

DIOPHANTINE EQUATIONS OF ERDÖS-MOSER TYPE

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Using an old result of Von Staudt on sums of consecutive integer powers, we shall show by an elementary method that the Diophantine equation $1^k + 2^k + \dots + (x-1)^k = ax^k$ has no solutions (a, x, k) with $k > 1$, $x < 10^{10^6}$. For $a = 1$ this equation reduces to the Erdős-Moser equation and the result to a result of Moser. Our method can also be used to deal with variants of the equation of the title, and two examples will be given. For one of them there are no integer solutions with $x < 10^{10^{11}}$.

1. INTRODUCTION

The Diophantine equation $1^k + 2^k + \dots + (x-1)^k = x^k$ is known as the *Erdős-Moser equation*. An obvious solution is $1 + 2 = 3$. Notice that there are no further solutions with $k = 1$. Erdős conjectured that there are no solutions with $k > 1$. This is Problem D7 in Guy [5], where some historical background is given. It has not been proved that the number of solutions of the Erdős-Moser equation is finite, but if solutions with $k > 1$ exist, both x and k must be larger than 10^{10^6} . For x this was proved by Moser [9]. Using Proposition 2 below the same bound can be deduced for k . Best and te Riele [1] proved that for every k there is at most one x (the trivial bound is $k+1$). The sum $1^k + 2^k + \dots + (x-1)^k$ cries out to be rewritten in terms of Bernoulli polynomials. Urbanowicz [13] seems to have been the first to attack the problem in this way. His approach is continued in [8], where the divisibility properties of solutions with $k > 1$ are investigated. It is shown there that $\text{lcm}(1, 2, \dots, 200)$ divides k and that x is not divisible by any regular prime nor by any prime < 10000 .

The present note is more in line with Moser's approach than with the approach of [8] and deals with Erdős-Moser type equations, that is equations of the form $1^k + 2^k + \dots + (x-1)^k = f(x, k)$. Moser's paper is remarkable in that he arrives at such a strong result using only elementary methods. His original paper is a gem of mathematics. Seemingly out of nothing Moser pulls the four inequalities given in (11) and (12) below

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with $a = 1$. Adding them together he finds Lemma 1 with $a = 1$. This is a strong inequality, for $\sum_{p|x} 1/p$ grows very slowly with x . Using a little bit of analytic number theory Moser then deduces that $x \geq 10^{10^6}$. Questions that arise are whether there are more inequalities like those in (11) and (12) (which could give rise to an even larger bound) and whether there is an interconnection between these inequalities. In this note a method will be presented that sheds some light on these questions. Furthermore, the method is sufficiently systematic to allow one to deal with equations of the form

$$(1) \quad 1^k + 2^k + \dots + (x - 1)^k = f(x, k).$$

We shall study the case $f(x, k) = ax^k$ in detail, where we consider a as a variable. Apparently Krzyzstofek [6] was the first to consider this equation. Our main result is the following generalisation of Moser’s result:

THEOREM 1. *The equation*

$$(2) \quad 1^k + 2^k + \dots + (x - 1)^k = ax^k$$

has no integer solutions (a, x, k) with

$$k > 1, x < \max(10^{10^6}, a \cdot 10^{22}).$$

Moser’s method can be easily extended to the case of fixed a , but it seems harder to generalise to variable a . As further examples of our method we shall present in Section 5 results on the equations

$$(3) \quad 1^k + 2^k + \dots + (x - 1)^k = ax^k + 1$$

and

$$(4) \quad 1^k + 2^k + \dots + (x - 1)^k = x^k + 1 + \frac{(x - 2)(x - 1)x(x + 1)(x + 5)}{3} 2^{k-6}.$$

Since our way of dealing with these equations is the same as for (2), we shall indicate only some fine points of the proofs. Unfortunately for most equations of the form (1), the analogue of Lemma 1 will contain a number on the right hand side of (6) which is too small in order to infer a strong result. Equation (4) is the simplest the author knows of for which a number $> 3\frac{1}{6}$ in the right hand side of (6) can be obtained and thus a larger bound for x than 10^{10^6} results ($10^{10^{11}}$ to be precise).

A key role in our method is played by the identity:

$$(5) \quad \frac{\sum_{j=1}^{y-1} j^r}{y} \equiv \begin{cases} 0 \pmod{\frac{1}{2}} & \text{odd } r > 1; \\ -\sum_{\substack{p-1|r \\ p|y}} \frac{1}{p} \pmod{1} & r \text{ even.} \end{cases}$$

This identity goes back to Von Staudt. We leave it as an exercise for the reader acquainted with Bernoulli numbers to prove (5) by using the Von Staudt-Clausen theorem. A combinatorial proof based on the theory of finite differences was given by Carlitz [3]. A proof of a more general form of (5) using only some results on primitive roots can be found in [7]. For appropriate choices of y in terms of x , where x is a solution of $1^k + 2^k + \dots + (x - 1)^k = x^k$, analysis of (5) yields the four inequalities of Moser.

It is perhaps a tribute to Moser, that the author found (to his surprise) that he could do without Bernoulli numbers. As a demonstration we shall give a more insightful proof of the result that $\text{ord}_2(x - 3) = \text{ord}_2(k) + 3$, first established in [8] by using Bernoulli polynomials.

In Section 2 we state and prove the key inequality in the proof of Theorem 1. In Section 3 we show that the fractions in (6) containing a are small and that this implies that k must be large (Lemma 2). In Section 4 we show that the proof of Theorem 1 easily follows from (6) and Lemma 2. In Section 5 we state analogues of Theorem 1 for equations (3) and (4). Finally in Section 6 we prove some further results for the solutions of (2).

2. THE KEY LEMMA IN THE PROOF OF THEOREM 1

By way of numerical computation Theorem 1 will be deduced from the following lemma:

LEMMA 1. Suppose (a, x, k) is a solution of (2) with $k > 1$. Then

$$(6) \quad \sum_{\substack{p-1|k \\ p|(x^2-1)(4x^2-1)}} \frac{1}{p} + \frac{a}{x-1} + \frac{a+1}{x+1} + \frac{2a}{2x-1} + \frac{2(a+1)}{2x+1} \geq 3\frac{1}{6}.$$

2.1 THE PROOF OF LEMMA 1.

PROPOSITION 1. Suppose (a, x, k) is a solution of (2) with $k > 1$. Then for $b = 0$

$$(7a) \quad \frac{ax^k}{x+b} \equiv \begin{cases} 0 \pmod{\frac{1}{2}} & \text{if } k \text{ is odd;} \\ - \sum_{\substack{p-1|k \\ p|x}} \frac{1}{p} \pmod{1} & \text{otherwise.} \end{cases}$$

For $1 \leq b < x$

$$(7b) \quad \frac{ab^k - 1^k - \dots - (b-1)^k}{x-b} \equiv \begin{cases} 0 \pmod{\frac{1}{2}} & \text{if } k \text{ is odd;} \\ - \sum_{\substack{p-1|k \\ p|x-b}} \frac{1}{p} \pmod{1} & \text{otherwise.} \end{cases}$$

For $b \geq 1$

$$(7c) \quad \frac{(a+1)(-b)^k + (-1)^k [1^k + \dots + (b-1)^k]}{x+b} \equiv \begin{cases} 0 \pmod{\frac{1}{2}} & \text{if } k \text{ is odd;} \\ -\sum_{\substack{p-1|k \\ p|x+b}} \frac{1}{p} \pmod{1} & \text{otherwise.} \end{cases}$$

PROOF: (7a) Put $y = x$, $r = k$ in (5) and use (2). (7b)+(7c) For $1 \leq b < x$ we have

$$\begin{aligned} \sum_{r=0}^{x-b-1} r^k &= ax^k - [(x-1)^k + \dots + (x-b)^k] \\ &\equiv ab^k - [1^k + 2^k + \dots + (b-1)^k] \pmod{x-b}. \end{aligned}$$

Similarly for $b \geq 1$ we have

$$\begin{aligned} \sum_{r=0}^{x+b-1} r^k &= ax^k + x^k + (x+1)^k \dots + (x+b-1)^k \\ &\equiv (a+1)(-b)^k + (-1)^k [1^k + \dots + (b-1)^k] \pmod{x+b}. \end{aligned}$$

On taking $y = x-b$ (respectively $x+b$ ($b \geq 1$)) in (5), (7b) (respectively (7c)) follows. \square

Proposition 1 suggests the importance of determining the parity of k . It turns out that k must be even. In the proof we shall use the following bound for x in terms of a and k that is important in its own right:

PROPOSITION 2. *If (a, x, k) is a solution of (2), then*

$$a(k+1) < x < (a+1)(k+1).$$

PROOF: By comparing the sum $1^k + \dots + (x-1)^k$ with $\int t^k dt$. \square

Krzysztofek [6, Lemat 4] has shown by more elaborate but elegant reasoning the stronger result that $a(k+1) + 1 < x < (a+1/2)(k+1)$.

PROPOSITION 3. *If (a, x, k) is a solution of (2) with $k > 1$, then k is even.*

PROOF: Clearly $x \neq 1$, so we may assume $x \geq 2$. Suppose $k > 1$ and k is odd. Taking $b = 1$ in (7b) we see that $2a/(x-1)$ is integral and in particular that $2a \geq x-1$. But then $2a \geq (k+1)a$, by Proposition 2. This contradiction shows that k must be even. \square

In the sequel Proposition 3 is used so often that it will not be referred to, but assumed implicitly.

PROPOSITION 4.

- (i) *If the denominator and numerator in (7b) are coprime for some $1 \leq b < x$, then $x - b$ is squarefree and $p|x - b$ implies $p - 1|k$.*
- (ii) *If the denominator and numerator in (7c) are coprime for some $b \geq 1$, then $x + b$ is squarefree and $p|x + b$ implies $p - 1|k$.*

PROOF: If p divides the denominator of the left hand side, the p -order of the fraction is ≤ -1 . Since the p -order of the right hand side is at least -1 , the p -order of both sides is exactly -1 . It follows that $p - 1|k$ and that the denominator of the left hand side is squarefree. □

EXAMPLES. (i) $(a, b) = (1, 1)$ (ii) $(a, b) = (2^m, x - 1)(m \geq 0), (2^m - 1, x + 1)(m \geq 1)$.

PROPOSITION 5. *Suppose (a, x, k) is a solution of (2) with $k > 1$. Then*

- (i) $\left(\frac{1}{2}\right)^k \equiv 1 \pmod{\frac{2x-1}{\alpha}}$ for some divisor α of a .
- (ii) $\left(\frac{1}{2}\right)^k \equiv 1 \pmod{\frac{2x+1}{\beta}}$ for some divisor β of $a + 1$.

In (i) and (ii) $1/2$ denotes the inverse of 2 modulo $2x - 1$, respectively $2x + 1$.

PROOF: Take $b = x - 1$ in (7c). On invoking (2) we obtain

$$(8) \quad \frac{a[x^k + (x - 1)^k]}{2x - 1} \equiv - \sum_{\substack{p-1|k \\ p|2x-1}} \frac{1}{p} \pmod{1}.$$

Now if $p|2x - 1$, p is odd and $x \equiv 1/2 \pmod{2x - 1}$. The numerator on the left hand side of (8) equals $2a(1/2)^k$ modulo $2x - 1$. So we have

$$(9) \quad \frac{2a(1/2)^k}{2x - 1} \equiv - \sum_{\substack{p-1|k \\ p|2x-1}} \frac{1}{p} \pmod{1}.$$

(i) If $p|2x - 1$ and $p \nmid a$, then from (9) we conclude that $p - 1|k$, (so $(1/2)^k \equiv 1 \pmod{p}$) and $p^2 \nmid 2x - 1$, so we can write

$$2x - 1 = \left(\prod_{\substack{p_i|a \\ p_i|2x-1}} p_i^{\alpha_i} \right) \prod_{\substack{p_i|a \\ p_i|2x-1}} p_i.$$

Let $a = \prod_{p_i|a} p_i^{\beta_i}$ be the prime factorisation of a . If $\alpha_i > \beta_i$, then from (9) it follows that $p_i - 1|k$ (so $(1/2)^k \equiv 1 \pmod{p_i}$) and $\alpha_i = \beta_i + 1$. By the Chinese remainder

theorem we have

$$\left(\frac{1}{2}\right)^k \equiv 1 \left(\text{mod } \prod_{\substack{p_i|a \\ \alpha_i > \beta_i \\ p_i|2x-1}} p_i \prod_{\substack{p|a \\ p|2x-1}} p\right).$$

Let α be the quotient of $2x - 1$ by the modulus of the latter congruence. We have

$$\alpha = \prod_{\substack{p_i|a \\ \alpha_i > \beta_i \\ p_i|2x-1}} p_i^{\alpha_i-1} \prod_{\substack{p_i|a \\ \alpha_i \leq \beta_i \\ p_i|2x-1}} p_i^{\alpha_i}.$$

Since $\alpha_i = \beta_i + 1$ whenever $\alpha_i > \beta_i$, α is a divisor of a .

(ii) Take $b = x + 1$ in (7c). Proceeding as in the proof of (i), we obtain

$$(10) \quad \frac{2(a+1)(1/2)^k}{2x+1} \equiv - \sum_{\substack{p-1|k \\ p|2x+1}} \frac{1}{p} \pmod{1}.$$

The rest of the proof is similar to the proof of part (i). □

PROOF OF LEMMA 1: Taking $b = 1$ in (7b) and (7c) we find that

$$(11) \quad \sum_{\substack{p-1|k \\ p|x-1}} \frac{1}{p} + \frac{a}{x-1} \geq 1, \text{ respectively } \sum_{\substack{p-1|k \\ p|x+1}} \frac{1}{p} + \frac{a+1}{x+1} \geq 1.$$

On combining (9) and Proposition 5(i), and (10) and Proposition 5(ii), we find that

$$(12) \quad \sum_{\substack{p-1|k \\ p|2x-1}} \frac{1}{p} + \frac{2a}{2x-1} \geq 1, \text{ respectively } \sum_{\substack{p-1|k \\ p|2x-1}} \frac{1}{p} + \frac{2(a+1)}{2x+1} \geq 1.$$

Adding the inequalities in (11) and (12) and noting that no prime > 3 can divide more than one of the numbers $x - 1, x + 1, 2x - 1$ and $2x + 1$ and that 2 and 3 can divide at most two of these numbers, we obtain

$$\sum_{\substack{p-1|k \\ p|(x^2-1)(4x^2-1)}} \frac{1}{p} + \frac{a}{x-1} + \frac{a+1}{x+1} + \frac{2a}{2x-1} + \frac{2(a+1)}{2x+1} \geq 4 - \frac{1}{2} - \frac{1}{3} = 3\frac{1}{6}.$$

□

3. THE NONEXISTENCE OF SOLUTIONS WITH SMALL k

In this section we shall show that (2) has no solutions with k ‘small’:

LEMMA 2. *Equation (2) has no solutions with $1 < k < 10^{22}$.*

The number 10^{22} is an approximation to twice the 200th highly composite number ($= 2^8 3^5 5^3 7^2 p_5 \cdots p_{15}$, where p_i denotes the i th successive prime). Recall that Ramanujan defined a number n to be highly composite if $d(n) > d(m)$ for $n > m$, where as usual $d(n)$ denotes the divisor function $\sum_{d|n} 1$. A list of the first 5000 highly composite numbers was independently computed by te Riele and Robin. The proof of Lemma 2 will follow on applying Propositions 2 and 6 to (6).

PROPOSITION 6. *Let $C > 0$ be an arbitrary real number. Suppose p_s , the s th prime, satisfies*

$$(13) \quad \sum_{p \leq p_s} \frac{1}{p} < C.$$

Let n be a highly composite number satisfying $d(n) \leq s$. Then if r satisfies

$$(14) \quad \sum_{p-1|2r} \frac{1}{p} \geq C,$$

the inequality $r \geq n$ holds true.

PROOF: The number of primes p such that $p - 1$ divides $2r$ is $\leq d(r) + 1$, so

$$(15) \quad \sum_{p-1|2r} \frac{1}{p} \leq \sum_{i=1}^{Pd(r)+1} \frac{1}{p_i}.$$

Now if $d(r) < s$ it follows that the sum on the left hand side is $< C$, so $d(r) \geq s \geq d(n)$. Since n is highly composite it follows that $r \geq n$. □

PROOF OF LEMMA 2: Using Proposition 2 we see that the sum of the fractions appearing in the left hand side of (6) is $< 6/(k + 1)$. Therefore

$$(16) \quad \sum_{p-1|k} \frac{1}{p} + \frac{6}{k+1} > 3\frac{1}{6}.$$

For $k = 2, \dots, 998$ it follows by direct computation that this inequality is not satisfied. Since $\sum_{p \leq 10^7} 1/p < 3.16$ [9, Lemma 2], $\pi(10^7) = 664579$ and the 200th highly composite number has divisor sum 663552 (the 201th has divisor sum 688128) and exceeds 10^{22} , the proof of Lemma 2 follows from Proposition 6 with $C = 3.16$, $s = 664579$ and n the 200th highly composite number. □

4. PROOF OF THEOREM 1

Recall that the a dependent terms in (6) are bounded above by $6/(k + 1)$. So on using Lemma 1 and Lemma 2 we find

$$\sum_{p|(x^2-1)(4x^2-1)} \frac{1}{p} > 3.16.$$

Using $\sum_{p \leq 10^7} 1/p < 3.16$ it follows that $(x^2 - 1)(4x^2 - 1) > \prod_{p \leq 10^7} p$. Proceeding as in Moser [9] it follows that $x < 10^{10^6}$. For the convenience of the reader we recall his (in essence) argument.

From [12] we have

$$|\theta(x) - x| < \frac{x}{40 \log x} \quad (x \geq 678407).$$

Hence

$$\begin{aligned} \log(4x^4) &> \log(x^2 - 1)(4x^2 - 1) > \log \prod_{p \leq 10^7} p = \sum_{p \leq 10^7} \log p \\ &> \left(1 - \frac{1}{280 \log 10}\right) 10^7 > (.99) 10^7, \text{ so } x > 10^{10^6}. \end{aligned}$$

On invoking Lemma 2 in combination with Proposition 2, Theorem 1 follows. □

5. ANALOGUES OF THEOREM 1 FOR EQUATIONS (3) AND (4)

Since, in dealing with Erdős-Moser type equations, Proposition 6 has to be applied for various C , it is convenient to have estimates for p_s and n in this proposition in terms of C .

Using the estimate .

$$\sum_{p \leq x} \frac{1}{p} < \log \log x + .2615 + \frac{1}{\log^2 x} \quad \text{for } x > 1,$$

due to Rosser and Schoenfeld [11, (3.20)], we find

$$\sum_{p \leq x} \frac{1}{p} < \log \log x + .267 \text{ for } x \geq 10^6.$$

From this inequality one easily deduces:

PROPOSITION 7. *Let $C > 2.9$. Inequality (13) is satisfied for any prime p_s with*

$$(17) \quad p_s \leq e^{e^{C-2.87}}.$$

Denote the right hand side of (17) by $f(C)$. Put

$$g(C) = 0.938\{\log f(C) - \log \log f(C)\} \text{ and } h(C) = e^{g(C)\log g(C)}.$$

Using the bound

$$\frac{\log d(n)}{\log 2} \leq 1.537 \frac{\log n}{\log \log n},$$

of Nicolas and Robin [10], the bound $\pi(x) > x/\log x$ ($x > 17$) of Rosser and Schoenfeld [11, (3.5)] and Proposition 7 we obtain:

PROPOSITION 8. *Let $C > 2.9$. Suppose r satisfies (14), then $r \geq e^{g(C)\log g(C)}$.*

Examples.

$$f(2.99) \geq 4 \cdot 10^6, \quad h(2.99) \geq 10^{12}.$$

$$f(3.16) \geq 6 \cdot 10^7, \quad h(3.16) \geq 10^{16}.$$

$$f(3.66) \geq 8 \cdot 10^{12}, \quad h(3.66) \geq 10^{34}.$$

The proofs of Theorems 2 and 3 are very similar to the proof of Theorem 1 and so we give only sketches. We leave it as an exercise to the reader to complete them.

THEOREM 2. *The equation*

$$1^k + 2^k + \dots + (x-1)^k = ax^k + 1$$

has no solutions $(a, x, k) \in \mathbb{N}^3$ with $k > 1$, $x < \max(10^{10^6}, a \cdot 10^{12})$.

PROOF: (Sketch.) If x is even we have

$$\sum_{\substack{p-1|k \\ p|x^3-x}} \frac{1}{p} + \frac{a+1}{x-1} + \frac{1}{x} + \frac{a+2}{x+1} \geq 3,$$

and if x is odd, we have

$$(18) \quad \sum_{\substack{p-1|k \\ p|(x^2-2x)(x^2-1)}} \frac{1}{p} + \frac{a}{x-2} + \frac{a+1}{x-1} + \frac{1}{x} + \frac{a+2}{x+1} \geq 3\frac{1}{6}.$$

After some numerical calculation we conclude that there are no solutions with $1 < k < 10^{12} < 2h(2.99)$. If $a \leq 10^{10^6}$, then $x < (a + 3/2)(k + 1) < 2^k$ and (18) is satisfied irrespective of whether x is even or odd. Now proceed as in the proof of Theorem 1. \square

THEOREM 3. *The equation*

$$(19) \quad 1^k + 2^k + \dots + (x - 1)^k = x^k + 1 + \frac{(x - 2)(x - 1)x(x + 1)(x + 5)}{3} 2^{k-6}$$

has no solutions $(x, k) \in \mathbb{N}^2$ with $x < 10^{10^{11}}$.

PROOF: (Sketch.) There are no solutions with $x \leq 4$. There are no solutions with $k = 1, 2$ or 4 . There are no solutions with k even, so we may assume $k \geq 6$. Notice that the last term in (19) remains an integer on division by $x + i$, $i = -2, -1, 1$. This remains true if $i = 0$ and $3 \nmid x$. We have $k + 1 < x < (k + 1)3^{1/4}$. Since $(k + 1)3^{1/4} \leq 2^k \leq 3^k$, it follows from $\sum_{\substack{p-1|k, \\ p|x-3}} 1/p + 3^k/(x - 3) \equiv 0 \pmod{1}$ that $3 \nmid x$

and from $\sum_{\substack{p-1|k, \\ p|x-2}} 1/p + 2^k/(x - 2) \equiv 0 \pmod{1}$ that x is odd. We find

$$\sum_{\substack{p-1|k \\ p|(x-3)(x-2)(x^3-x)}} \frac{1}{p} + \frac{1}{x-3} + \frac{1}{x-2} + \frac{2}{x-1} + \frac{1}{x} + \frac{3}{x+1} \geq 5 - 1 - \frac{1}{3} = 3\frac{2}{3}.$$

Proceeding as in the proof of Theorem 1 we find

$$\log(x^5) > \sum_{p \leq f(3.66)} \log p > 7.99 \cdot 10^{12},$$

and so $x > 10^{10^{11}}$. \square

Theorems 1, 2 and 3 suggest the following conjecture:

CONJECTURE. Equations (2), (3) and (4) do not have integer solutions with $k > 1$.

6. FURTHER RESULTS ON (2)

In this section we derive some further results on the integer solutions of (2) and prove some additional results in the case $a = 1$ in (2), that is the Erdős-Moser equation.

PROPOSITION 9. *Suppose (a, x, k) with $k > 1$ satisfies (2). If $p|x$, then $p - 1 \nmid k$.*

PROOF: From (7a) it follows that $\sum_{\substack{p-1|k, \\ p|x}} 1/p$ is either zero or a positive integer.

Since the latter is clearly impossible, the proposition follows. \square

When $a = 1$ this result is already known [8, Lemma 10a)].

From Proposition 9 it follows that x must be odd. Using the fact that k is even, the proposition yields that $3 \nmid x$. Take $b = 1$ in (7b). Using the fact that x is odd, we see that the 2-order of the right hand side is -1 . So the 2-order of the left hand side must be -1 too, that is, $\text{ord}_2(x - 1) = \text{ord}_2(a) + 1$. Suppose a is odd. Then $x \equiv 3 \pmod{4}$. Take $b = 3$ in (7b). The left hand side is

$$\frac{a3^k - 1 - 2^k}{x - 3}.$$

Since an easy induction shows that $\text{ord}_2(3^r - 1) = \text{ord}_2(r) + 2$ for even r (alternatively one may invoke Proposition 1 of Beyl [2]), we have

$$a3^k - 1 - 2^k = a - 1 + am2^{\text{ord}_2(k)+2} - 2^k,$$

where m is odd. Since $k \geq 6$, by Lemma 2, $\text{ord}_2(k) + 3 < k$. We conclude:

PROPOSITION 10. *Suppose (a, x, k) satisfies (2), a is odd and $k > 1$. Then*

$$\text{ord}_2(x - 3) = \begin{cases} \text{ord}_2(a - 1) + 1 & \text{if } \text{ord}_2(a - 1) < \text{ord}_2(k) + 2; \\ \text{ord}_2(k) + 3 & \text{if } \text{ord}_2(a - 1) > \text{ord}_2(k) + 2; \\ \geq \text{ord}_2(k) + 4 & \text{otherwise.} \end{cases}$$

In the case $a = 1$ we find $\text{ord}_2(x - 3) = \text{ord}_2(k) + 3$, which is Lemma 12 of [8]. In the case $a \not\equiv 1 \pmod{8}$, we have $\text{ord}_2(x - 3) = 3$ if $a \equiv 5 \pmod{8}$ and 2 otherwise.

For the Erdős-Moser equation the following new result can be proved by comparing orders of the left hand side and right hand side of (7b) with $b = 2$:

PROPOSITION 11. *Suppose (x, k) is a solution of the Erdős-Moser equation with $k > 1$. Let p be a prime divisor of $x - 2$. If 2 is a primitive root modulo p , then $p - 1 \mid k$ and*

$$\text{ord}_p(x - 2) = \text{ord}_p(2^{p-1} - 1) + \text{ord}_p(k) + 1 \geq 2.$$

The following argument works for equations (3) and (4). Take a fixed. By reasoning as in the proof of Proposition 2, we see that there exist positive constants c_1 and c_2 such that $c_1 \leq k/x \leq c_2$. In this case it follows from [4, Théorème 2] that

$$\sum_{r=1}^{x-1} r^k = \frac{(x-1)^k}{1 - e^{-((k+1)/(x-1))}} \left(1 + O\left(\frac{1}{\sqrt{k}}\right) \right) \quad (k \rightarrow \infty).$$

From this it is easily deduced that the solutions of equations (3) and (4) satisfy

$$\lim_{x \rightarrow \infty} \frac{k}{x} = \log\left(1 + \frac{1}{a}\right),$$

that is, when there are infinitely many solutions, the ratio of k and x tends to $\log(1 + 1/a)$. When there are only finitely many, since we know that k and x have to be very large the ratio of k to x will be very close to $\log(1 + 1/a)$. For $a = 1$ this was proved using continued fractions by Best and te Riele [1].

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