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ABSTRACT

In this paper we study the subgroup of the Picard group of Voevodsky’s category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(k; \mathbb{Z}/2)$ generated by the reduced motives of affine quadrics. Our main tools here are the functors of Bachmann [*On the invertibility of motives of affine quadrics*, Doc. Math. **22** (2017), 363–395], but we also provide an alternative method. We show that the group in question can be described in terms of indecomposable direct summands in the motives of projective quadrics over k . In particular, we describe all the relations among the reduced motives of affine quadrics. We also extend the criterion of motivic equivalence of projective quadrics.

1. Introduction

The study of the Picard group of the motivic category in the algebro-geometric context was initiated by Hu in [Hu05], who considered the case of the \mathbb{A}^1 -stable homotopy category of Morel and Voevodsky. It was established there that the reduced classes of affine Pfister quadrics $\{\langle\langle a_1, \dots, a_r \rangle\rangle = b\}$ of small foldness represent invertible objects in $\mathcal{SH}(k)$, and some relations among these classes in $\mathrm{Pic}(\mathcal{SH}(k))$ were found. It was conjectured that the same should hold for arbitrary r .

The topic was picked up by Bachmann in [Bac17]. Here, instead of the \mathbb{A}^1 -stable homotopic category $\mathcal{SH}(k)$, Voevodsky’s category of motives $\mathrm{DM}(k; \mathbb{Z}/2)$ was considered. This simplified the task somewhat. As a result, not only the conjectures of Po Hu were proven in this context, but it was shown that the reduced motive $\widetilde{M}(A_q)$ of any affine quadric $A_q = \{q = 1\}$ is invertible in $\mathrm{DM}(k; \mathbb{Z}/2)$. This was established with the help of functors Φ^E of Bachmann. These tensor triangulated functors, defined for every finitely generated field extension E/k , map the tensor triangulated category $\mathrm{DQM}^{\mathrm{sm}}$ generated by the motives of smooth projective quadrics to the category $K^b(\mathrm{Tate}(\mathbb{Z}/2))$ of bigraded $\mathbb{Z}/2$ -vector spaces. They are characterized by the property that $\Phi^E(T(i)[j]) = T(i)[j]$ (the one-dimensional vector space of the specified bidegree), where T is the monoidal unit, while $\Phi^E(M(Q)) = 0$, for every projective quadric Q/k which stays anisotropic over E . It was shown in [Bac17] that the collection of these functors (for all finitely generated E/k) is conservative, detects invertible objects, and is injective on the Pic.

In [BV18] it was proven that the map $q \mapsto \widetilde{M}(A_q)$ defines an embedding of sets $\mathrm{GW}(k) \hookrightarrow \mathrm{Pic}(\mathrm{DM}(k; \mathbb{Z}/2))$ of the Grothendieck–Witt ring of quadratic forms (or, by Morel’s result [Mor04], of $\pi_{(0)[0]}^s(\mathbb{S})$) into the Picard group of the motivic category.

In the current paper, we study the relations among these elements in Pic. Or, equivalently, the subgroup $\mathrm{Pic}_{\mathrm{qua}}$ of $\mathrm{Pic}(\mathrm{DM}(k; \mathbb{Z}/2))$ generated by the reduced motives of affine quadrics.

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It appears that these can be described in terms of motives of projective quadrics and the direct sum operation. Inside Pic_{qua} there is a subgroup $\mathbb{T} \cong \mathbb{Z} \oplus \mathbb{Z}$ consisting of Tate motives $T(i)[j]$. It is enough to describe $\text{Pic}_{\text{qua}}/\mathbb{T}$. This group is generated by our (shifted) reduced motives $e^q := \widetilde{M}(A_q)[1]$ of affine quadrics. First of all, in Proposition 2.1, we complement Bachmann’s invertibility result by observing that our set of generators is closed under inverses, $(e^q)^{-1} = e^{q'}$ in $\text{Pic}_{\text{qua}}/\mathbb{T}$, where $q' = \langle 1 \rangle \perp -q$ (note that the operation $q \mapsto q'$ is a ‘square root’ of $q \mapsto q \perp \mathbb{H}$). In particular, this gives that $e^{\langle\langle \alpha \rangle\rangle}$ is the inverse of the reduced Rost motive \widetilde{M}_α (here α is a pure symbol in $K_*^M(k)/2$). In Theorem 3.1 we provide a large supply of linearly independent elements in Pic_{qua} . Namely, the collection $\{e^{q_i}\}_i$ will be linearly independent as long as all the projective quadrics Q'_i are anisotropic and pairwise not stably birationally equivalent. Moreover, it is shown in Proposition 3.17 that if Question 3.16 has positive answer, then a maximal such collection will form a \mathbb{Z} -basis of $\text{Pic}_{\text{qua}}/\mathbb{T}$ (note that the fact that this group is torsion free follows from [Bac17]). Every smooth projective quadric Q can be cut into affine ones (using some flag of plane sections). Multiplying the respective elements e^q , we get the new element $\det(Q)$. From some basic relations among the e^q it follows that this does not depend on the choice of a flag, and is an invariant of Q , and even of the motive of Q . The set $\{\det(Q)\}_Q$, where Q runs over all smooth projective quadrics over k , provides another set of generators of Pic_{qua} . In Theorem 3.12 we establish all the relations among these elements in $\text{Pic}_{\text{qua}}/\mathbb{T}$. Namely, $\prod_i \det(P_i) = \prod_j \det(Q_j) \in \text{Pic}_{\text{qua}}/\mathbb{T}$ if and only if $\bigoplus_i M(P_i)$ and $\bigoplus_j M(Q_j)$ are Tate equivalent, that is, if we ignore the Tate summands in both, then the respective indecomposable (anisotropic) direct summands of both sides can be identified up to Tate shift. This embeds Pic_{qua} into the free abelian group with the basis consisting of indecomposable direct summands in the motives of k -quadrics considered up to Tate shift. What is remarkable here is that the question about Voevodsky’s triangulated motives and the tensor product operation is reduced to the one about classical Chow motives and the direct sum operation. As a small by-product we can complement the classical criterion of motivic equivalence of projective quadrics ([Vis98, Vis04]; see also [Kar00]) with the equality $\det(P) = \det(Q) \in \text{Pic}_{\text{qua}}$. Thus, in Pic_{qua} we have two generating subsets: one identified with the isomorphism classes of quadratic forms, another with the isomorphism classes of motives of projective quadrics.

All the above results are obtained with the help of Bachmann’s functors Φ^E which provide a very effective tool for comparing elements of Pic_{qua} . In § 4 we introduce an alternative method which allows us to perform the same calculations. Here we use the Čech simplicial schemes and some ideas from [Vis98]. The idea is very simple: for a smooth projective P , the motive \mathcal{X}_P of the Čech simplicial scheme is an idempotent, $\mathcal{X}_P^{\otimes 2} \cong \mathcal{X}_P$, and so is its ‘complement’, $\widetilde{\mathcal{X}}_P = \text{Cone}(\mathcal{X}_P \rightarrow T)$. As a result, we get two orthogonal projections $\otimes \mathcal{X}_P$ and $\otimes \widetilde{\mathcal{X}}_P$ on $\text{DM}(k)$ which define a semi-orthogonal decomposition of this category. For different varieties, these projectors naturally commute, and we can consider a poly-semi-orthogonal decomposition corresponding to a finite collection $\mathbf{X} = \{X_i\}_{i \in I}$ of smooth projective varieties. The resulting functor is obviously conservative and so detects invertible geometric objects. It is shown in Proposition 4.3 that it is also injective on the Pic. It follows from the results of [Vis98] that, for any object A of DQM^{gm} , there is an appropriate collection \mathbf{X} , for which all the projections of A will be extensions of Tate motives. And if A represents an element of Pic_{qua} , then there is a collection where all the projections are the Tate motives $T(i)[j]$. In particular, we re-prove Bachmann’s result on the invertibility of the reduced motives of affine quadrics (see Proposition 4.6). Moreover, two elements of Pic_{qua} are equal if and only if the respective functions $(i)[j]$ on the set of (non-trivial) projectors are the same. This creates an environment which permits us to substitute the functors

of Bachmann in the study of Pic_{qua} . The new approach is not restricted to the subcategory DQM^{gm} only, but allows us to study the whole Pic of $\text{DM}(k)$. We will address this question in a sequel to this paper.

2. Motives of affine quadrics

2.1 Notations and some basic facts

Let k be a field of characteristic different from 2, and q be a (non-degenerate) quadratic form of dimension n over k . We denote by A_q the affine quadric $\{q = 1\}$. Then A_q can be considered as a (not necessarily split) sphere. In particular, over \bar{k} , the motive $M(A_q)_{\bar{k}} = T \oplus T([n/2])[n - 1]$ is a sum of just two Tate motives. This motive is a complete invariant of q (see [BV18, Theorem 2.1]). We have a natural projection $A_q \rightarrow \text{Spec}(k)$, and it was shown in [BV18] that the reduced motive $\widetilde{M}(A_q) = \text{Cone}[-1](M(A_q) \rightarrow T)$ of A_q determines q as well. This reduced motive is a form of a Tate motive, as over \bar{k} it becomes isomorphic to $T([n/2])[n - 1]$. It belongs to the category $\text{DM}_{\text{gm}}(k; \mathbb{Z}/2)$ of geometric motives of Voevodsky (see [Voe00]). Moreover, it was shown by Bachmann in [Bac17] that this motive is invertible there, that is, it represents an element of $\text{Pic}(\text{DM}_{\text{gm}}(k; \mathbb{Z}/2))$. And for $p = q \perp \mathbb{H}$, one has $\widetilde{M}(P) \cong \widetilde{M}(Q)(1)[2]$ (see [Bac17, Lemma 34]). Hence, we get an embedding

$$\begin{aligned} GW(k) &\hookrightarrow \text{Pic}(\text{DM}_{\text{gm}}(k; \mathbb{Z}/2)) \\ q - r\mathbb{H} &\mapsto \widetilde{M}(A_q)(-r)[-2r + 1] \end{aligned}$$

of sets of the Grothendieck–Witt ring of quadratic forms into the Picard of the category of geometric motives. In other words, we get a complete invariant of the $(0)[0]$ -stable \mathbb{A}^1 -homotopy group of spheres (as, by the result of Morel [Mor04, Theorem 6.2.1] this group coincides with the $GW(k)$). The corresponding map in topology is the map $\mathbf{Z} = \pi_0^s(\mathbb{S}) \rightarrow \text{Pic}(D(\mathbb{A}b))$ sending n to $T[n]$, which happens to be an isomorphism. The aim of the current paper is to study the motivic variant of such a map.

We start by introducing some notation. Let us denote by $e^q \in \text{Pic}(\text{DM}_{\text{gm}}(k; \mathbb{Z}/2))$ the shifted reduced motive $\widetilde{M}(A_q)[1]$ and by Pic_{qua} the subgroup of $\text{Pic}(\text{DM}_{\text{gm}}(k; \mathbb{Z}/2))$ generated by e^q , for all quadratic forms q/k .

For a quadratic form q , let q' be the quadratic form $\langle 1 \rangle \perp -q$, and Q, Q' be the respective smooth projective quadrics. Then $A_q = Q' \setminus Q$, and we have the Gysin triangle

$$M(Q') \rightarrow M(Q)(1)[2] \rightarrow M(A_q)[1] \rightarrow M(Q')[1]. \tag{1}$$

When both quadratic forms q and q' are split, $M(Q')$ and $M(Q)$ are sums of (pure) Tate motives, which implies that $M(A_q) = T \oplus T([n/2])[n - 1]$ and $e^q = T([n/2])[n]$ (one can also see this from [Bac17, Lemma 34]). In particular, Pic_{qua} contains a subgroup $\mathbb{T} \cong \mathbb{Z} \times \mathbb{Z}$ consisting of Tate motives $T(i)[j], i, j \in \mathbb{Z}$, which coincides with the entire Pic_{qua} for an algebraically closed field (since all quadrics are split there). Restriction to \bar{k} , together with the projection, provides an isomorphism $\text{Pic}_{\text{qua}} \xrightarrow{\cong} \mathbb{T} \times (\text{Pic}_{\text{qua}}/\mathbb{T})$. Thus, the description of Pic_{qua} is reduced to that of $\text{Pic}_{\text{qua}}/\mathbb{T}$, which amounts to describing the relations among the e^q there.

First, we will describe the inverse of e^q .

PROPOSITION 2.1. *Let $q' = \langle 1 \rangle \perp -q$. Then in $\text{Pic}_{\text{qua}}/\mathbb{T}$,*

$$(e^q)^{-1} = e^{q'}.$$

Proof. Let $q'' = \langle 1 \rangle \perp -q' = \mathbb{H} \perp q$ and $\dim(q) = n$. Considering the $\text{Cone}[-1]$ of the map of triangles

$$\begin{array}{ccccccc} M(A_q) & \longrightarrow & M(Q') & \longrightarrow & M(Q)(1)[2] & \longrightarrow & M(A_q)[1] \\ \downarrow & & \downarrow & & \downarrow 0 & & \downarrow \\ T & \longrightarrow & 0 & \longrightarrow & T[1] & \xlongequal{\quad} & T[1] \end{array}$$

where $M(A_q) \rightarrow T$ is the standard projection, we obtain a distinguished triangle

$$\widetilde{M}(A_q) \longrightarrow M(Q') \xrightarrow{(a,b)} M(Q)(1)[2] \oplus T \xrightarrow{(c,d)} \widetilde{M}(A_q)[1],$$

where c is the unique lifting of the map $M(Q)(1)[2] \rightarrow M(A_q)[1]$ from (1), d is the canonical map from the definition of $\widetilde{M}(A_q)$, a is a map from (1), and from the diagram chase one can see that the standard projection $M(Q') \rightarrow T$ factors through b , which means that these two maps coincide. The same applies to the pair $Q' \subset Q''$. We get exact triangles (after shifting):

$$\begin{aligned} M(Q') &\rightarrow M(Q)(1)[2] \oplus T \rightarrow \widetilde{M}(A_q)[1] \rightarrow M(Q')[1], \\ M(Q'') &\rightarrow M(Q')(1)[2] \oplus T \rightarrow \widetilde{M}(A_{q'})[1] \rightarrow M(Q'')[1]. \end{aligned}$$

Since $q'' = \mathbb{H} \perp q$, Q'' is isotropic, and Q can be identified with the quadric of lines l on Q'' passing through a fixed rational point p . This gives the decomposition $M(Q'') = T \oplus M(Q)(1)[2] \oplus T(n)[2n]$, where the map $M(Q)(1)[2] \rightarrow M(Q'')$ is given by the cycle $A = \{(l, x) | x \in l\} \subset Q \times Q''$. The map $M(Q'') \rightarrow M(Q')(1)[2]$ from the Gysin triangle is dual to the embedding $M(Q') \rightarrow M(Q'')$ and given by the cycle $B = \Delta_{Q''} \cap (Q' \times Q'')$. Taking p outside $Q' \subset Q''$, we obtain that the composition $M(Q)(1)[2] \rightarrow M(Q'') \rightarrow M(Q')(1)[2]$ is given by the cycle $C = \Delta_{Q'} \cap (Q \times Q')$, where the embedding $Q \subset Q'$ is given by the choice of p . In other words, this composition is dual to the embedding $M(Q) \rightarrow M(Q')$. The map $T(n)[2n] \rightarrow M(Q'')$ is given by the generic cycle of Q'' , so the composition $T(n)[2n] \rightarrow M(Q'') \rightarrow M(Q')(1)[2]$ is given by the generic cycle of Q' and, hence, is dual to the projection $M(Q') \rightarrow T$. Thus, the resulting map $M(Q)(1)[2] \oplus T(n)[2n] \rightarrow M(Q')(1)[2]$ is dual to the map $M(Q') \rightarrow M(Q)(1)[2] \oplus T$, and we obtain that $e^{q'} = \underline{\text{Hom}}(e^q, T(n)[2n+1])$, where $\underline{\text{Hom}}(-, -)$ is the internal Hom in $\text{DM}_{\text{gm}}(k; \mathbb{Z}/2)$. By [Bac17, Theorem 33] (see also Proposition 4.6 below), e^q is an invertible object. It follows from the standard properties of duality that the dual of an invertible object is its inverse. Thus,

$$e^q \cdot e^{q'} = T(n)[2n+1] \in \text{Pic}_{\text{qua}}. \quad \square$$

Example 2.2. Let $\alpha \in K_*^M(k)/2$ be some pure symbol, and $\langle\langle \alpha \rangle\rangle$ be the respective Pfister form. Then $\langle\langle \alpha \rangle\rangle = q_\alpha = \langle 1 \rangle \perp -\tilde{q}_\alpha$, and the motive of the affine quadric $Q_\alpha \setminus \tilde{Q}_\alpha$ is the Rost motive M_α [Ros98]. Hence, in $\text{Pic}_{\text{qua}}/\mathbb{T}$, $e^{\langle\langle \alpha \rangle\rangle} = (e^{\tilde{q}_\alpha})^{-1}$ is the inverse of the reduced Rost motive $\widetilde{M}_\alpha = \text{Cone}[-1](M_\alpha \rightarrow T)$.

2.2 Bachmann’s functors

In [Bac17] Bachmann considers DQM^{gm} , the thick tensor triangulated subcategory of $\text{DM}_{\text{gm}}(k; \mathbb{Z}/2)$ generated by motives of smooth projective quadrics over k . Then, for any field extension E/k , he constructs a tensor triangulated functor

$$\Phi^E : \text{DQM}^{\text{gm}} \longrightarrow K^b(\text{Tate}(\mathbb{Z}/2)),$$

where $K^b(\text{Tate}(\mathbb{Z}/2))$ is the category of finite-dimensional bigraded $\mathbb{Z}/2$ -vector spaces (which we can view as direct sums of Tate motives $T(i)[j]$). This functor is essentially defined by the following two properties:

- (1) $\Phi^E(T(i)[j]) = T(i)[j]$;
- (2) if smooth projective quadric Q_E is anisotropic, then $\Phi^E(Q) = 0$.

The main result of [Bac17] is the following theorem.

THEOREM 2.3 (Bachmann [Bac17, Theorem 31]). *The collection of functors $\{\Phi^E\}$ for all finitely generated extensions E/k is conservative, and it is injective on the Picard.*

Since $\text{Pic}(K^b(\text{Tate}(\mathbb{Z}/2))) = \mathbb{Z} \times \mathbb{Z}$, in particular, this implies the following proposition.

PROPOSITION 2.4 (Bachmann [Bac17, Corollary 32]). *The group Pic_{qua} has no torsion.*

The above functors of Bachmann will be the main tool in our calculations.

3. Structure of Pic_{qua}

3.1 Linearly independent elements

We will identify the elements of $\text{Pic}(K^b(\text{Tate}(\mathbb{Z}/2)))$ with the Tate motives $T(i)[j]$ (with identification given by Φ^E). It follows from Bachmann’s results [Bac17] that, for any quadratic form q/k and any extension E/k , the value of Φ on e^q is a single Tate motive $T(f(q, E))[g(q, E)]$. Thus, we get two functions: $(q, E) \mapsto f(q, E), g(q, E) \in \mathbb{Z}$. We can identify targets of various Φ^E , and since $\Phi^E(e^q)$ is invertible, we can consider expressions like $(\Phi^E/\Phi^F)(e^q)$ which is still a single Tate motive.

We would like to describe the group $\text{Pic}_{\text{qua}}/\mathbb{T}$. Since Φ^E maps \mathbb{T} isomorphically to $\text{Pic}(K^b(\text{Tate}(\mathbb{Z}/2)))$, this quotient-group is still torsion-free. The following theorem provides a large supply of linearly independent elements there.

THEOREM 3.1. *Let $\{q_i\}_{i \in I}$ be a collection of quadratic forms over k , such that q'_i is anisotropic, for all i , and for $i \neq j$, the forms q'_i and q'_j are not stably birationally equivalent. Then the collection of elements $\{e^{q_i}\}_{i \in I}$ is linearly independent in $\text{Pic}_{\text{qua}}/\mathbb{T}$.*

Proof. Recall that two quadrics P and Q are stably birationally equivalent if and only if there are rational maps $P \dashrightarrow Q$ and $Q \dashrightarrow P$ (see [Lam05, Theorem X.4.25]). Suppose we have some linear relation in Pic_{qua} ,

$$\prod_i (e^{q_i})^{m_i} = T(*)[*'].$$

Consider a directed graph whose vertices are q'_i (for q_i appearing in the above equation), and where we have an arrow $q'_i \rightarrow q'_j$ if and only if there exists a rational map $Q'_i \dashrightarrow Q'_j$ (note that this condition just means that $Q'_j|_{k(Q'_i)}$ has a rational point, or in other words, that $i_W(q'_j|_{k(Q'_i)}) > 0$). Since this property is transitive (by the valuative criterion of properness [Har77, Theorem II.4.7]), and all our forms q'_i are pairwise not stably birationally equivalent, we obtain that our graph has no oriented cycles. Hence, there is (at least) one final vertex q'_l . Consider $F_l = k(Q'_l)$. Then $i_W(q'_j|_{F_l}) = 0$, for $j \neq l$ (as the vertex is final), while $i_W(q'_l|_{F_l}) \neq 0$ (because any quadric is isotropic over its own function field). Since the forms q'_j , and so q_j , for $j \neq l$, stay anisotropic over F_l , it follows that Φ^{F_l} acts in the same way on e^{q_j} as Φ^k . At the same time, since q'_l is anisotropic over k and isotropic over F_l , these functors act differently on e^{q_l} (see [BV18, § 3] or Proposition 3.3 below). As a result, we obtain that

$$\frac{\Phi^{F_l}}{\Phi^k} \left(\prod_i (e^{q_i})^{m_i} \right) = T(x \cdot m_l)[y \cdot m_l] \quad \text{where } T(x)[y] \neq T.$$

And this must be equal to T , since all Φ^E act the same way on Tate motives. Hence, $m_l = 0$, and we managed to exclude one term from our relation. Then we argue by induction. \square

COROLLARY 3.2. *Let $\{\langle\langle\alpha\rangle\rangle\}_\alpha$ be the collection of all Pfister forms (of various foldness) for all non-zero pure symbols $\alpha \in K_*^M(k)/2$. Then the collection $\{e^{\langle\langle\alpha\rangle\rangle}\}_\alpha$ is linearly independent in $\text{Pic}_{\text{qua}}/\mathbb{T}$.*

Proof. Indeed, let $\langle\langle\alpha\rangle\rangle = q_\alpha = \langle 1 \rangle \perp -\tilde{q}_\alpha$, where \tilde{q}_α is a pure part of a Pfister form. Then the collection $\{\tilde{q}_\alpha\}_\alpha$ satisfies the conditions of Theorem 3.1, since different Pfister forms are not stably birationally equivalent (because the existence of a rational map $Q_\alpha \dashrightarrow Q_\beta$ means that $Q_\beta|_{k(Q_\alpha)}$ is isotropic and hence hyperbolic (being a Pfister form), which implies that q_β is divisible by q_α ; see [EKM08, Corollary 23.6]). Finally, by Proposition 2.1, $e^{\langle\langle\alpha\rangle\rangle} = (e^{\tilde{q}_\alpha})^{-1}$. \square

3.2 The new generators

Let $Q \supset P$ be a codimension-one embedding of smooth projective quadrics. We will use the notation $e^{Q \setminus P}$ for the (shifted) reduced motive of the affine quadric $Q \setminus P$.

Let us explicitly describe the value of the functor Φ^E on $e^{Q \setminus P}$ in terms of the Witt indices of both projective quadrics (cf. [BV18, §3]). Below we will use the additive notation $(x)[y]$ for the elements of the abelian group \mathbb{Z}^2 .

PROPOSITION 3.3. *Let $P' \supset P$ be a codimension-one embedding of smooth projective quadrics, $\dim(P') = m'$, $\dim(P) = m$ (of course, $m' = m + 1$), E/k be some field extension and $j_{P'} = i_W(P'_E)$, $j_P = i_W(P_E)$ be the Witt indices of P' and P over E . Then $\Phi^E(e^{P' \setminus P}) = T(x)[y]$, where $(x)[y] = (f(P') - f(P))[g(P') - g(P)]$, for some functions f and g . More precisely,*

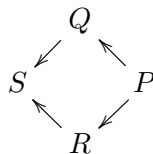
$$(x)[y] = \sum_{l'=0}^{j_{P'}-1} (m' - 2l')[2m' - 4l' + 1] - \sum_{l=0}^{j_P-1} (m - 2l)[2m - 4l + 1].$$

Proof. We have an exact triangle

$$M(P') \rightarrow M(P)(1)[2] \oplus T \rightarrow \widetilde{M}(P' \setminus P)[1] \rightarrow M(P')[1].$$

Our Witt indices are related as follows: $j_P \leq j_{P'} \leq j_P + 1$ (since $p \subset p' \subset p \perp \mathbb{H}$). From the defining property of Bachmann’s functors (as well as from [Bac17]) we see that $\Phi^E(e^{P' \setminus P})$ will be a single Tate motive $T(x)[y]$ whose grading depends only on the above Witt indices. It remains to determine the exact shape of such a dependence. If $(j_{P'}, j_P) = (l, l)$, then the ‘non-cancelled’ Tate motive is on the P -side and $(x)[y] = (l)[2l]$, while if $(j_{P'}, j_P) = (l + 1, l)$, then the ‘non-cancelled’ Tate motive is on the P' -side and $(x)[y] = (m' - l)[2m' - 2l + 1]$. Consider $(x)[y]$ as a function of (l', l) (for $l \leq l' \leq l + 1$) defined by these formulas. We can move from the pair $(0, 0)$ to $(j_{P'}, j_P)$ in $j_{P'} + j_P$ steps: $(0, 0) \rightarrow (1, 0) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow \dots$. When we move $(l', l) \rightarrow (l' + 1, l')$, $(x)[y]$ jumps up by $(m' - 2l')[2m' - 4l' + 1]$. When we move $(l + 1, l) \rightarrow (l + 1, l + 1)$, $(x)[y]$ jumps down by $(m - 2l)[2m - 4l + 1]$. Finally, for $(l', l) = (0, 0)$, $(x)[y] = (0)[0]$. Hence the formula. \square

PROPOSITION 3.4. *Suppose we have codimension-one embeddings*



of smooth projective quadrics. Then in Pic_{qua} ,

$$e^{S \setminus Q} \cdot e^{Q \setminus P} = e^{S \setminus R} \cdot e^{R \setminus P}.$$

Proof. Let E/k be some field extension, and j_S, j_R, j_Q, j_P be the Witt indices of our quadrics over E . Then, by Proposition 3.3, we have that both $\Phi^E(e^{S \setminus Q} \cdot e^{Q \setminus P})$ and $\Phi^E(e^{S \setminus R} \cdot e^{R \setminus P})$ are isomorphic to $T(x)[y]$, where

$$(x)[y] = \sum_{i=0}^{j_S-1} (\dim(S) - 2i)[2 \dim(S) - 4i + 1] - \sum_{l=0}^{j_P-1} (\dim(P) - 2l)[2 \dim(P) - 4l + 1].$$

By Theorem 2.3, $e^{S \setminus Q} \cdot e^{Q \setminus P} = e^{S \setminus R} \cdot e^{R \setminus P}$ in Pic_{qua} . □

The above relations among e^q allow us to introduce new generators of Pic_{qua} .

DEFINITION 3.5. Let Q be an m -dimensional smooth projective quadric with the complete flag of subquadrics $Q = Q_m \supset Q_{m-1} \supset \dots \supset Q_0$. Define

$$\det(Q) := e^{Q_m \setminus Q_{m-1}} \cdot e^{Q_{m-1} \setminus Q_{m-2}} \cdot \dots \cdot e^{Q_0} \in \text{Pic}_{\text{qua}}.$$

One can express the shifted reduced motive $e^{Q \setminus P}$ of an affine quadric $Q \setminus P$ as $\det(Q)/\det(P)$. Thus, determinants of smooth projective quadrics form another system of generators of Pic_{qua} . It follows from Proposition 3.4 that $\det(Q)$ does not depend on the choice of a complete flag in Q and is an invariant of Q . Moreover, it actually depends on $M(Q)$ only.

PROPOSITION 3.6. *Let Q be a smooth projective quadric of dimension m . Then,*

(1) *for any E/k , $\Phi^E(\det(Q)) = T(x)[y]$, where*

$$(x)[y] = \sum_{i=0}^{i_W(Q_E)-1} (m - 2i)[2m - 4i + 1];$$

(2) *$\det(Q)$ depends on $M(Q)$ only.*

Proof. (1) Follows straight from the Definition 3.5 and Proposition 3.3.

(2) It follows from part (1) that $\Phi^E(\det(Q))$ depends only on m and $i_W(Q_E)$. By the criterion of motivic equivalence of projective quadrics ([Vis98, Proposition 5.1] or [Vis04, Theorem 4.18]; see also [Kar00]), the motives $M(P)$ and $M(Q)$ of two smooth projective quadrics are isomorphic if and only if $\dim(P) = \dim(Q)$ and $i_W(P_E) = i_W(Q_E)$, for any field extension E/k . Thus, $\Phi^E(\det(Q))$ depends only on $M(Q)$, and by Theorem 2.3, so does $\det(Q)$ itself. □

Let N be a direct summand of the (possibly shifted) motive $M(Q)(i)[2i]$ of an anisotropic quadric. Then, over \bar{k} , it splits into a direct sum of pure Tate motives $T(l)[2l]$. These Tate motives are of two kinds: *lower* and *upper*. The lower ones are characterized by the property that the splitting map $N \rightarrow T(l)[2l]$ is defined already over the ground field k , while for the upper ones the splitting map $T(l)[2l] \rightarrow N$ is defined over k (see [Vis11, Appendix] for details). Similarly, for an extension E/k , we denote by $\text{Tate}^{\text{up}}(N_E)$ the collection of upper Tate motives splitting from N_E , and by $\text{Tate}_{\text{lo}}(N_E)$ the collection of lower Tate motives splitting from N over E . Let us define certain auxiliary elements of $\text{Pic}(K^b(\text{Tate}(\mathbb{Z}/2)))$ we will use.

DEFINITION 3.7. Let N be a direct summand of $M(Q)(i)[2i]$ with Q anisotropic. Define

$$\frac{\Phi^E}{\Phi^k}(\det(N)) := \left(\bigotimes_{T(l)[2l] \in \text{Tate}^{\text{up}}(N_E)} T(l)[2l]\right) \otimes \left(\bigotimes_{T(l)[2l] \in \text{Tate}_{\text{lo}}(N_E)} T(l)[2l-1]\right)^{-1}.$$

If N is a direct summand of the motive of a possibly isotropic quadric Q , then we can define $(\Phi^E/\Phi^k)(\det(N))$ by the same formula, where we ignore those Tate summands of N_E which already split over k . Note that the number of upper Tate motives splitting from N_E is equal to the number of the lower Tate motives splitting there (by [Vis04, Theorem 4.19]), for any anisotropic N . Hence, the Tate shift of N does not affect the result:

$$\frac{\Phi^E}{\Phi^k}(\det(N(i)[2i])) = \frac{\Phi^E}{\Phi^k}(\det(N)).$$

Our elements behave multiplicatively with respect to the direct sum of motives:

$$\frac{\Phi^E}{\Phi^k}(\det(N_1 \oplus N_2)) = \frac{\Phi^E}{\Phi^k}(\det(N_1)) \otimes \frac{\Phi^E}{\Phi^k}(\det(N_2)).$$

Consequently, it can be extended to arbitrary direct sums of direct summands as above.

In the case of $N = M(Q)$ with $\dim(Q) = m$, we have the decomposition over E :

$$M(Q_E) = \bigoplus_{i=0}^{i_W(Q_E)-1} (T_{\text{lo}}(i)[2i] \oplus T^{\text{up}}(m-i)[2m-2i]) \oplus M((Q_E)_{\text{anis}})(i_W(Q_E))[2i_W(Q_E)].$$

From Proposition 3.6(1) we get

$$\frac{\Phi^E}{\Phi^k}(\det(M(Q))) = \frac{\Phi^E}{\Phi^k}(\det(Q)).$$

This explains the notation. *A priori*, it is unclear if the element $(\Phi^E/\Phi^k)(\det(N))$ comes from some element ‘ $\det(N)$ ’ in Pic_{qua} (so the notation is somewhat misleading). But, in certain cases, we can produce $\det(N) \in \text{Pic}_{\text{qua}}$.

Example 3.8. Let $\alpha \in K_r^M(k)/2$ be a pure symbol, and $\langle\langle \alpha \rangle\rangle$ be the respective Pfister form. Then $\langle\langle \alpha \rangle\rangle = q_\alpha = \langle 1 \rangle \perp -\tilde{q}_\alpha$, the Gysin triangle

$$M(Q_\alpha) \rightarrow M(\tilde{Q}_\alpha)(1)[2] \rightarrow M(Q_\alpha \setminus \tilde{Q}_\alpha)[1] \rightarrow M(Q_\alpha)[1]$$

is split, and the motive $M(Q_\alpha \setminus \tilde{Q}_\alpha)$ is the Rost motive M_α . Also, $e^{Q_\alpha \setminus \tilde{Q}_\alpha} = \det(Q_\alpha)/\det(\tilde{Q}_\alpha)$. Since $M(Q_\alpha) = M(\tilde{Q}_\alpha)(1)[2] \oplus M_\alpha$, we obtain that

$$\frac{\Phi^E}{\Phi^k}(\det(Q_\alpha)/\det(\tilde{Q}_\alpha)) = \frac{\Phi^E}{\Phi^k}(\det(M_\alpha)).$$

So, we can define $\det(M_\alpha) \in \text{Pic}_{\text{qua}}$ as $\det(Q_\alpha)/\det(\tilde{Q}_\alpha) = e^{Q_\alpha \setminus \tilde{Q}_\alpha}$. It is nothing more than the shifted reduced Rost motive $\tilde{M}_\alpha[1]$. By Rost’s result [Ros90, Ros98],

$$M(Q_\alpha) = \bigoplus_{i=0}^{2^{r-1}-1} M_\alpha(i)[2i].$$

Due to the multiplicativity property of the $(\Phi^E/\Phi^k)(\det(-))$ and Bachmann’s injectivity result, Theorem 2.3 (comparing also Φ^k of both parts), we obtain

$$\det(Q_\alpha) = (e^{Q_\alpha \setminus \tilde{Q}_\alpha})^{2^{r-1}}.$$

In particular, by Proposition 2.1, in $\text{Pic}_{\text{qua}}/\mathbb{T}$ we have an identity,

$$\det(Q_\alpha) = (e^{\langle\langle\alpha\rangle\rangle})^{-2^{r-1}}.$$

Consider the full additive subcategory $\text{Chow}_{\text{qua}}(k, \mathbb{Z}/2)$ of $\text{Chow}(k, \mathbb{Z}/2)$ which is the pseudo-abelian envelope of the subcategory generated by the motives of smooth projective quadrics. Then in this category the Krull–Schmidt principle holds, that is, the decomposition into irreducible objects is unique (see [Vis04, CM06]). Let us introduce the following equivalence relation on the set of objects of $\text{Chow}_{\text{qua}}(k, \mathbb{Z}/2)$.

DEFINITION 3.9. Suppose N and M are objects of $\text{Chow}_{\text{qua}}(k, \mathbb{Z}/2)$. We say that $N \stackrel{T}{\sim} M$ if anisotropic indecomposable direct summands of N can be identified up to (reordering and) Tate shifts with such summands of M . More precisely, if $N \cong (\oplus \text{Tates}) \oplus \bigoplus_{i=1}^r N_i$, $M \cong (\oplus \text{Tates}) \oplus \bigoplus_{i=1}^r M_i$, where $M_i \cong N_i(a_i)[2a_i]$, for some $a_i \in \mathbb{Z}$ and some choice of ordering.

In particular, the T -equivalence ignores Tate motives, but it keeps the total rank of anisotropic direct summands, and it is stable under field extensions.

We will use a minor modification of Bachmann’s injectivity theorem (Theorem 2.3). Observe that, for any E/k , the map $\Phi^E/\Phi^k : \text{Pic}_{\text{qua}}/\mathbb{T} \rightarrow \text{Pic}(K^b(\text{Tate}(\mathbb{Z}/2)))$ is well defined.

PROPOSITION 3.10. *The collection of maps Φ^E/Φ^k , for all finitely generated E/k , is injective on $\text{Pic}_{\text{qua}}/\mathbb{T}$.*

Proof. Suppose all the homomorphisms Φ^E/Φ^k vanish on a certain element $x \in \text{Pic}_{\text{qua}}$. This means that $\Phi^E(x) = \Phi^k(x) = T(a)[b]$, for any E/k and some fixed $a, b \in \mathbb{Z}$. But $\Phi^E(T(a)[b])$ is also equal to $T(a)[b]$. By Bachmann’s injectivity theorem, $x = T(a)[b]$. \square

Let us recall some facts about indecomposable direct summands in the motives of quadrics. Suppose N is such a summand in the motive of a quadric Q , such that the smallest Tate motive in the decomposition of $N_{\bar{k}}$ is $T(n)[2n]$. Then we can assign to N the Grassmannian $X_N = G(Q, n)$ of n -dimensional projective subspaces on Q , and to the latter variety we can assign the motive \mathcal{X}_{X_N} of the respective Čech simplicial scheme $\check{\text{Cech}}(X_N)$ (where all our motives are with $\mathbb{Z}/2$ -coefficients). Recall that this is a *form* of a Tate motive which becomes isomorphic to the Tate motive if and only if our variety has a zero-cycle of degree 1 (see [Vis98, § 2.3]). For quadratic Grassmannians the latter condition is equivalent to the existence of a rational point (by Springer’s theorem).

PROPOSITION 3.11. *With notation as above, let N and L be indecomposable direct summands in the motives of quadrics. Then the following are equivalent:*

- (1) N is isomorphic to L up to Tate shift;
- (2) X_N and X_L are stably birationally equivalent;
- (3) $\mathcal{X}_{X_N} \cong \mathcal{X}_{X_L}$ (in this case, this isomorphism is unique).

Proof. The fact that the lowest Tate motive $T(n)[2n]$ splits from N is equivalent to the fact that it splits from the motive of the respective quadric Q . This, in turn, is equivalent to the fact that $i_W(Q) > n$ (by [Ros90, Proposition 1] and [Vis04, Proposition 2.6]), which means exactly that the Grassmannian X_N has a rational point. Thus (1) implies that X_N and X_L have rational points simultaneously, that is, there are rational maps both ways. For quadratic Grassmannians the latter condition is equivalent to stable birational equivalence. Indeed, clearly, $X_N = G(Q, n)$ is stably birationally equivalent to the flag variety $F(Q, n)$ of subspaces of dimensions from 0 to n (as the latter variety is a consecutive projective bundle over the former). And flag variety is rational as soon as it has a rational point (as a consecutive quadric fibration $F(Q, n) \rightarrow F(Q, n - 1) \rightarrow \dots \rightarrow Q \rightarrow \text{Spec}(k)$). If $F(P, l)$ is the flag variety corresponding to the motive L , then $F(Q, n) \times F(P, l)$ is stably birationally equivalent to both $F(Q, n)$ and $F(P, l)$, since each variety is rational over the generic point of the other (recall that these varieties have rational points simultaneously). Thus, (1) implies (2). The opposite implication follows from [Vis04, Theorem 4.17], taking into account [Vis04, Corollary 4.4].

Finally, (2) \Leftrightarrow (3) by [Vis98, Theorem 2.3.4] and above considerations, since for quadratic Grassmannians the existence of a zero-cycle of degree 1 is equivalent to that of a rational point (by the Theorem of Springer). \square

Now we can describe the relations among $\det(Q)$ in $\text{Pic}_{\text{qua}}/\mathbb{T}$. Any such relation can be reduced to the form $\prod_i \det(P_i) = \prod_j \det(Q_j)$.

THEOREM 3.12. *Let P_i, Q_j be some smooth projective quadrics. The following conditions are equivalent:*

- (1) $\prod_i \det(P_i) = \prod_j \det(Q_j) \in \text{Pic}_{\text{qua}}/\mathbb{T}$;
- (2) $\bigoplus_i M(P_i) \stackrel{T}{\sim} \bigoplus_j M(Q_j)$.

Proof. (2 \Rightarrow 1) Let $\bigoplus_i M(P_i) = (\oplus \text{Tates}) \oplus \bigoplus_l N_l$ and $\bigoplus_j M(Q_j) = (\oplus \text{Tates}) \oplus \bigoplus_l M_l$ be the decompositions of both sides of (2) into Tates and anisotropic irreducibles. Then (after reordering) we have isomorphisms $M_l \cong N_l(a_l)[2a_l]$, for some $a_l \in \mathbb{Z}$. Then as Φ^E/Φ^k ignores Tate motives and Tate shifts, we get:

$$\begin{aligned} \frac{\Phi^E}{\Phi^k} \left(\prod_i \det(P_i) \right) &= \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_i M(P_i))) = \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_l N_l)) \\ &= \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_l M_l)) = \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_j M(Q_j))) = \frac{\Phi^E}{\Phi^k} \left(\prod_j \det(Q_j) \right). \end{aligned}$$

By Proposition 3.10, $\prod_i \det(P_i) = \prod_j \det(Q_j) \in \text{Pic}_{\text{qua}}/\mathbb{T}$.

(1 \Rightarrow 2) Let $\bigoplus_i M(P_i) = (\oplus \text{Tates}) \oplus \bigoplus_l N_l$ and $\bigoplus_j M(Q_j) = (\oplus \text{Tates}) \oplus \bigoplus_r M_r$ be the decomposition into a direct sum of Tates and anisotropic irreducibles. From (1) we know that

$$\begin{aligned} \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_l N_l)) &= \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_i M(P_i))) = \frac{\Phi^E}{\Phi^k} \left(\prod_i \det(P_i) \right) \\ &= \frac{\Phi^E}{\Phi^k} \left(\prod_j \det(Q_j) \right) = \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_j M(Q_j))) = \frac{\Phi^E}{\Phi^k} (\det(\bigoplus_r M_r)). \end{aligned}$$

Let us cancel as many isomorphic (up to Tate shift) direct summands from both sides as possible. Suppose the remaining relation is non-trivial. So we can assume that N_l is not isomorphic to M_r

for any l, r , while we have

$$\frac{\Phi^E}{\Phi^k}(\det(\bigoplus_l N_l)) = \frac{\Phi^E}{\Phi^k}(\det(\bigoplus_r M_r)).$$

Each such irreducible is a direct summand in the (possibly shifted) motive of some smooth projective anisotropic quadric. To each such summand N_l (respectively, M_r), we can associate the field extension F_l/k (respectively, E_r/k) as follows. Let N_l be a direct summand of $M(R)$, such that the smallest Tate in $(N_l)_{\bar{k}}$ is $T(m)[2m]$. Then take $F_l = k(X_l)$, the function field of the Grassmannian $X_l = G(R, m)$ of m -dimensional projective subspaces on R , and similarly for $E_r = k(Y_r)$.

If any two of the above extensions are stably birationally equivalent, then the respective indecomposable direct summands are isomorphic (up to shift) by Proposition 3.11. Consider the directed graph whose vertices are isomorphism (up to shift) classes of $\{N_l\}_l, \{M_r\}_r$ and where we have an arrow $N_l \rightarrow M_r$ (respectively, $N_l \rightarrow N_{l'}, M_r \rightarrow M_{r'}$) if and only if there is a rational map $X_l \dashrightarrow Y_r$ ($X_l \dashrightarrow X_{l'}$, etc.). Since the existence of such a map is a transitive property, our graph has no directed cycles (as we have chosen a single representative from each isomorphism class of direct summands). Hence, our graph has (at least one) final vertex. Suppose it is N_l . Then every indecomposable summand (from our list) which is not isomorphic to N_l will stay anisotropic over F_l . Indeed, if L is this other summand, and some Tate motive would split from L_{F_l} (by [Vis04, Theorem 4.19] we can always assume it to be the ‘lower’ one; see [Vis11, Appendix]), then the lowest Tate motive (from which L ‘starts’) will split there as well (since splitting from L is equivalent to the splitting from the motive of the respective quadric L is part of, and for quadrics the splitting of the larger ‘lower’ Tate motive implies the splitting of the smaller one). But the lowest such Tate motive cannot split from the motive of the quadric mentioned, since the respective Grassmannian has no rational point over F_l . In particular, since no M_r were isomorphic to N_l , we get that all M_r stay anisotropic over F_l . Hence, $(\Phi^{F_l}/\Phi^k)(\det(\bigoplus_r M_r)) = T$. Similarly, $(\Phi^{F_l}/\Phi^k)(\det(N_{l'})) = T$, for all $N_{l'}$ not isomorphic to N_l . At the same time, some (actually, exactly two) Tate motives split from N_l over F_l , since the Tate motive $T(m)[2m]$ splits from $M(R)|_{F_l}$. The fact that the number of ‘lower’ Tate summands split from $N_l|_{F_l}$ is equal to the number of ‘upper’ Tate summands split follows from [Vis04, Theorem 4.19] (while the fact that there is only one of each kind follows from [Vis04, Theorem 4.17], but we do not need this). Hence $(\Phi^{F_l}/\Phi^k)(\det(N_l)) = T(x)[y] \neq T$ (this follows from the Definition 3.7, taking into account that no ‘lower’ Tate motive of N_l can be ‘above’ an ‘upper’ one (since it is so for the quadric)). Then $(\Phi^{F_l}/\Phi^k)(\det(\bigoplus_n N_n)) = T(d \cdot x)[d \cdot y]$, where d is the number of indecomposable $N_{l'}$ isomorphic (up to shift) to N_l . We obtain a contradiction: $T = T(d \cdot x)[d \cdot y]$. Thus, we can cancel all the terms, and so $\bigoplus_i M(P_i) \stackrel{T}{\sim} \bigoplus_j M(Q_j)$. □

In particular, we can see that $\det(Q) \in \text{Pic}_{\text{qua}}$ is a complete invariant of $M(Q)$. This extends the criterion of motivic equivalence of projective quadrics ([Vis98, Proposition 5.1] or [Vis04, Theorem 4.18]; see also [Kar00]).

COROLLARY 3.13. *Let P and Q be smooth projective quadrics. Then the following conditions are equivalent:*

- (1) $M(P) \cong M(Q)$;
- (2) $\det(P) = \det(Q) \in \text{Pic}_{\text{qua}}$.

Proof. (1 \Rightarrow 2) This is Proposition 3.6(2).

(2 \Rightarrow 1) Since $\det(P_{\bar{k}}) = \det(Q_{\bar{k}})$, we obtain that $\dim(P) = \dim(Q)$. It follows from (2) and Theorem 3.12 that $M(P) \stackrel{T}{\sim} M(Q)$. Since T -equivalence preserves the rank of anisotropic summand, and the total rank is the same (since dimensions are the same), we obtain that the number of Tate summands in $M(P_E)$ and $M(Q_E)$ is the same, hence, $i_W(Q_E) = i_W(P_E)$, for all E/k . By the criterion of motivic equivalence, $M(P) \cong M(Q)$. \square

Let $\{X_l\}_l$ be the collection of all anisotropic quadratic Grassmannians for all quadrics from the set $\{P_i\}_i$ (considered with multiplicities), and $\{Y_r\}_r$ be the collection of all anisotropic quadratic Grassmannians for all quadrics from the set $\{Q_j\}_j$. Let $\{\mathcal{X}_{X_l}\}_l$ be the collection of the motives of the respective Čech simplicial schemes $\check{C}ech(X_l)$ (again with multiplicity), and similarly for $\{\mathcal{X}_{Y_r}\}_r$. Repeating the arguments of the proof of [Vis98, Proposition 5.1], we obtain the following result.

PROPOSITION 3.14. *Conditions (1) and (2) of Theorem 3.12 are equivalent to:*

$$(3) \quad \{\mathcal{X}_{X_l}\}_l \cong \{\mathcal{X}_{Y_r}\}_r.$$

Proof. Note that, by [Vis98, Theorem 3.1], the anisotropic part of $\bigoplus_i M(P_i)$ is an extension of (shifted) \mathcal{X} s from the union of the two copies of the set $\{\mathcal{X}_{X_l}\}_l$ (corresponding to the ‘upper’ and ‘lower’ Tate motives, respectively) which I will denote by ‘2’ $\{\mathcal{X}_{X_l}\}_l$, and similarly for the right-hand side.

Let N_l be some indecomposable (anisotropic) irreducible summand of the $\bigoplus_i M(P_i)$, and X_l be the respective Grassmannian (corresponding to the lowest Tate motive in N_l). Then (3) implies that \mathcal{X}_{X_l} is isomorphic to some \mathcal{X}_{Y_r} , and it follows from [Vis04, Theorem 4.17] that $\bigoplus_j M(Q_j)$ contains an irreducible summand isomorphic up to shift to N_l . By [Vis98, Theorems 3.1, 3.7], N_l is an extension of (shifted) motives of Čech simplicial schemes of Grassmannians and the respective \mathcal{X} s are determined uniquely by N_l (since such an \mathcal{X} is trivial if and only if the respective Tate motive splits from N_l , and this information determines \mathcal{X} by [Vis98, Theorem 2.3.4]). Hence, we can identify the respective subsets in ‘2’ $\{\mathcal{X}_{X_l}\}_l$ and ‘2’ $\{\mathcal{X}_{Y_r}\}_r$. Cancelling N_l s on both sides as well as the subsets mentioned, we reduce to smaller identical collections in ‘2’(3) and to shorter sums of indecomposables in (2). Continuing in this way, we cancel all the indecomposables in (2). This shows that (3) implies (2). The converse is clear, since N_l is an extension of \mathcal{X} s which are uniquely determined by N_l as explained above. Thus, (2) implies that ‘2’ $\{\mathcal{X}_{X_l}\}_l \cong$ ‘2’ $\{\mathcal{X}_{Y_r}\}_r$. \square

Denote by $\mathcal{N}(k)$ the set of isomorphism (up to shift) classes of indecomposable anisotropic direct summands in the motives of smooth projective quadrics over k . Then Theorem 3.12 shows that we have an embedding of $\text{Pic}_{\text{qua}}/\mathbb{T}$ into a free abelian group generated by the set $\mathcal{N}(k)$,

$$\text{Pic}_{\text{qua}}/\mathbb{T} \hookrightarrow \bigoplus_{N \in \mathcal{N}(k)} \mathbb{Z} \cdot \det(N),$$

mapping $\det(Q)$ to the sum of determinants of anisotropic irreducible summands $M(Q)$ is a direct sum of (Tate summands are ignored). Indeed, since $\text{Pic}_{\text{qua}}/\mathbb{T}$ is a quotient of a free abelian group generated by $\det(Q)$ where Q runs over (isomorphism classes of) smooth projective quadrics over k modulo relations in (1) of Theorem 3.12, this theorem (together with Definition 3.9 and the fact that \det is stable under Tate shift) ensures that the map is well defined and injective, since $\prod_i \det(P_i) / \prod_j \det(Q_j) = 1$ in $\text{Pic}_{\text{qua}}/\mathbb{T}$ if and only if the respective element $\sum_l \det(N_l) - \sum_r \det(M_r)$ is zero in $\bigoplus_{N \in \mathcal{N}(k)} \mathbb{Z} \cdot \det(N)$. Note also that our map maps $\det(Q)$ the same way as the determinant of its anisotropic part. So we obtain the following corollary.

COROLLARY 3.15. *The group $\text{Pic}_{\text{qua}}/\mathbb{T}$ is isomorphic to the image of the map*

$$\varphi : \bigoplus_{Q \in \mathcal{Q}} \mathbb{Z} \cdot \det(Q) \longrightarrow \bigoplus_{N \in \mathcal{N}(k)} \mathbb{Z} \cdot \det(N),$$

where \mathcal{Q} is the set of isomorphism classes of smooth anisotropic projective quadrics over k . In particular, Pic_{qua} is a free abelian group.

It is an interesting question whether φ is surjective or not. It is related to an old and non-trivial motivic question.

Question 3.16 [Vis04, Question 4.16]. Let Q be a smooth projective quadric and N be an indecomposable direct summand in $M(Q)$. Is it true that there exists a smooth projective quadric P over k with direct summand M of $M(P)$, such that M is isomorphic to N up to Tate shift, and $M_{\bar{k}}$ contains T ?

We obtain the following proposition.

PROPOSITION 3.17. *Suppose Question 3.16 has a positive answer. Then*

$$\text{Pic}_{\text{qua}}/\mathbb{T} \cong \bigoplus_{N \in \mathcal{N}(k)} \mathbb{Z} \cdot \det(N).$$

Alternatively, in $\text{Pic}_{\text{qua}}/\mathbb{T}$ we can choose a basis $\{e^{q_i}\}_i$, where we take exactly one representative q_i in every class of stable birational equivalence of anisotropic quadratic forms over k .

Proof. If Question 3.16 has a positive answer, then any $N \in \mathcal{N}(k)$ will be represented by a direct summand in $M(Q'_i)$, for some i , a direct summand starting from T (over \bar{k}). By [Vis04, Theorem 4.13] (taking into account [Vis04, Corollary 4.4] and Proposition 3.11) the multiplicity (up to shift) of such a summand in $M(Q'_i)$ is equal to the first higher Witt index $i_1(q'_i)$ of our quadric (see [Vis04, § 7]). Since Q_i is a subquadric of codimension 1 in Q'_i , by [Vis04, Corollary 4.9], the multiplicity of N in $M(Q'_i)$ will be exactly 1 more than that in $M(Q_i)$. Indeed, recall that $\text{size}(N) = \max\{i \mid \text{Ch}^i(N_{\bar{k}}) \neq 0\} - \min\{i \mid \text{Ch}^i(N_{\bar{k}}) \neq 0\}$, [Vis04, Definition 4.6]. Then either $i_1(q'_i) = 1$, in which case $\text{size}(N) = \dim(Q'_i)$ [Vis04, Proposition 4.5] and $M(Q_i)$ does not contain such summands at all, or $i_1(q'_i) > 1$, in which case N is also a direct summand of $M(Q_i)$ of multiplicity $i_1(Q_i) = i_1(q'_i) - 1$. Also, by the same results, the remaining indecomposable direct summands of $M(Q_i)$ and $M(Q'_i)$ will be of strictly smaller size. Indeed, the ‘outer shell’ (all the Tate motives split over the generic point) of $M(Q'_i)$ as well as that of $M(Q_i)$ (if $i_1(q'_i) > 1$) are completely covered by the copies of N , by [Vis04, Theorem 4.13]. Hence, remaining direct summands start and end in ‘higher shells’, and so have smaller size, [Vis04, Corollary 4.14]. Hence, $\det(Q'_i) = \det(N)^{i_1(q'_i)} \cdot \det(\text{smaller motives})$, while $\det(Q_i) = \det(N)^{i_1(q'_i)-1} \cdot \det(\text{smaller motives})$ and so, $e^{q_i}/\det(N) = \det(Q'_i)/(\det(Q_i) \cdot \det(N))$ is expressible in terms of $\det(M)$, for $\text{size}(M) < \text{size}(N)$. By induction on $\text{size}(N)$, we get that the map φ is surjective and the collection $\{e^{q_i}\}_i$ generates $\text{Pic}_{\text{qua}}/\mathbb{T}$. The fact that it is linearly independent follows from Theorem 3.1. \square

Example 3.18. Let $k = \mathbb{R}$. Then the only indecomposable anisotropic direct summands in the motives of real quadrics are Rost motives corresponding to the pure symbols $\{-1\}^r, r \in \mathbb{N}$, and so it follows from Example 3.8 that the map φ is surjective and

$$(\text{Pic}_{\text{qua}}/\mathbb{T})(\mathbb{R}) = \bigoplus_{r \in \mathbb{N}} \mathbb{Z} \cdot e^{\langle -1 \rangle^r}.$$

4. An alternative to the method of Bachmann

In this section I present an approach which can serve as an alternative to that used by Bachmann. Here I instead employ some ideas of [Vis98]. This allows us to look at the same questions from a slightly different point of view. That said, there are certain similarities in both approaches. At the same time, the new method is applicable not just to the category generated by the motives of quadrics but to arbitrary geometric motives, and can be used for the study of the entire Pic group of $DM_{gm}(k)$. This approach in the end leads to the ‘motivic category of an extension’ which has many remarkable properties and allows us to study Voevodsky motives ‘locally’. It will be discussed in detail in a separate paper.

Let $DM(k; F)$ be the Voevodsky triangulated category of motives with coefficients in an arbitrary commutative (unital) ring F . Let Q be a smooth projective variety (not necessarily connected) over k . Let $\check{C}ech(Q)$ be the respective Čech simplicial scheme, where $(\check{C}ech(Q))_n = Q^{\times n+1}$ with faces and degeneracy maps being partial diagonals and partial projections. Denote its motive by \mathcal{X}_Q . We get the natural projection $\check{C}ech(Q) \rightarrow Spec(k)$, which gives a distinguished triangle Δ_Q in $DM(k; F)$:

$$\mathcal{X}_Q \longrightarrow T \longrightarrow \tilde{\mathcal{X}}_Q \longrightarrow \mathcal{X}_Q[1]. \tag{2}$$

Motives \mathcal{X}_Q and $\tilde{\mathcal{X}}_Q$ are mutually orthogonal idempotents in $DM(k; F)$:

$$\mathcal{X}_Q^{\otimes 2} \xrightarrow{\cong} \mathcal{X}_Q, \quad \tilde{\mathcal{X}}_Q^{\otimes 2} \xleftarrow{\cong} \tilde{\mathcal{X}}_Q, \quad \mathcal{X}_Q \otimes \tilde{\mathcal{X}}_Q \cong 0.$$

Denote by $\pi_Q : DM(k; F) \rightarrow DM(k; F)$ and $\tilde{\pi}_Q : DM(k; F) \rightarrow DM(k; F)$ the projection functors given by $\otimes \mathcal{X}_Q$ and $\otimes \tilde{\mathcal{X}}_Q$, respectively. Then the image of π_Q is the full localizing subcategory $DM_Q(k; F)$ consisting of objects which are stable under $\otimes \mathcal{X}_Q$, while the image of $\tilde{\pi}_Q$ is the full localizing subcategory $DM_{\tilde{Q}}(k; F)$ consisting of objects which are killed by $\otimes \mathcal{X}_Q$. It follows from [Vis98, Theorem 2.3.2] (which is, basically, [Voe95, Lemma 4.9]) that

$$Hom_{DM(k;F)}(U, V) = 0, \quad \text{for any } U \in Ob(DM_Q(k; F)) \text{ and } V \in Ob(DM_{\tilde{Q}}(k; F)). \tag{3}$$

At the same time, any object W in $DM(k; F)$ has a functorial decomposition

$$\pi_Q(W) \longrightarrow W \longrightarrow \tilde{\pi}_Q(W) \longrightarrow \pi_Q(W)[1].$$

If P and Q are two smooth projective varieties over k , then there are natural identifications

$$\mathcal{X}_{P \times Q} \xrightarrow{\cong} \mathcal{X}_P \otimes \mathcal{X}_Q, \quad \tilde{\mathcal{X}}_{P \sqcup Q} \xleftarrow{\cong} \tilde{\mathcal{X}}_P \otimes \tilde{\mathcal{X}}_Q.$$

The respective functors $\pi_Q, \tilde{\pi}_Q, \pi_P, \tilde{\pi}_P$ commute up to isomorphism, and we have identifications of functors $\pi_{Q \times P} \cong \pi_Q \circ \pi_P$ and $\tilde{\pi}_{Q \sqcup P} \cong \tilde{\pi}_Q \circ \tilde{\pi}_P$. It is also worth recalling [Vis98, Theorem 2.3.4] that there is an identification $\mathcal{X}_Q \cong \mathcal{X}_P$ if and only if P has a zero-cycle of degree 1 over every generic point of Q , and vice versa.

Now let $\mathbf{X} = \{X_i\}_{i \in I}$ be some finite collection of smooth projective varieties. For any subset $J \subset I$, write

$$\mathcal{X}_{\mathbf{X}_J} := \left(\otimes_{i \in J} \mathcal{X}_{X_i} \right) \otimes \left(\otimes_{i \notin J} \tilde{\mathcal{X}}_{X_i} \right).$$

Note that these are still idempotents: $\mathcal{X}_{\mathbf{X}_J}^{\otimes 2} \cong \mathcal{X}_{\mathbf{X}_J}$. Denote by $\pi_{\mathbf{X}_J}$ the respective projector. Its image is a full localizing subcategory $DM_{\mathbf{X}_J}(k; F)$ of $DM(k; F)$ made out of objects which are stable under $\otimes \mathcal{X}_{X_i}$, for $i \in J$, and are killed by $\otimes \mathcal{X}_{X_i}$, for $i \notin J$.

Tensoring the distinguished triangles Δ_{X_i} from (2) for every $X_i, i \in I$, we obtain what I will call a *distinguished poly-triangle* $\Delta_{\mathbf{X}}$. It presents the unit T of the tensor structure of $DM(k; F)$

as an extension of graded pieces $\mathcal{X}_{\mathbf{X}_J}$, for all $J \subset I$. Note that it follows from (3) that, for $J_1, J_2 \subset I$, we have

$$\mathrm{Hom}_{\mathrm{DM}(k;F)}(U, V) = 0, \quad \text{for } U \in \mathrm{Ob}(\mathrm{DM}_{\mathbf{X}_{J_1}}(k; F)), V \in \mathrm{Ob}(\mathrm{DM}_{\mathbf{X}_{J_2}}(k; F)), \text{ if } J_1 \not\subset J_2. \quad (4)$$

As any object W of $\mathrm{DM}(k)$ is an extension of $\pi_{\mathbf{X}_J}(W)$, for $J \subset I$, we obtain the following proposition.

PROPOSITION 4.1. *The functor*

$$\times_{J \subset I} \pi_{\mathbf{X}_J} : \mathrm{DM}(k; F) \longrightarrow \times_{J \subset I} \mathrm{DM}_{\mathbf{X}_J}(k; F)$$

is conservative.

Since $\pi_{\mathbf{X}_J}$ is symmetric monoidal, it preserves duals. Hence, using Proposition 4.1, we obtain our next result.

PROPOSITION 4.2. *The functor*

$$\times_{J \subset I} \pi_{\mathbf{X}_J} : \mathrm{DM}_{\mathrm{gm}}(k; F) \longrightarrow \times_{J \subset I} \mathrm{DM}_{\mathbf{X}_J}(k; F)$$

detects invertible objects.

Moreover, we have the following proposition.

PROPOSITION 4.3. *The functor $\times_{J \subset I} \pi_{\mathbf{X}_J}$ is injective on the Pic.*

Proof. Denote by T_P (respectively, $T_{P \times \tilde{Q}}$) the unit objects of the category $\mathrm{DM}_P(k; F)$ (respectively, $\mathrm{DM}_{P \times \tilde{Q}}(k; F)$).

LEMMA 4.4. *Let P and Q be smooth projective varieties, and $V \in \mathrm{Pic}(\mathrm{DM}_P(k; F))$ be such an invertible object that $V \otimes \mathcal{X}_Q \cong T_{P \times Q}$ and $V \otimes \tilde{\mathcal{X}}_Q \cong T_{P \times \tilde{Q}}$. Then $V \cong T_P$. Moreover, this identification $\otimes \mathcal{X}_Q$ coincides with the original identification of $V \otimes \mathcal{X}_Q$ and $T_{P \times Q}$.*

Proof. Let $U \in \mathrm{Pic}(\mathrm{DM}_P(k; F))$ be the inverse of V . Tensoring V and T_P with Δ_Q , we get two exact triangles in $\mathrm{DM}_P(k; F)$, where we can identify the $T_{P \times Q}$ -terms:

$$\begin{array}{ccccccc} T_P & \longrightarrow & T_{P \times \tilde{Q}} & \longrightarrow & T_{P \times Q}[1] & \longrightarrow & T_P[1] \\ & & & & \psi \uparrow & \downarrow \varphi & \\ & & & & T_{P \times \tilde{Q}} & & \\ V & \longrightarrow & T_{P \times \tilde{Q}} & \longrightarrow & T_{P \times Q}[1] & \longrightarrow & V[1] \end{array} \quad (5)$$

Since $U = V^{-1}$ is invertible, we have

$$\mathrm{Hom}(T_{P \times \tilde{Q}}, V[1]) = \mathrm{Hom}(U \otimes T_{P \times \tilde{Q}}, T_P[1]) = \mathrm{Hom}(T_{P \times \tilde{Q}}, T_P[1]).$$

By [Vis98, Theorem 2.3.2] (or [Voe95, Lemma 4.9]), we can identify

$$\begin{aligned} \mathrm{Hom}(T_P, T_P[1]) &= \mathrm{Hom}(T_P, T[1]) = H_{\mathcal{M}}^{1,0}(\check{\mathrm{Cech}}(P), F) = H_{\mathrm{Zar}}^1(\check{\mathrm{Cech}}(P), F), \\ \mathrm{Hom}(T_{P \times Q}, T_P[1]) &= \mathrm{Hom}(T_{P \times Q}, T[1]) = H_{\mathcal{M}}^{1,0}(\check{\mathrm{Cech}}(P \times Q), F) = H_{\mathrm{Zar}}^1(\check{\mathrm{Cech}}(P \times Q), F). \end{aligned}$$

And since $\check{C}ech(P)$ and $\check{C}ech(P \times Q)$ in étale topology are both contractable to $\text{Spec}(k)$, by the Beilinson–Lichtenbaum conjecture in weight zero (classical), the groups $H_{\mathcal{M}}^{1,0}(\check{C}ech(P), F)$ and $H_{\mathcal{M}}^{1,0}(\check{C}ech(P \times Q), F)$ both embed into $H_{\text{ét}}^1(\text{Spec}(k), F)$, as the map from Zariski to étale cohomology is injective on the first diagonal [Voe03, Corollary 6.9]. In particular, the natural map $\text{Hom}(T_P, T_P[1]) \hookrightarrow \text{Hom}(T_{P \times Q}, T_P[1])$ is injective. On the other hand, $\text{Hom}(T_{P \times Q}[1], T_P[1]) = \text{Hom}(T_{P \times Q}, T) = F$, $\text{Hom}(T_P[1], T_P[1]) = \text{Hom}(T_P, T) = F$ and the map $\text{Hom}(T_P[1], T_P[1]) \rightarrow \text{Hom}(T_{P \times Q}[1], T_P[1])$ is an isomorphism. Considering long exact sequences of Homs from the triangle $T_{P \times Q} \rightarrow T_P \rightarrow T_{P \times \tilde{Q}} \rightarrow T_{P \times Q}[1]$ to T_P and V , we get that both groups $\text{Hom}(T_{P \times \tilde{Q}}, T_P[1])$ and $\text{Hom}(T_{P \times \tilde{Q}}, V[1])$ are trivial. Hence the above mutually inverse isomorphisms φ, ψ can be extended to maps of exact triangles. In particular, we get maps $T_P \xrightleftharpoons[g]{f} V$. Moreover, since by (3) there are no homs from $T_{P \times Q}[1]$ to $T_{P \times \tilde{Q}}$, it follows that $(g \circ f) \otimes T_{P \times Q} = (\psi \circ \varphi)$. Indeed, let $u : T_{P \times Q}[1] \rightarrow T_P[1]$; then $u \circ (\psi \circ \varphi) = (g \circ f) \circ u$. On the other hand, $u \circ ((g \circ f) \otimes T_{P \times Q}) = (g \circ f) \circ u$, and so, the difference $(\psi \circ \varphi) - (g \circ f) \otimes T_{P \times Q}$ lifts to a map to $T_{P \times \tilde{Q}}$ which must be zero. Since $\text{End}(T_P) = F = \text{End}(T_{P \times Q})$ with the isomorphism (from left to right) given by $\otimes T_{P \times Q}$, and $(\psi \circ \varphi)$ is invertible, we obtain that $(g \circ f)$ is invertible as well. That means that T_P is a direct summand in V . By [Bac18, Lemma 30], this implies that $V \cong T_P$.¹ Alternatively, applying the same considerations to the bottom row of (5) instead of the top, one obtains that $(f \circ g) \otimes T_{P \times Q} = (\varphi \circ \psi)$ and so $(f \circ g)$ is invertible too, which gives the same isomorphism $V \cong T_P$. \square

LEMMA 4.5. *If P and Q are smooth projective varieties, $V \in \text{Pic}(\text{DM}(k; F))$ is such that $V \otimes \mathcal{X}_P \cong T_P$, $V \otimes \mathcal{X}_Q \cong T_Q$ and $V \otimes \mathcal{X}_{P \times Q} \cong T_{P \times Q}$, and these identifications commute with the maps $T_P \leftarrow T_{P \times Q} \rightarrow T_Q$, then there is an identification $V \otimes \mathcal{X}_{P \sqcup Q} \cong T_{P \sqcup Q}$ and this commutes with the maps $T_P \rightarrow T_{P \sqcup Q} \leftarrow T_Q$.*

Proof. The natural projections provide a complex

$$T_{P \times Q} \longrightarrow T_P \oplus T_Q \longrightarrow T_{P \sqcup Q}.$$

I claim that it extends to a distinguished triangle. Indeed, tensoring it with \mathcal{X}_Q and $\tilde{\mathcal{X}}_Q$, we get split exact complexes $T_{P \times Q} \rightarrow T_{P \times Q} \oplus T_Q \rightarrow T_Q$ and $0 \rightarrow T_{P \times \tilde{Q}} \rightarrow T_{P \times \tilde{Q}}$. Since the unit T is an extension of \mathcal{X}_Q and $\tilde{\mathcal{X}}_Q$, we obtain the Mayer–Vietoris type distinguished triangle

$$T_{P \times Q} \longrightarrow T_P \oplus T_Q \longrightarrow T_{P \sqcup Q} \longrightarrow T_{P \times Q}[1].$$

Tensoring it with V and identifying both in $T_{P \times Q}$ and $T_P \oplus T_Q$ terms

$$\begin{array}{ccccccc} T_{P \times Q} & \longrightarrow & T_P \oplus T_Q & \longrightarrow & T_{P \sqcup Q} & \longrightarrow & T_{P \times Q}[1] \\ \downarrow \cong & & \downarrow \cong & & & & \downarrow \cong \\ T_{P \times Q} & \longrightarrow & T_P \oplus T_Q & \longrightarrow & V_{P \sqcup Q} & \longrightarrow & T_{P \times Q}[1] \end{array}$$

we extend it to an isomorphism of distinguished triangles. In particular, we get an identification $T_{P \sqcup Q} \cong V_{P \sqcup Q}$ commuting with the needed maps. \square

Returning to the proof of Proposition 4.3, let $\mathbf{X}^J = \prod_{i \in J} X_i$, and $T_{\mathbf{X}^J} = \mathcal{X}_{\mathbf{X}^J}$. We now prove by decreasing induction on $|J|$ that $V \otimes T_{\mathbf{X}^J} \cong T_{\mathbf{X}^J}$ and these identifications commute with the canonical maps $T_{\mathbf{X}^L} \rightarrow T_{\mathbf{X}^J}$, for $J \subset L$.

¹I am grateful to the referee for pointing out that connectivity of coefficients is not needed in this statement.

For $J = I$ we have $\mathcal{X}_{\mathbf{X}^I} = \mathcal{X}_{\mathbf{X}_I}$, and we get the identification from the conditions of Proposition 4.3.

Suppose we want to prove the inductive step for a given J . By the inductive assumption, we have (coherent) isomorphisms for all $T_{\mathbf{X}^{J'}}$ with $J \subsetneq J'$. It follows from the inductive application of Lemma 4.5 that we can add to this coherent collection the isomorphism for $T_{\mathbf{X}^J \times (\sqcup_{i \notin J} X_i)}$. Since $T_{\mathbf{X}^J \times (\widetilde{\sqcup_{i \notin J} X_i})} = \mathcal{X}_{\mathbf{X}_J}$, from the conditions of Proposition 4.3 and Lemma 4.4, we get that $T_{\mathbf{X}^J}$ can be added to our collection as well. The induction step is proven. For $J = \emptyset$ we get an isomorphism $V \cong T$. Hence our map is injective on Pic. \square

To start with, let us re-prove Bachmann’s result claiming that the reduced motives of affine quadrics are invertible.

PROPOSITION 4.6 (Bachmann [Bac17, Theorem 33]). *Let A_q be an affine quadric $\{q = 1\}$. Then the reduced motive $\widetilde{M}(A_q)$ is invertible in $DM_{\text{gm}}(k; \mathbb{Z}/2)$.*

Proof. Let $\dim(q) = n$ and $q' = \langle 1 \rangle \perp -q$. Then $A_q = Q' \setminus Q$, and we have an exact triangle in $DM(k; \mathbb{Z}/2)$:

$$M(Q') \rightarrow M(Q)(1)[2] \oplus T \rightarrow \widetilde{M}(A_q)[1] \rightarrow M(Q')[1].$$

For a smooth projective quadric R let us denote by R^i the Grassmannian of the i -dimensional projective planes on R . Since Q is a codimension-one subquadric of Q' , we have that, for any extension E/k , the following inequalities on the Witt indices hold: $i_W(q|_E) \leq i_W(q'|_E) \leq i_W(q|_E) + 1$. In particular, we have a chain of rational maps between the respective Grassmannians

$$Q'^0 \leftarrow \dots \leftarrow Q^0 \leftarrow \dots \leftarrow Q'^1 \leftarrow \dots \leftarrow Q^1 \leftarrow \dots$$

which induces a chain of morphisms of motives of their Čech simplicial schemes (cf. [BV18, proof of Proposition 2.3])

$$\mathcal{X}_{Q'^0} \leftarrow \mathcal{X}_{Q^0} \leftarrow \mathcal{X}_{Q'^1} \leftarrow \mathcal{X}_{Q^1} \leftarrow \mathcal{X}_{Q'^2} \leftarrow \mathcal{X}_{Q^2} \leftarrow \dots$$

Let us rewrite it as:

$$\mathcal{X}_{X_1} \leftarrow \mathcal{X}_{X_2} \leftarrow \mathcal{X}_{X_3} \leftarrow \dots \leftarrow \mathcal{X}_{X_n}.$$

Consider the collection $\mathbf{X} = \{X_i\}_{i=1}^n$. We can use the standard poly-binary approach as above, but since our motives of Čech simplicial schemes are ordered, we can substitute it by an ‘ordered’ version. We have a Postnikov system in $DM(k; \mathbb{Z}/2)$:

$$\begin{array}{ccccccc} & \mathcal{X}_{X_{0/1}} & & \mathcal{X}_{X_{1/2}} & & \mathcal{X}_{X_{n-1/n}} & & \mathcal{X}_{X_{n/n+1}} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow \\ T & \star & \downarrow [1] & \star & \downarrow [1] & \star & \downarrow [1] & \star & \downarrow [1] \\ & \mathcal{X}_{X_1} & \longleftarrow & \mathcal{X}_{X_2} & \dots & \mathcal{X}_{X_{n-1}} & \longleftarrow & \mathcal{X}_{X_n} & \longleftarrow & 0 \end{array}$$

where $\mathcal{X}_{X_{i/i+1}} = \mathcal{X}_{X_i} \otimes \widetilde{\mathcal{X}}_{X_{i+1}}$. Thus, the unit T is an extension of idempotents $\mathcal{X}_{X_{i/i+1}}$, $i = 1, \dots, n$. Note that $\mathcal{X}_{X_i} \otimes \mathcal{X}_{X_j} = \mathcal{X}_{\max(i,j)}$, while $\widetilde{\mathcal{X}}_{X_i} \otimes \widetilde{\mathcal{X}}_{X_j} = \widetilde{\mathcal{X}}_{\min(i,j)}$, and so $\mathcal{X}_{X_{i/i+1}} \otimes \mathcal{X}_{X_j} = \mathcal{X}_{X_{i/i+1}}$, for $j \leq i$, and is zero otherwise. Let $N' = n - 1 = \dim(Q')$ and $N = n - 2 = \dim(Q)$. By [Vis98, Proposition 3.6], for a quadric R of dimension M , we have an exact triangle in $DM(k; \mathbb{Z}/2)$:

$$\bigoplus_{l=0}^k \mathcal{X}_{R^{k/k+1}}(M - l)[2M - 2l] \longrightarrow M(R) \otimes \mathcal{X}_{R^{k/k+1}} \longrightarrow \bigoplus_{l=0}^k \mathcal{X}_{R^{k/k+1}}(l)[2l] \longrightarrow \dots,$$

where the maps are induced by the plane section cycles $T(M - l)[2M - 2l] \rightarrow M(R)$ and the dual ones $M(R) \rightarrow T(l)[2l]$ (in the same proposition it is stated for odd-dimensional quadrics only as motives were integral there, but the same arguments work for all quadrics for motives with $\mathbb{Z}/2$ -coefficients). Considering $R = Q'$ and $R = Q$, and tensoring the above triangle with the appropriate $\mathcal{X}_{X_{i/i+1}}$ (with $k = [i - 1/2]$ (respectively, $k = [i - 2/2]$)), we obtain exact triangles in $\text{DM}(k; \mathbb{Z}/2)$:

$$\begin{array}{c} \bigoplus_{l=0}^{[i-1/2]} \mathcal{X}_{X_{i/i+1}}(N' - l)[2N' - 2l] \longrightarrow M(Q') \otimes \mathcal{X}_{X_{i/i+1}} \longrightarrow \bigoplus_{l=0}^{[i-1/2]} \mathcal{X}_{X_{i/i+1}}(l)[2l] \longrightarrow \dots, \\ \bigoplus_{l=0}^{[i-2/2]} \mathcal{X}_{X_{i/i+1}}(N - l)[2N - 2l] \longrightarrow M(Q) \otimes \mathcal{X}_{X_{i/i+1}} \longrightarrow \bigoplus_{l=0}^{[i-2/2]} \mathcal{X}_{X_{i/i+1}}(l)[2l] \longrightarrow \dots. \end{array}$$

Thus $\widetilde{M}(A_q) \otimes \mathcal{X}_{X_{i/i+1}} \cong \mathcal{X}_{X_{i/i+1}}(i/2)[i - 1]$, for i even, and $\cong \mathcal{X}_{X_{i/i+1}}(N' - (i - 1/2))[2N' - i + 1]$, for i odd. In any case, $\widetilde{M}(A_q) \otimes \mathcal{X}_{X_{i/i+1}}$ is invertible in $\text{DM}_{X_{i/i+1}}(k; \mathbb{Z}/2)$, for every i . By the ‘ordered’ version of Proposition 4.2, the functor

$$\times (\otimes \mathcal{X}_{X_{i/i+1}}) : \text{DM}_{\text{gm}}(k; \mathbb{Z}/2) \rightarrow \times_i \text{DM}_{X_{i/i+1}}(k; \mathbb{Z}/2)$$

is conservative and detects invertible objects. Hence, $\widetilde{M}(A_q)$ is invertible in $\text{DM}(k; \mathbb{Z}/2)$. \square

Denote by $\text{Tate}_{\mathbf{X}_J} = \text{Tate}(\text{DM}_{\mathbf{X}_J}(k; F))$ the thick tensor triangulated subcategory of $\text{DM}_{\mathbf{X}_J}(k; F)$ generated by the Tate motives $T_{\mathbf{X}_J}(a)$. Then $\text{Pic}(\text{Tate}_{\mathbf{X}_J})$ contains the subgroup \mathbb{T} consisting of Tate motives $T_{\mathbf{X}_J}(a)[b]$, $a, b \in \mathbb{Z}$.

PROPOSITION 4.7. *Suppose the category $\text{DM}_{\mathbf{X}_J}(k; F)$ is non-zero. Then $\mathbb{T} \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Suppose that $T_{\mathbf{X}_J} \cong T_{\mathbf{X}_J}(a)[b]$, for some $(a)[b] \neq (0)[0]$.

Our projector $\mathcal{X}_{\mathbf{X}_J}$ has the form $\mathcal{X}_P \otimes \widetilde{\mathcal{X}}_Q$ for some smooth projective P and Q . This projector will be non-zero if and only if Q has no zero-cycle of degree 1 over the function field E of some connected component of P . Indeed, it follows from [Vis98, Theorems 2.3.4 and 2.3.5] that if Q has a zero-cycle of degree 1 over each such function field, then $M(P) \otimes \widetilde{\mathcal{X}}_Q = 0$. Since \mathcal{X}_P belongs to the localizing subcategory generated by $M(P)$, $\mathcal{X}_P \otimes \widetilde{\mathcal{X}}_Q$ is zero as well. For the converse, it is sufficient to restrict to E and observe that $\widetilde{\mathcal{X}}_Q|_E$ is non-zero by [Vis98, Theorem 2.3.3]. Hence, in our situation, $\mathcal{X}_P \otimes \widetilde{\mathcal{X}}_Q|_E = \widetilde{\mathcal{X}}_{Q_E}$ is still non-zero. Thus, by passing to E , we can assume that our projector is $\widetilde{\mathcal{X}}_Q$. As we have $(a)[b]$ -periodicity, we obtain

$$0 \neq \text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}) = \text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}(a)[b]) = \text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}(-a)[-b]).$$

From the fact that $\text{Hom}(T_Q, T_{\widetilde{Q}}(*)[*']) = 0$, we have $\text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}(c)[d]) = \text{Hom}(T, T_{\widetilde{Q}}(c)[d])$, and we have an exact sequence $\text{Hom}(T, T_Q(c)[d + 1]) \leftarrow \text{Hom}(T, T_{\widetilde{Q}}(c)[d]) \leftarrow \text{Hom}(T, T(c)[d])$. The group $\text{Hom}(T, T(c)[d])$ is zero, for $d > c$ and for $c < 0$, while the group $\text{Hom}(T, T_Q(c)[d + 1])$ is zero, for $d \geq c$. Thus, one of the groups $\text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}(a)[b])$ or $\text{Hom}(T_{\widetilde{Q}}, T_{\widetilde{Q}}(-a)[-b])$ will be zero – a contradiction. Hence, all the Tate motives $T_{\mathbf{X}_J}(a)[b]$ are different. \square

For $\mathbf{X} = \{X_i\}_{i \in I}$, denote by $\underline{2}^I$ the set of those $J \subset I$, for which the projector $\mathcal{X}_{\mathbf{X}_J}$ is non-zero.

COROLLARY 4.8. *Let A be an object of $\text{DM}_{\text{gm}}(k; F)$ and $\mathbf{X} = \{X_i\}_{i \in I}$ be some collection such that $\pi_{\mathbf{X}_J}(A) \cong T_{\mathbf{X}_J}(a_J)[b_J]$, for every $J \subset I$. Then A is invertible. Moreover, two such objects are isomorphic if and only if the respective functions $(a_J)[b_J] : \underline{2}^I \rightarrow (\mathbb{Z})[\mathbb{Z}]$ are the same.*

Proof. This follows directly from Propositions 4.2, 4.3 and 4.7. □

Now let A be some object of DQM^{gm} (the category considered by Bachmann). Then A is obtained from motives of finitely many smooth projective quadrics $\{Q_l\}_{l \in L}$ using cone operations and tensor products (as well as direct summands and Tate shifts). Let $Q_l^r, r = 0, \dots, [\dim(Q_l)/2]$, be the quadratic Grassmannians of Q_l . Consider the collection $\mathbf{X}(A) = \{X_i\}_{i \in I}$ of all these Grassmannians.

PROPOSITION 4.9. *Let $F = \mathbb{Z}/2$, and A be an object of DQM^{gm} . Let $\mathbf{X} = \mathbf{X}(A)$. Then $\pi_{\mathbf{X}_J}(A)$ belongs to $\text{Tate}_{\mathbf{X}_J}$, for every $J \subset I$.*

Proof. From [Vis98, Proposition 3.6] we have exact triangles in $\text{DM}(k; \mathbb{Z}/2)$:

$$\bigoplus_{j=0}^r \mathcal{X}_{Q_l^{r/r+1}}(N-j)[2N-2j] \longrightarrow M(Q_l) \otimes \mathcal{X}_{Q_l^{r/r+1}} \longrightarrow \bigoplus_{j=0}^r \mathcal{X}_{Q_l^{r/r+1}}(j)[2j] \longrightarrow \dots,$$

where $\mathcal{X}_{Q_l^{r/r+1}} = \mathcal{X}_{Q_l^r} \otimes \tilde{\mathcal{X}}_{Q_l^{r+1}}$ and $N = \dim(Q_l)$. Hence, $\pi_{Q_l^{r/r+1}}(M(Q_l))$ belongs to the Tate motivic category. And every projector $\pi_{\mathbf{X}_J}$ factors through some $\pi_{Q_l^{r/r+1}}, r = 0, \dots, [\dim(Q_l)/2]$ (for a fixed l). Indeed, if $Q_l^t \in J$ and $Q_l^s \notin J$, for some $t > s$, then the projector $\pi_{\mathbf{X}_J}$ contains the factor $\mathcal{X}_{Q_l^t} \otimes \tilde{\mathcal{X}}_{Q_l^s}$, and so is zero (as $\mathcal{X}_{Q_l^t} \geq \mathcal{X}_{Q_l^s}$; see below). Hence, for a non-zero projector (and fixed l), there exists r such that $Q_l^t \in J$, for $t \leq r$, and $Q_l^s \notin J$, for $s > r$. Then $\pi_{\mathbf{X}_J}$ has the factor $\pi_{Q_l^{r/r+1}}$. Thus, $\pi_{\mathbf{X}_J}(M(Q_l))$ belongs to $\text{Tate}_{\mathbf{X}_J}$, and so does $\pi_{\mathbf{X}_J}(A)$, as $\text{Tate}_{\mathbf{X}_J}$ is closed under the operations mentioned. □

Remark 4.10. The restriction on coefficients is caused by even-dimensional quadrics. For integral coefficients one has to use, in addition, Artin motives corresponding to quadratic extensions. But if we restrict ourselves to odd-dimensional quadrics only, then the statement is true with integral (and so, any) coefficients.

For a collection \mathbf{X} , let the *reduced* collection $\underline{\mathbf{X}}$ be the subset of those X_i for which there are no zero-cycles of degree 1 over k . For a coefficient ring F of prime characteristic, the collection is reduced exactly when $\emptyset \in \underline{2}^I$. If the collection \mathbf{Y} contains \mathbf{X} , we call it a *refinement* of \mathbf{X} .

The following result shows that, at least in working with Pic_{qua} , both approaches are equivalent. For $A \in \text{Pic}_{\text{qua}}$, let $\mathbf{X}(A) = \mathbf{X}$ be the collection such that $\pi_{\mathbf{X}_J}(A) \in \mathbb{T}$, for all J . We know from the proof of Proposition 4.6 that such a collection exists.

PROPOSITION 4.11. *Let $A \in \text{Pic}_{\text{qua}}$, and $\underline{\mathbf{Y}}$ be any refinement of $\mathbf{X}(A)$ reduced. Then $\Phi^k(A) = T(a)[b] \in K^b(\text{Tate}(\mathbb{Z}/2))$ and $\pi_{\underline{\mathbf{Y}}_\emptyset}(A) = T(a)[b] \in \text{DM}_{\underline{\mathbf{Y}}_\emptyset}(k; \mathbb{Z}/2)$, for the same $(a)[b]$.*

Proof. It is sufficient to check this for $A = e^q$, where we get from Proposition 3.3 and the proof of Proposition 4.6 that in both cases

$$(a)[b] = \sum_{l'=0}^{i_W(Q')-1} (m' - 2l')[2m' - 4l' + 1] - \sum_{l=0}^{i_W(Q)-1} (m - 2l)[2m - 4l + 1],$$

where $q' = \langle 1 \rangle \perp -q$, $m' = \dim(Q')$ and $m = \dim(Q)$. □

Thus, to recover $\Phi^k(A)$, we do not need the whole collection \mathbf{Y} , but only $\pi_{\mathbf{Y}_0}$. It is enough to take $Q = \bigsqcup_l Q_l$ the disjoint union of sufficiently many anisotropic quadrics, so that $\tilde{\mathcal{X}}_Q \otimes A \cong T_{\tilde{Q}}(a)[b]$. Then $\Phi^k(A) \cong T(a)[b]$ as well. One can define $\Phi^E(A)$ similarly.

Let us introduce an ordering on Čech simplicial schemes: $\mathcal{X}_R \geq \mathcal{X}_S$ if and only if the projection $\mathcal{X}_R \rightarrow T$ factors through a map $\mathcal{X}_R \rightarrow \mathcal{X}_S$ in $\text{DM}(k; \mathbb{Z}/2)$. This is equivalent to the map $\mathcal{X}_R \otimes \mathcal{X}_S \rightarrow \mathcal{X}_R$ being an isomorphism, or, equivalently, $\tilde{\mathcal{X}}_S \cong \tilde{\mathcal{X}}_R \otimes \tilde{\mathcal{X}}_S$. Since the automorphism group of $\tilde{\mathcal{X}}_R$ is trivial, we can choose these identifications simultaneously for all inequalities in an associative way. Then $\mathcal{X}_R \cong \mathcal{X}_S$ if and only if there are inequalities in both directions. Let E be a finitely generated field extension, and P be a smooth projective variety with $k(P) = E$. Let \mathbf{Q} be the disjoint union of all connected varieties Q with $\mathcal{X}_Q \geq \mathcal{X}_P$ (so it is a smooth variety, but with infinitely many components). Let $\mathcal{X}_{\mathbf{Q}}$ be the motive of the respective Čech simplicial scheme, and $\tilde{\mathcal{X}}_{\mathbf{Q}}$ be the complementary projector. Define the ‘motivic category of an extension’:

$$\text{DM}(E/k; \mathbb{Z}/2) = \mathcal{X}_P \otimes \tilde{\mathcal{X}}_{\mathbf{Q}} \otimes \text{DM}(k; \mathbb{Z}/2). \tag{6}$$

Note that the category $\text{DM}(E/k; \mathbb{Z}/2)$ is non-zero, and by the arguments from the proof of Proposition 4.7, Pic of it contains the subgroup of Tate motives isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let $\varphi_E : \text{DM}(k; \mathbb{Z}/2) \rightarrow \text{DM}(E/k; \mathbb{Z}/2)$ be the natural projection functor. For $A \in \text{Pic}_{\text{qua}}$, we know that $\varphi_E(A) \cong T_E(a)[b]$, for some (unique) $a, b \in \mathbb{Z}$. Then $\Phi^E(A) \cong T(a)[b]$ as well. Indeed, we can reduce to the case of a trivial field extension using the natural functor

$$\text{DM}(E/k; \mathbb{Z}/2) \rightarrow \text{DM}(E/E; \mathbb{Z}/2).$$

Now all the calculations we did in § 3 can be performed with the help of Corollary 4.8 instead of [Bac17, Theorem 31]. Actually, the whole functor Φ^E of Bachmann can be alternatively introduced along these lines.

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