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Maximal ideals of semigroups of endomorphisms

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The maximal left ideal, the maximal right ideal and the maximal two-sided ideal are all characterized for quite general semigroups of endomorphisms. These results are then applied to such semigroups as the semigroup of all continuous selfmaps of a topological space, the semigroup of all closed selfmaps of a topological space, and the semigroup of all linear endomorphisms of a vector space.

1. Introduction

The word ideal means two-sided ideal. In semigroups, unions of ideals are ideals, so that any semigroup with identity has a unique maximal proper ideal if it has proper ideals at all. It is simply the union of all of the proper ideals of the semigroup. It therefore makes sense to refer to this ideal as *the* maximal ideal of the semigroup and we will do this. In this paper we characterize these ideals for quite general semigroups of endomorphisms. We are then able to treat, as special cases, such semigroups as the semigroup of all continuous selfmaps of a topological space and the semigroup of all linear transformations of a vector space. In order to give some idea of the nature of the results, we quote several. For example, let $S(I^N)$ denote the semigroup, under composition, of all continuous selfmaps of the N-dimensional euclidean cube I^N. Then the maximal ideal of $S(I^N)$ consists of all those functions which are not

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injective on any subspace which is homeomorphic to I^N . It follows from this that the maximal ideal of $S(I)$ (I is the closed unit interval) certainly contains all those functions which are continuous but nowhere differentiable on I . The case for $S(E^N)$ where E^N is the euclidean N-space is quite different. The maximal ideal consists of those continuous selfmaps which are not units of $S(E^N)$; that is, those continuous self-maps which are *not* autohomeomorphisms of E^N . Thus, $S(E^N)$ is the union of its maximal ideal and its group of units. The maximal ideal of $L(V)$, the semigroup of all linear transformations on a vector space V consists of all those linear transformations whose ranks are less than the dimension of V . It turns out that $L(V)$ is the union of its maximal ideal and group of units if and only if V is finite dimensional.

The previous remarks should be sufficient to give some indication of the type of result we get. In addition to the semigroups we have discussed, we also consider semigroups of closed functions and in every instance we get results about the maximal one-sided ideals as well. Before we conclude with the introduction, some other remarks are perhaps in order. In any semigroup with identity, the maximal ideal consists of all elements which are not $\mathcal{J}$-equivalent to the identity where $\mathcal{J}$ is the relation of Green in which two elements are equivalent if they generate the same ideal. Consequently, the problem of determining the maximal ideal is equivalent to the problem of determining the $\mathcal{J}$-class which contains the identity. We also discuss the analogous problem for Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{D}$. Of course, the $\mathcal{H}$-class containing the identity in any semigroup is just the group of units of that semigroup. It follows from one of our previous remarks that the $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, and $\mathcal{J}$-classes of $S(E^N)$ which contain the identity all coincide and they also coincide for $L(V)$ when V is finite dimensional. They are all distinct, however, for the semigroups $S(I^N)$.

2. Δ-structures

We recall some definitions and notation from [3]. The symbols $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ will be used to denote respectively the domain and range of a function f . Composition of functions will be denoted by juxtaposition and f/A will denote the restriction of a function f to a subset A of its domain.

DEFINITION 2.1. A Δ-structure on a nonempty set X is a pair (A, M) where A is a family of subsets of X containing X itself and

$$M = \{\hom(A, B) : (A, B) \in \mathsf{A} \times \mathsf{A}\}$$

where $\hom(A, B)$ is a collection of functions with domains equal to A and ranges contained in B, and the following conditions are satisfied:

(2.1.1) $\mathrm{end}X = \hom(X, X)$ is a semigroup under composition which contains id_X, the identity map on X ;

(2.1.2) $\mathrm{ran}f \in \mathsf{A}$ for each f in $\mathrm{end}X$;

(2.1.3) if $f \in \mathrm{end}X$ and $g \in \hom(\mathrm{ran}(f), B)$, then $gf \in \mathrm{end}X$;

(2.1.4) suppose $f, g \in \mathrm{end}X$; $A, B \in \mathsf{A}$; $f(B) \subset A$, and $g(A) \subset B$. Suppose also that $fg/A = \mathrm{id}_A$ and $gf/B = \mathrm{id}_B$. Then $g/A \in \hom(A, B)$ and $f/B \in \hom(B, A)$.

We will refer to $\mathrm{end}X$ as the semigroup of the Δ-structure (A, M).

DEFINITION 2.2. A function f in $\hom(A, B)$ is a Δ-isomorphism if there exists a g in $\hom(B, A)$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$. When $\hom(A, B)$ contains a Δ-isomorphism, we say that A and B are Δ-isomorphic.

DEFINITION 2.3. A Δ-retract of X is any subset which is the range of an idempotent map in $\mathrm{end}X$.

Now suppose we review the definitions of Green's relations. Two elements a and b of a semigroup S are $\mathcal{L}$-related if they generate the same principal left ideal, $\mathcal{R}$-related if they generate the same principal right ideal, and $\mathcal{J}$-related if they generate the same principal ideal. The relation $\mathcal{L} \cap \mathcal{R}$ is denoted by $\mathcal{H}$ and $\mathcal{L} \circ \mathcal{R}$ is denoted by $\mathcal{D}$. Since $\mathcal{L}$ and $\mathcal{R}$ commute, $\mathcal{D}$ is also an equivalence relation. These five relations are referred to as Green's relations and a fairly extensive discussion of them can be found in [1].

In what follows, $\mathrm{end}X$ will be the semigroup of some Δ-structure on

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a nonempty set X and its identity will be denoted by i . Now, in any semigroup, the maximal subgroups are precisely the H-classes which contain idempotents and if the semigroup has an identity, the H-class containing the identity is just the group of units of the semigroup. Thus, the H-class H_i of end$X$ is just the collection of all Δ-isomorphisms from X onto X . In the next four results, we determine L_i, R_i, J_i , and D_i , and the L-class, R-class, J-class, and D-class respectively which contain the identity i . As we mentioned previously, we are interested in L_i, R_i , and J_i primarily because we want to characterize the maximal left, right, and two-sided ideals of end$X$. For the sake of completeness, we include the result on D_i as well. As will be seen, the results on L_i, R_i , and D_i follow immediately from results in [3] and the well known fact that if one element in a D-class is regular then all the elements in that D-class are regular. The result on J_i does not follow from results in [3] since theorems in that paper are about regular elements and J_i in general contains many irregular elements.

THEOREM 2.4. *A function* f *in* end$X$ *belongs to* L_i *if and only if* ran(f) *is a* Δ*-retract of* X *and* f *is a* Δ*-isomorphism from* X *onto* ran(f) .

THEOREM 2.5. *A function* f *in* end$X$ *belongs to* R_i *if and only if there exists a set* $A \in \mathsf{A}$ *such that* f/A *is a* Δ*-isomorphism from* A *onto* X .

THEOREM 2.6. *A function* f *in* end$X$ *belongs to* D_i *if and only if* ran(f) *is a* Δ*-retract of* X *which is* Δ*-isomorphic to* X *and there exists a* Δ*-retract* A *of* X *such that* f/A *is a* Δ*-isomorphism from* A *onto* ran(f) .

Proofs. We recall once again that if one element in a D-class is regular, then all of the elements in that D-class are regular. Moreover, both L_i and R_i are subsets of D_i . Theorems 2.4 and 2.5 follow readily from these facts and Theorems (2.4) and (2.5) of [3]. In a similar manner, Theorem 2.6 follows from Theorems (2.4) and (2.6) of [3].

Next we characterize the elements in J_i .

THEOREM 2.7. *A function f in* end$X$ *belongs to J_i if and only if there exist two Δ-retracts A and B of X, both Δ-isomorphic to X such that f/A is a Δ-isomorphism from A onto B.*

Proof. Suppose first that $f \in J_i$. Then there exist functions h and g in end$X$ such that $hfg = i$. Take $A = \text{ran}(g)$ and $B = \text{ran}(fg)$. For any $y \in A$, we have $y = g(x)$ for some x and hence

$$ghf(y) = ghfg(x) = gi(x) = g(x) = y \ .$$

For any $y \in B$, we have $y = fg(x)$ for some x and it follows that

$$fgh(y) = fghfg(x) = fgi(x) = fg(x) = y \ .$$

Thus $ghf/A = \text{id}_A$ and $fgh/B = \text{id}_B$, and it follows that f/A is a Δ-isomorphism from A onto B .

Next, we need that g is a Δ-isomorphism from X onto A which, as we recall, is Ran(g) . We have already observed above that $ghf/A = \text{id}_A$ and since $hfg = i$, it follows immediately that g is indeed a Δ-isomorphism from X onto A . Furthermore, since f/A is a Δ-isomorphism from A onto B , it follows that fg is a Δ-isomorphism from X onto B , that is, B is Δ-isomorphic to X .

Now we want to show that both A and B are Δ-retracts of X . Since $hfg = i$, hf must map X onto X . Thus,

$$\text{ran}(ghf) = ghf(X) = g(X) = \text{ran}(g) = A \ .$$

Since ghf is idempotent, A is a Δ-retract of X . In a similar manner, since $hfg = i$, we may conclude that h maps X onto X and it follows that

$$\text{ran}(fgh) = fgh(X) = fg(X) = \text{ran}(fg) = B \ .$$

But fgh is also idempotent so that B is also a Δ-retract of X .

Conversely, suppose that A and B are Δ-retracts of X , both Δ-isomorphic to X and that f/A is a Δ-isomorphism from A onto B . Then there exist idempotents v and w of end$X$ such that $\text{ran}(v) = A$ and $\text{ran}(w) = B$ and there exist functions $h \in \text{hom}(X, A)$, $k \in \text{hom}(A, X)$,

$t \in \hom(B, X)$, $r \in \hom(X, B)$, and $p \in \hom(B, A)$, such that $hk = \mathrm{id}_A$, $kh = i$, $tr = i$, $rt = \mathrm{id}_B$, $(f/A)p = \mathrm{id}_B$, and $p(f/A) = \mathrm{id}_A$. Then, by condition (2.1.3), the functions tw, kv , and pw all belong to end$X$. One can then verify that

$$(kv)(pw)r(tw)fh = i \ .$$

In fact, since v and w are idempotent, $v/A = \mathrm{id}_A$ and $w/B = \mathrm{id}_B$. These two facts, together with the previous statements result in the following

$$(kv)(pw)r(tw)fh = kprtfh = kp(\mathrm{id}_B)fh = kpfh = k(\mathrm{id}_A)h = kh = i \ .$$

Thus $f \in J_i$, and the theorem is proved.

We conclude this section by essentially restating Theorems 2.4, 2.5, and 2.7. If end$X$ has a proper left ideal, then it has a unique maximal (proper) left ideal. It is simply the union of all proper left ideals of end$X$ and it consists precisely of those functions which are not L-related to i . We therefore get from Theorem 2.4 the following

THEOREM 2.8. *Suppose* end$X$ *has a proper left ideal. Then a function* f *belongs to the maximal left ideal of* end$X$ *if and only if either* ran(f) *is not a* Δ*-retract of* X *or* f *is not a* Δ*-isomorphism from* X *onto* ran(f) .

The next two results follow immediately from Theorems 2.5 and 2.7 respectively and are the analogous results for the maximal right ideal and the maximal ideal of end$X$.

THEOREM 2.9. *Suppose* end$X$ *has a proper right ideal. Then a function* f *belongs to the maximal right ideal if and only if its restriction to no* $A \in \mathrm{A}$ *is a* Δ*-isomorphism onto* X .

THEOREM 2.10. *Suppose* end$X$ *has a proper ideal. Then a function* f *belongs to the maximal ideal of* end$X$ *if and only if for no pair of* Δ*-retracts* A *and* B *of* X *, both* Δ*-isomorphic to* X *, is the restriction of* f *to* A *a* Δ*-isomorphism onto* B .

It happens at times that a semigroup is the union of its maximal ideal and its group of units. These two sets must, of course, be disjoint. We give such semigroups a name.

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DEFINITION 2.11. A semigroup with proper ideals which is the union of its maximal ideal and its group of units will be referred to as a *separated* semigroup.

As we mentioned previously, $H_\iota = L_\iota \cap R_\iota$ is, in any semigroup with identity ι, just the group of units of that semigroup. Of course, both L_ι and R_ι contain H_ι and both are contained in D_ι which, in turn, is contained in J_ι. These classes are generally all distinct but, as the following result shows, not always.

THEOREM 2.12. *Suppose* end$X$ *contains a proper ideal. Then the following statements are equivalent:*

(2.12.1) end$X$ *is a separated semigroup;*

(2.12.2) *there is a Rees-factor semigroup of* end$X$ *which is a group with zero;*

(2.12.3) *if the product of two functions in* end$X$ *is a unit, then each of the functions is a unit;*

(2.12.4) *the classes* H_ι, L_ι, R_ι, D_ι, *and* J_ι *all coincide;*

(2.12.5) *if* $A \in \mathrm{A}$ *is a* Δ*-retract of* X *and is also* Δ*-isomorphic to* X, *then* $A = X$.

Proof. Suppose end$X$ is separated and M is its maximal ideal. Then $(\mathrm{end}X)/M$ is a group with zero and we see that (2.12.1) implies (2.12.2). Now assume (2.12.2) holds and suppose fg is a unit of end$X$. Then $(\mathrm{end}X)/N$ is a group with zero for some ideal N of end$X$. Since fg is a unit $fg \notin N$, that is fg is a nonzero element of $(\mathrm{end}X)/N$. It readily follows that both f and g are units of end$X$ and hence that (2.12.2) implies (2.12.3).

Now we show that (2.12.3) implies (2.12.4) and to do this, it is sufficient to observe that $J_\iota \subset H_\iota$. Suppose $f \in J_\iota$. Then $\iota = hfg$ for appropriate h and g in end$X$. But (2.12.3) implies that h, g, and f must all be units. Thus $J_\iota \subset H_\iota$.

Now assume (2.12.4) holds and let $A \in \mathrm{A}$ be a Δ-retract of X which is also Δ-isomorphic to X. Then there exist functions v and

$f \in \text{hom}(X, A)$, and $g \in \text{hom}(A, X)$ such that $fg = \text{id}_A$, $gh = i$, $v^2 = v$, and $\text{ran}(v) = A$. Thus both f and gv belong to $\text{end}X$ and $(gv)f = gf = i$. Thus $f \in J_i$ and since $J_i = H_i$, we conclude that f is a unit in $\text{end}X$ and hence that A and X must coincide. Thus (2.12.4) implies (2.12.5).

Now suppose (2.12.5) holds and let f be any element which is not in the maximal ideal of $\text{end}X$. Then according to Theorem 2.10, there are two Δ-retracts A and B of X both Δ-isomorphic to X such that the restriction of f to A is a Δ-isomorphism onto B . But (2.12.5) assures us that $A = X = B$. Thus, f is a unit of $\text{end}X$. That is to say, $\text{end}X$ is a separated semigroup. This verifies that (2.12.5) implies (2.12.1) and completes the proof.

REMARK. Conditions (2.12.1), (2.12.2), (2.12.3), and (2.12.4) are equivalent in any semigroup with identity which contains a proper ideal. Condition (2.12.5), of course, does not even make sense for an arbitrary semigroup.

COROLLARY 2.13. *In a separated semigroup with a proper ideal, the maximal left ideal, the maximal right ideal, and the maximal ideal all coincide.*

Proof. The maximal left ideal is the complement of L_i , the maximal right ideal is the complement of R_i , and the maximal ideal is the complement of J_i . The conclusion now follows from the fact that in any semigroup with identity and a proper ideal, conditions (2.12.1) and (2.12.4) are equivalent.

In the next several sections, we apply these results to particular semigroups of endomorphisms.

3. Semigroups of continuous functions

Let X be a topological space and let $S(X)$ denote the semigroup, under composition, of all continuous functions mapping X into X . We will assume throughout this section and the next that X has more than one point so that the semigroups under discussion do have proper ideals. The semigroup $S(X)$ is really $\text{end}X$ for a very natural Δ-structure on X .

Simply take $\mathcal{A}$ to be the collection of all nonempty subspaces of X and $\hom(A, B)$ $(A, B \in \mathcal{A})$ to be the collection of all continuous functions from A into B. It is immediate that this results in a Δ-structure, so that all the results of the previous section apply. In this case, a Δ-retract is just a retract in the usual topological sense, that is, the range of an idempotent continuous selfmap, and a Δ-isomorphism is simply a homeomorphism. The results of the previous section then translate immediately into the following results.

THEOREM 3.1. *A function f in $S(X)$ belongs to L_i if and only if it is a homeomorphism from X into X and its range is a retract. Consequently, the maximal left ideal consists of all functions in $S(X)$ with the property that either the range is not a retract or the function is not a homeomorphism from X into X.*

THEOREM 3.2. *A function f in $S(X)$ belongs to R_i if and only if it maps some subspace of X homeomorphically onto X. Consequently, the maximal right ideal of $S(X)$ consists of all those functions which map no subspace of X homeomorphically onto X.*

THEOREM 3.3. *A function f in $S(X)$ belongs to J_i if and only if there exist two retracts of X both homeomorphic to X such that f maps one homeomorphically onto the other. Consequently, the maximal ideal of $S(X)$ consists of all those functions in $S(X)$ with the property that they map no retract homeomorphically onto another when the retracts are both homeomorphic to X.*

THEOREM 3.4. *A function f in $S(X)$ belongs to D_i if and only if its range is a retract which is homeomorphic to X and it maps some retract of X homeomorphically onto its range.*

Now let E^N denote the euclidean N-space and let I^N denote the compact N-dimensional cube. We apply these latter results to $S(E^N)$ and $S(I^N)$. First of all, it follows from a well known theorem of Brouwer [2, p. 95] that no euclidean N-space is homeomorphic to a proper closed subspace and since any retract of a Hausdorff space is closed, condition (2.12.5) is satisfied, so from Theorem 2.12 we immediately get

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THEOREM 3.5. *The following statements about* $S(E^N)$ *are valid for each euclidean* N*-space* E^N *:*

(3.5.1) $S(E^N)$ *is a separated semigroup;*

(3.5.2) *there is a Rees-factor semigroup of* $S(E^N)$ *which is a group with zero;*

(3.5.3) *if the composition of two functions in* $S(E^N)$ *is a homeomorphism mapping* E^N *onto* E^N *, then each of the functions is a homeomorphism mapping* E^N *onto* E^N *;*

(3.5.4) *the classes* H_i, L_i, R_i, D_i *, and* J_i *all coincide in* $S(E^N)$.

This latter result together with Corollary 2.13 yields

COROLLARY 3.6. *The maximal left ideal, the maximal right ideal, and the maximal ideal all coincide in* $S(E^N)$.

The next result shows that the situation is far different for the semigroups $S(I^N)$.

THEOREM 3.7. *The classes* H_i, L_i, R_i, D_i *, and* J_i *are all distinct in* $S(I^N)$ *. Moreover, the maximal left ideal of* $S(I^N)$ *consists of all those functions in* $S(I^N)$ *which are not injective. The maximal right ideal consists of all those functions which map no subspace of* I^N *homeomorphically onto* I^N *and the maximal ideal consists of all those functions which are not injective on any subspace which is homeomorphic to* I^N .

Proof. One appeals to Theorems 3.1, 3.2, 3.3, and 3.4 and easily produces functions which show that the various classes are indeed distinct. Now suppose that $f \in S(I^N)$ is not injective. Then in view of Theorem 3.1, f belongs to the maximal left ideal of $S(I^N)$. Suppose, on the other hand, that f belongs to the maximal left ideal of $S(I^N)$. Then f

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cannot possibly be injective for f injective implies that it is a homeomorphism into I^N, which in turn implies that $\operatorname{ran}(f)$ is a retract (in fact, an absolute retract for normal spaces) and all this contradicts Theorem 3.1. Consequently, the maximal left ideal of $S(I^N)$ does consist of all those functions in $S(I^N)$ which are not injective. In a similar manner, the assertions for the maximal right ideal and the maximal ideal follow from Theorems 3.2 and 3.3 respectively.

REMARK. The maximal ideal of $S(I)$ (I is the closed unit interval) contains all continuous selfmaps of I which are nowhere differentiable. To get nontrivial functions in $S(I^N)$ $(N > 1)$ which belong to the maximal ideal of $S(I^N)$, map I^N onto I with any continuous function f and map I into I^N with any nonconstant continuous function g. The function f cannot possibly be injective on any subspace of I^N which is homeomorphic to I^N. Consequently, gf cannot be either and it follows from the previous corollary that gf belongs to the maximal ideal of $S(I^N)$.

4. Semigroups of closed functions

By a closed function from a topological space X into a topological space Y, we mean any function f such that $f(A)$ is closed in Y for each closed subset A of X. We emphasize that continuity is not required. *It will be assumed that all topological spaces discussed in this section are* T_1 *and we will also assume that* X *has more than one point.* The symbol $\Gamma(X)$ will denote the semigroup, under composition, of all closed functions which map the topological space X into itself. Then $\Gamma(X)$ is the semigroup of a very natural Δ-structure on the space X. Simply take A to be the collection of all nonempty closed subsets and for $A, B \in \mathrm{A}$, take $\hom(A, B)$ to be the collection of all closed functions from A into B. This results in a Δ-structure where the Δ-isomorphisms are simply homeomorphisms. The Δ-retracts of X in this case are precisely the nonempty closed subsets of X. Certainly, every Δ-retract is closed. To see that the converse is true, let A be any nonempty closed subset of X and choose $a \in A$. Define a function v

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from X into X by

$$\upsilon(x) = x \quad \text{for} \quad x \in A ,$$

$$\upsilon(x) = a \quad \text{for} \quad x \in X - A .$$

Then υ is idempotent since it is the identity on its range. Moreover, for any closed subset B of X we have

$$\upsilon(B) = \upsilon(B\cap A) \cup \upsilon\big(B\cap(X-A)\big) = (B\cap A) \cup \upsilon\big(B\cap(X-A)\big) .$$

Now $\upsilon\big(B\cap(X-A)\big)$ is either empty or consists of the single point a. In either event, $\upsilon(B)$ is closed since $B \cap A$ is closed and X is T_1. Among other things, this means that $\text{ran}(f)$ is a Δ-retract for each f in $\Gamma(X)$. With all this in mind, one sees that various results of Section 2 translate into the following results.

THEOREM 4.1. *A function f in $\Gamma(X)$ belongs to L_i if and only if it is a homeomorphism from X into X. Consequently, the maximal left ideal of $\Gamma(X)$ consists of all functions in $\Gamma(X)$ which do not map X homeomorphically into X.*

THEOREM 4.2. *A function f in $\Gamma(X)$ belongs to R_i if and only if it maps some closed subset of X homeomorphically onto X. Consequently, the maximal right ideal of $\Gamma(X)$ consists of all those functions in $\Gamma(X)$ which map no closed subset of X homeomorphically onto X.*

THEOREM 4.3. *A function f in $\Gamma(X)$ belongs to J_i if and only if it maps some closed subspace which is a homeomorphic copy of X homeomorphically onto some closed subspace which is also a homeomorphic copy of X. Consequently, the maximal ideal of $\Gamma(X)$ consists of all those functions in $\Gamma(X)$ which do not map homeomorphic copies of X homeomorphically onto homeomorphic copies of X.*

THEOREM 4.4. *A function f in $\Gamma(X)$ belongs to D_i if and only if its range is homeomorphic to X and it maps some closed subset of X homeomorphically onto its range.*

Again we appeal to Theorem 2.12 and Brouwer's Theorem [2, p. 95] which tells us that no euclidean N-space E^N is homeomorphic to a proper closed subspace and we get the following analogue of Theorem 3.5.

THEOREM 4.5. *The following statements about* $\Gamma(E^N)$ *are valid for each euclidean* N*-space* E^N *:*

(4.5.1) $\Gamma(E^N)$ *is a separated semigroup;*

(4.5.2) *there is a Rees-factor semigroup of* $\Gamma(E^N)$ *which is a group with zero;*

(4.5.3) *if the composition of two functions in* $\Gamma(E^N)$ *is a homeomorphism mapping* E^N *onto* E^N *, then each of the functions is a homeomorphism mapping* E^N *onto* E^N *;*

(4.5.4) *the classes* H_i, L_i, R_i, D_i *, and* J_i *all coincide in* $\Gamma(E^N)$.

We will omit the statements of the analogues for $\Gamma(I^N)$ of Theorems 3.6 and 3.7.

5. Semigroups of linear transformations

One can take any algebraic system such as a group, ring, module, and so on, and then define an appropriate Δ-structure so that the semigroup of all endomorphisms of the system is the semigroup of the Δ-structure and then, of course, the results of Section 2 apply. We illustrate this with a vector space V over a division ring. We denote by $L(V)$ the semigroup, under composition, of all linear transformations mapping V into V . We now define a Δ-structure whose semigroup will be $L(V)$. Take $\mathcal{A}$ to be the collection of all nonempty subspaces of V and for $A, B \in \mathcal{A}$, take $\hom(A, B)$ to be the collection of all linear transformations from A into B . The Δ-isomorphisms are, in this case, the usual linear isomorphisms and every $A \in \mathcal{A}$ is a Δ-retract. To see this, let $\mathcal{B}$ be any basis for A and extend it to a basis $\mathcal{C}$ for V . Take any function mapping $\mathcal{C}$ into $\mathcal{B}$ which is the identity on $\mathcal{B}$ and extend it linearly to an element f in $L(V)$. Then $\operatorname{ran}(f) = A$ and v is the identity on A and hence is idempotent. Thus A is a Δ-retract of V . The results of Section 2 now combine with well known results about vector spaces to produce the following three theorems.

THEOREM 5.1. *A linear transformation on* V *belongs to* L_i *if and only if it is an isomorphism from* V *into* V. *Consequently, the maximal left ideal of* $L(V)$ *consists of all linear transformations on* V *which are not injective.*

THEOREM 5.2. *A linear transformation on* V *belongs to* R_i *if and only if its range is all of* V. *Consequently, the maximal right ideal of* $L(V)$ *consists of all those linear transformations whose ranges are proper subspaces of* V.

THEOREM 5.3. *A linear transformation on* V *belongs to* J_i *if and only if its rank (dimension of its range) is equal to the dimension of* V. *Consequently, the maximal ideal of* $L(V)$ *consists of all those linear transformations whose ranks are less than the dimension of* V.

THEOREM 5.4. *A linear transformation on* V *belongs to* D_i *if and only if its rank is equal to the dimension of* V.

It follows from the latter two theorems that $D_i = J_i$ in any $L(V)$ whatsoever. This also follows from Theorem (5.3) of [3] which states, among other things, that $\mathcal{D}$ and $\mathcal{J}$ coincide for $L(V)$.

Now, a vector space V is isomorphic to a proper subspace if and only if it is infinite dimensional, so that condition (2.12.5) is satisfied if and only if V is finite dimensional, and we have

THEOREM 5.5. *The following statements about* $L(V)$ *are equivalent:*

(5.5.1) $L(V)$ *is a separated semigroup;*

(5.5.2) *if the composition of two linear transformations on* V *is a linear isomorphism from* V *onto* V, *then both transformations are also linear isomorphisms from* V *onto* V;

(5.5.3) *the classes* H_i, L_i, R_i, D_i, *and* J_i *all coincide in* $L(V)$;

(5.5.4) V *is finite dimensional.*

We conclude with the analogue of Corollary 3.6. Its proof is an immediate consequence of Corollary 2.13 and the latter theorem.

COROLLARY 5.6. *Let V be finite dimensional. Then its maximal left ideal, its maximal right ideal, and its maximal ideal all coincide in* $L(V)$.

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