n such that

$$(Ax_1, x_1) + \ldots + (Ax_k, x_k) = \alpha^2 S_1 + \beta^2 S_2 + \theta^2 S_3.$$

We now give an example to show that the set of all numbers $g_r(x_1, \ldots, x_k)$ for orthonormal $x_1, \ldots, x_k$ need not be a convex set if $r > 1$. Let $r = k = 2$, $n = 4$, and take $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = i = (-1)^{\frac{1}{2}}$. Let $p_{i,j} = |B[1, 2|i, j]|$, where B is the matrix $((x_i, e_j))$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant 4$. Then, from (5),

$$g_2(x_1, x_2) = |p_{1,2}|^2 - |p_{3,4}|^2 + i(|p_{1,3}|^2 + |p_{1,4}|^2 + |p_{2,3}|^2 + |p_{2,4}|^2).$$

Now $g_2(e_1, e_2) = 1$ and $g_2(e_3, e_4) = -1$. If $g_2(x_1, x_2) = 0$, then we must have $|p_{1,2}| = |p_{3,4}|$, $p_{1,3} = p_{1,4} = p_{2,3} = p_{2,4} = 0$. However, it is known **(1)** that $p_{1,2}p_{3,4} = p_{1,3}p_{2,4} - p_{1,4}p_{2,3}$. Combining these facts, it follows that also $p_{1,2} = p_{3,4} = 0$. This is a contradiction, since

$$\sum_{1 \leqslant i < j \leqslant 4} |p_{i,j}|^2 = (x_1 \wedge x_2, x_1 \wedge x_2) = 1$$

if x_1 and x_2 are orthonormal.

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