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SEMIREGULAR MODULES AND RINGS

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Introduction. Mares [**9**] has called a projective module *semiperfect* if every homomorphic image has a projective cover and has shown that many of the properties of semiperfect rings can be extended to these modules. More recently Zelmanowitz [**16**] has called a module *regular* if every finitely generated submodule is a projective direct summand. In the present paper a class of *semiregular* modules is introduced which contains all regular and all semiperfect modules. Several characterizations of these modules are given and a structure theorem is proved. In addition several theorems about regular and semiperfect modules are extended.

In Section 2 these results are applied to the study of rings R (called *semiregular rings*) such that ${}_RR$ is semiregular. The basic properties of these rings are derived and theorems about regular and semiperfect rings are extended. It is shown that R is semiregular if and only if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.

Finally endomorphism rings are studied. Conditions under which a module has a semiregular endomorphism ring are given and applied to injective and semiperfect modules. It is shown that a ring is left perfect if and only if every projective left module has a semiregular endomorphism ring. Turning to semiregular modules, it is shown that the endomorphism ring of a semiregular module contains a semiregular ideal provided epimorphisms have left inverses.

Unless stated otherwise, all rings are associative and have identity, all modules are unital left modules and module endomorphisms are written on the right of their arguments.

1. Semiregular modules.

If M is an R-module the *dual* of M will be denoted by $M^* = \mathrm{Hom}_R(M, R)$. A *dual basis* for M is a pair of subsets $\{x_i | i \in I\} \subseteq M$ and $\{\varphi_i | i \in I\} \subseteq M^*$ (indexed by the same set I) such that, for each $x \in M$, $x\varphi_i = 0$ for all but finitely many $i \in I$ and $x = \sum_i (x\varphi_i)x_i$. It is well known that M is (finitely generated) projective if and only if it has a (finite) dual basis. An element x in a module M is called *regular* if $(x\alpha)x = x$ for some $\alpha \in M^*$. Zelmanowitz calls a module *regular* if each of its elements is regular.

LEMMA 1.1. *Let M be a module and let $x \in M$ be a regular element. If $\alpha \in M^*$ satisfies $(x\alpha)x = x$ and if we write $e = x\alpha$, then:*

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(1) $e^2 = e$ *and* $x = ex$.

(2) $Rx \xrightarrow{\alpha} Re$ *is an isomorphism so* Rx *is projective.*

(3) $M = Rx \oplus W$ *where* $W = \{w \in M | (w\alpha)x = 0\}$.

Proof. (1) is obvious and $Rx \xrightarrow{\alpha} Re$ is clearly an epimorphism. If $(rx)\alpha = 0$ then $rx = r(x\alpha)x = 0$ proving (2). Since $y - (y\alpha)x \in W$ for all $y \in M$ we have $M = Rx + W$. This sum is direct since $rx \in W$ implies $rx = r(x\alpha)x = 0$.

Let M be any module. A submodule K of M is said to be *small* in M if $K + N \neq M$ for every submodule $N \neq M$. The *Jacobson radical* of a ring R will be denoted by $J(R)$ and it is easily verified that $J(R)x$ is small in M for each $x \in M$. The following submodules of M are equal: (1) the intersection of all the maximal submodules of M, (2) the sum of all the small submodules of M, and (3) $\{x \in M | Rx$ is small in $M\}$. This submodule is called the *radical* of M and will be denoted by rad M. If $\alpha : M \to N$ is an R-homomorphism it is well known that (rad $M)\alpha \subseteq$ rad N.

A submodule N of a module M is said to *lie over a summand* of M it there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $Q \cap N$ small in M. A *projective cover* of a module K is an R-epimorphism $P \to K \to 0$ with small kernel where P is projective. The following fact will be needed (Bass [2, Lemma 2.3]).

LEMMA 1.2. *If M is projective, a submodule N lies over a summand of M if and only if M/N has a projective cover.*

PROPOSITION 1.3. *If M is any module, the following conditions are equivalent for $x \in M$:*

(1) *Rx lies over a projective summand of M.*

(2) *There exists $\alpha \in M^*$ such that $(x\alpha)^2 = (x\alpha)$ and $x - (x\alpha)x \in$ rad M.*

(3) *There exists a regular element $y \in Rx$ such that $x - y \in$ rad M and $Rx = Ry \oplus R(x - y)$.*

(4) *There exists a regular element $y \in M$ such that $x - y \in$ rad M.*

(5) *There exists $\gamma : M \to Rx$ such that $\gamma^2 = \gamma$, $M\gamma$ is projective and $x - x\gamma \in$ rad M.*

Proof. (1) $\Rightarrow$ (2). Let $M = P \oplus Q$ where $P \subseteq Rx$ is projective and $Rx \cap Q$ is small in M. Then P is finitely generated, so if $\varphi_1, \ldots, \varphi_n; x_1, \ldots, x_n$ is a dual basis of P, write $x_i = r_i x$ and define $\alpha \in P^*$ by $\alpha = \sum_i \varphi_i r_i$. Extend α to M by setting $Q\alpha = 0$. If $x = p + q$, $p \in P$, $q \in Q$ then $(x\alpha)x = (p\alpha)x = p$. Consequently $(x\alpha)^2 = p\alpha = x\alpha$ and $x - (x\alpha)x = q \in Rx \cap Q \subseteq$ rad M.

(2) $\Rightarrow$ (3). Given x and α as in (2) write $y = (x\alpha)x$. Then $(y\alpha)y = y$ so, by Lemma 1.1, $M = Ry \oplus W$ where $W = \{w | (w\alpha)y = 0\}$. The result follows since $Rx \cap W = R(x - y)$.

(3) $\Rightarrow$ (4). This is obvious.

(4) $\Rightarrow$ (5) Let $x - y \in \text{rad } M$ and suppose $(y\alpha)y = y$, $\alpha \in M^*$. If $e = y\alpha$ then $x - ex = (1 - e)(x - y) \in \text{rad } M$ and we claim ex is regular. Indeed $e - x\alpha = (y - x)\alpha \in J(R)$ so, if $(1 - e + x\alpha)b = 1$, then $\alpha b \in M^*$ and $[(ex)\alpha b)](ex) = e(ex) = ex$. Hence, we may assume $y \in Rx$. Write $M = Ry \oplus W$ where Ry is projective and $W = \{w \in M | (w\alpha)y = 0\}$ (Lemma 1.1). If $\gamma : M \to Ry$ is the projection, it remains to show $x - x\gamma \in \text{rad } M$. Write $x = ry + w$, $r \in R$, $w \in W$. Then $0 = (x - ry)\alpha y = (x\alpha)y - ry$ so $x\gamma = ry = (x\alpha)y$. Hence $x - x\gamma = (x - y) - [(x - y)\alpha]y \in \text{rad } M + J(R)y \subseteq \text{rad } M$.

(5) $\Rightarrow$ (1). This is clear.

Definition. An element x in a module M is said to be *semiregular* (in M) if the conditions in Proposition 1.3 are satisfied. A module is called a *semiregular* module if each of its elements is semiregular.

COROLLARY 1.4. *Let M be a module and let $x, y \in M$. If $x - y \in$ rad M and y is semiregular then x is semiregular.*

The regular modules of Zelmanowitz [**16**] are precisely the semiregular modules with zero radical. On the other hand, Lemma 1.2 implies that every semiperfect module is semiregular. A ring is called *local* if it has a unique maximal left ideal and it is clear that, if R is local, then ${}_RR$ is a semiregular module. Many other examples of semiregular modules appear below.

If M is a projective module and $N \subset M$ it is well known that M/N is flat if and only if, given $x \in N$, there exists $\gamma : M \to N$ such that $x = x\gamma$ (see Ware [**13**, Lemma 2.2]).

COROLLARY 1.5 (Ware [**13**, Proposition 2.1]). *A projective module is regular if and only if every homomorphic image is flat.*

Proof. If M is projective, $x \in M$ and M/Rx is flat, let $\gamma : M \to Rx$ satisfy $x = xy$. Clearly $\gamma = \gamma^2$ so $Rx = R\gamma$ is a (projective) direct summand. It follows that M is semiregular and rad $M = 0$. The converse is immediate from Proposition 1.3.

The next result gives some important characterizations of semiregular modules.

THEOREM 1.6. *The following conditions are equivalent for a module M:*

(1) *M is semiregular.*

(2) *If $N \subseteq M$ is a finitely generated submodule there exists $\gamma : M \to N$ such that $\gamma^2 = \gamma$, $M\gamma$ is projective and $N(1 - \gamma) \subseteq$ rad M.*

(3) *Every finitely generated submodule of M lies over a projective summand of M.*

Proof. It is clear that (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) since, if N and γ are as in (2), $N \cap M(1 - \gamma) = N(1 - \gamma)$ and $N(1 - \gamma)$ is small in M (being a finitely generated submodule of rad M). To prove (1) $\Rightarrow$ (2), observe that Proposition 1.3 starts an induction on the number of generators of N. Suppose $N = Rx_0 + \ldots + Rx_n$. Choose $\beta : M \to Rx_n$ such that $\beta^2 = \beta$, $M\beta$ is projective and $x_n(1 - \beta) \in \text{rad } M$. Write $K = Rx_0(1 - \beta) + \ldots + Rx_{n-1}(1 - \beta)$ and, by induction, choose $\delta : M \to K$ such that $\delta^2 = \delta$, $M\delta$ is projective and $K(1 - \delta) \subseteq \text{rad } M$. Define $\gamma = \beta + \delta - \beta\delta$. Then $\gamma^2 = \gamma$ and $M\gamma = M\beta \oplus M\delta$ (since $\delta\beta = 0$). Hence $M\gamma$ is projective and, since $N = K + Rx_n$, it follows that $M\gamma \subseteq N$ and $N(1 - \gamma) = N(1 - \beta)(1 - \delta) \subseteq \text{rad } M$.

COROLLARY 1.7. *A projective module M is semiregular if and only if M/N has a projective cover for every finitely generated (cyclic) sub-module N.*

It is clear that a module has zero radical if each cyclic sub-module is a summand. Hence:

COROLLARY 1.8. (Zelmanowitz [**16**, Theorem 2.2]). *A module is regular if and only if every finitely generated (cyclic) submodule is a projective summand.*

The next result will be used with Corollary 1.4 to prove that a direct sum of semiregular modules is semiregular.

LEMMA 1.9. *Let M be a module and let $x \in M$. If $\alpha \in M^*$ is such that $(x\alpha)^2 = (x\alpha)$ and $x - (x\alpha)x$ is semiregular, then x is semiregular.*

Proof. Write $e = x\alpha$ and choose $\beta \in M^*$ such that $(x - ex)\beta = f$ is an idempotent and $(x - ex) - f(x - ex) \in \text{rad } M$. Then $e + f - fe$ is an idempotent (since $ef = 0$) and $x - (e + f - fe)x \in \text{rad } M$. Define $\gamma = \alpha + (\beta - \alpha(x\beta))(1 - e) \in M^*$. Since $x\gamma = e + f - fe$, the lemma is proved.

THEOREM 1.10. *If $M = \oplus_{i \in I} M_i$ is a direct sum of modules then M is semiregular if and only if each M_i is semiregular.*

Proof. Let N be a direct summand of M and let $x \in N$. Since $\text{rad } N = N \cap \text{rad } M$, it follows easily that x is semiregular in M if and only if x is semiregular in N. Consequently it suffices to prove the theorem for two summands. Hence, let $M = N \oplus K$ where N and K are semiregular modules. Consider $x + y \in M$ where $x \in N$, $y \in K$. Choose $\alpha \in N^*$ such that $(x\alpha)^2 = x\alpha$ and $x - (x\alpha)x \in \text{rad } N$. Extend α to M by defining $K\alpha = 0$. Then $(x + y)\alpha = x\alpha$ is an idempotent so, by Lemma 1.9, it suffices to show that

$$(x + y) - [(x + y)\alpha](x + y) = (x - (x\alpha)x) + (y - (x\alpha)y)$$

is semiregular in M. But $x - (x\alpha)x \in \text{rad } N \subseteq \text{rad } M$ and $y - (x\alpha)y$ is semiregular in K (and hence in M). The result follows by Corollary 1.4.

COROLLARY 1.11. (Zelmanowitz [**16**, Theorem 2.8]). *A direct sum $\oplus_{i \in I} M_i$ is regular if and only if each M_i is regular.*

The next result contains structure theorems for regular and semiperfect modules and for finitely generated semiregular modules.

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THEOREM 1.12. *Let M be a countably generated semiregular module. If* rad M *is small in M then $M \cong \oplus_{i=1}^{\infty} Re_i$ where $e_i^2 = e_i \in R$. In particular, M is projective.*

Proof. By Lemma 1.1 it suffices to show that $M = \oplus_{i=1}^{\infty} Ry_i$ where each y_i is a regular element of M. Let $x_1, x_2, \ldots$ be a generating set for M. Since rad is small, it suffices to find regular elements y_i, submodules $W_i \subset M$ and submodules $K_i \subset \operatorname{rad} M$ such that

(1) $M = Ry_1 \oplus \ldots \oplus Ry_n \oplus W_n$

(2) $Rx_1 + \ldots + Rx_n \subseteq [Ry_1 \oplus \ldots \oplus Ry_n] + K_n$

hold for each n. By Proposition 1.3 write $x_1 = y_1 + z_1$ where $y_1 \in Rx_1$ is regular in M, $z_1 \in \operatorname{rad} M$ and $Rx_1 = Ry_1 \oplus Rz_1$. Since $M = Ry_1 \oplus W_1$ by Lemma 1.1, this starts an inductive construction. Now suppose y_i, W_i and K_i have been constructed, $1 \leqq i \leqq n$. Write $P_n = Ry_1 \oplus \ldots \oplus Ry_n$ and let $\pi : M \rightarrow M$ be the projection with $M\pi = P_n$ and $\ker \pi = W_n$. Then

$$P_n + Rx_{n+1} = P_n \oplus Rx_{n+1}(1 - \pi)$$

so write $t_{n+1} = x_{n+1}(1 - \pi)$. Then $t_{n+1} \in W_n$ and W_n is semiregular so $t_{n+1} = y_{n+1} + z_{n+1}$ where $y_{n+1} \in Rt_{n+1}$ is regular in W_n (and so in M), $z_{n+1} \in \operatorname{rad} W_n \subseteq \operatorname{rad} M$ and $Rt_{n+1} = Ry_{n+1} \oplus Rz_{n+1}$. If we write $W_n = Ry_{n+1} \oplus W_{n+1}$ it is clear that $M = Ry_1 \oplus \ldots \oplus Ry_{n+1} \oplus W_{n+1}$ and

$$Rx_1 + \ldots + Rx_{n+1} \subseteq [Ry_1 \oplus \ldots \oplus Ry_{n+1}] + (K_n + Rz_{n+1}).$$

This completes the construction with $K_{n+1} = K_n + Rz_{n+1}$.

COROLLARY 1.13. *Every finitely generated semiregular module is projective of the form $Re_1 \oplus \ldots \oplus Re_n$, $e_i^2 = e_i \in R$.*

It is well known that every projective module is the direct sum of countably generated submodules. Hence:

COROLLARY 1.14. *A projective semiregular module with small radical is a direct sum of cyclic submodules.*

It was observed by Mares [**9**] that the radical of a semiperfect module is small (this follows from Lemma 3.6 below). The following structure theorems are now apparent:

COROLLARY 1.15. (1) (Zelmanowitz [**16**, Theorem 1.6]). *Every countably generated regular module is projective.*

(2) (Ware [**13**, Theorem 2.12]). *Every projective regular module is a direct sum of cyclic left ideals.*

(3) (Mares [**9**]). *Every semiperfect module is a direct sum of cyclic left ideals.*

We do not know whether the hypothesis that the radical is small can be deleted from Corollary 1.14. The hypothesis is unnecessary if the ring R is semiregular (that is $_RR$ is a semiregular module) by a result of Warfield (see

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Theorem 2.11 below). We conjecture that it holds if idempotents can be lifted modulo $J(R)$.

The next result will be used below and may have some independent interest.

LEMMA 1.16. *Let M be a projective module. Suppose $M = P + K$ where P and K are submodules and P is a direct summand of M. There exists a submodule $Q \subseteq K$ such that $M = P \oplus Q$.*

Proof. Let $\gamma^2 = \gamma : M \rightarrow M$ be any projection with $M\gamma = P$. If $\varphi : M \rightarrow M/K$ is the natural map let $\alpha : M \rightarrow M$ satisfy $\alpha\gamma\varphi = \varphi$ (see diagram). Define $\delta = \gamma + (1 - \gamma)\alpha\gamma$. Then $\delta^2 = \delta$, $M\delta = M\gamma = P$ and $\ker \delta = M(1 - \delta) = M(1 - \gamma)(1 - \alpha\gamma) \subseteq \ker \varphi = K$. The proof is complete with $Q = \ker \delta$.

$$\begin{tikzcd}
 & & M \arrow[dll, dashed, "\alpha"'] \arrow[d, "\varphi"] & \\
M \arrow[r, "\gamma"] & P \arrow[r, "\varphi|P"] & M/K \arrow[r] & 0
\end{tikzcd}$$

Mares [**9**] proves that a projective module M is semiperfect if and only if it has the following three properties: (1) $M/\text{rad } M$ is semisimple (that is every submodule is a summand), (2) each indempotent in end$(M/\text{rad } M)$ is induced by an idempotent in end M, and (3) rad M is small in M. The analog for semiregular modules follows (recall that the radical need not be small).

PROPOSITION 1.17. *Let M be a projective module such that* rad M *is small in M. Let $\varphi : M \rightarrow M/\text{rad } M$ be the natural map. Then M is semiregular if and only if it satisfies the following two conditions:*

(1) *Every finitely generated submodule of $M\varphi$ is a direct summand.*

(2) *If $M\varphi = A \oplus B$ where A is finitely generated, there exists a decomposition $M = P \oplus Q$ such that $P\varphi = A$ and $Q\varphi = B$.*

Proof. Suppose M is semiregular and let $A \subseteq M\varphi$ be finitely generated. Write $A = N\varphi$ where $N \subseteq M$ is finitely generated. By Theorem 1.6 choose $\gamma^2 = \gamma : M \rightarrow N$ such that $N(1 - \gamma) \subseteq \text{rad } M$. Clearly $M = N + \ker \gamma$ and it follows easily that $M\varphi = A \oplus (\ker \gamma)\varphi$. This proves (1).

Now assume $M\varphi = A \oplus B$ where A is finitely generated. Choose N and γ as above. If $y \in M$ choose $x \in N$ and $b \in B$ such that $y\varphi = x\varphi + b$. Since $x\varphi = x\gamma\varphi$, this means $M = N\gamma + B\varphi^{-1}$ where $B\varphi^{-1} = \{x \in M | x\varphi \in B\}$. Since $N\gamma = M\gamma$ is a direct summand of M, apply Lemma 1.16 to write $M = N\gamma \oplus Q$ where $Q \subseteq B\varphi^{-1}$. Then (2) follows because $N\gamma\varphi = N\varphi = A$ and $Q\varphi \subseteq B$.

Conversely, assume (1) and (2) hold. If $N \subseteq M$ is finitely generated, there exists a direct summand Q of M such that $M\varphi = N\varphi \oplus Q\varphi$. Since rad M is small, this means $N \cap Q$ is small and $M = N + Q$. The proof is completely by Lemma 1.16.

The proof shows that every semiregular module has property (1) and every

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projective semiregular module has property (2). We do not know an example showing that the radical must be small for the converse to hold.

COROLLARY 1.18. *A finitely generated projective module M is semiregular if and only if it satisfies the following conditions:*

(1) *Every finitely generated submodule of $M/$*rad *M is a direct summand.*

(2) *Direct decompositions of $M/$*rad *M can be lifted to M.*

A variation of this was announced by Jansen [**6**]. He replaces (1) by the condition that $M/$rad M is regular in the sense of Fieldhouse ([**5**, Section 8]).

The next result characterizes the semiperfect modules among the projective semiregular ones.

PROPOSITION 1.19. *A projective module M is semiperfect if and only if it is semiregular,* rad *M is small in M and $M/$*rad *M is semisimple.*

Proof. Mares ([**9**, Theorem 5.2]) shows that a direct sum of semiperfect modules is semiperfect if and only if the radical is small. If M is semiperfect, $M/$rad M is semisimple by the proof of Proposition 1.17. For the converse we may assume M is finitely generated by Mares' theorem and Theorem 1.12. But then, if the conditions hold, every submodule of $M/$rad M is finitely generated so the proof of Proposition 1.17 goes through.

Note that the theorem of Mares used in the proof is in fact a consequence of Proposition 1.19.

If idempotents can be lifted modulo the Jacobson radical a better version of Proposition 1.17 is possible.

PROPOSITION 1.20. *Let R be a ring in which indempotents can be lifted modulo $J(R)$. The following conditions are equivalent for a projective R-module M:*

(1) *M is semiregular.*

(2) *Every finitely generated submodule of $M/$*rad *M is a direct summand.*

(3) *Every cyclic submodule of $M/$*rad *M is direct summand.*

Proof. We prove (3) $\Rightarrow$ (1). Let $x \in M$ and let Q be a submodule such that $M = Rx + Q$ and $Rx \cap Q \subseteq$ rad M. Define $\psi : R \rightarrow Rx$ by $r\psi = rx$ and let $\varphi : M \rightarrow M/Q$ be the natural map (see diagram). If $\alpha \in M^*$ is such that $\alpha\psi\varphi = \varphi$ then $x - (x\alpha)x \in Rx \cap Q \subseteq$ rad M and so $x\alpha - (x\alpha)^2 \in J(R)$. By hypothesis choose $e^2 = e \in R$ such that $e - x\alpha \in J(R)$. Then $u = 1 - e + xa$ is a unit and $u^{-1}(x\alpha)e = e$. Define $\beta \in M^*$ by $\beta = \alpha eu^{-1}$. Then $(x\beta)^2 = x\beta$ and $x\beta = (x\alpha)eu^{-1} \equiv (x\alpha)e \equiv x\alpha$ modulo $J(R)$. Hence $x - (x\beta)x = (x - (x\alpha)x) + (x\alpha - x\beta)x \in$ rad M.

$$\begin{tikzcd} & & M \arrow[dll, dashed, "\alpha"'] \arrow[d, "\varphi"] & \\ R \arrow[r, "\psi"] & Rx \arrow[r, "\varphi|Rx"] & M/Q \arrow[r] & 0 \end{tikzcd}$$

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Let $N \subseteq M$ be modules. A submodule K is said to be a *complement* of N in M if $M = N + K$ and $N \cap K$ is small in K. Kasch and Mares [**8**] have shown that a projective module is semiperfect if and only if every submodule has a complement.

PROPOSITION 1.21. *A finitely generated module M is semiregular if and only if it is projective and every finitely generated (cyclic) submodule has a complement in M.*

Proof. The necessity of the conditions is clear. Conversely, let $N \subseteq M$ be finitely generated and let K be a complement of N in M. If $\varphi : M \to M/N$ is the natural map there exists $\alpha : M \to K$ such that $\alpha\varphi = \varphi$ (see diagram). This

$$\begin{tikzcd}
& M \arrow[dl, dashed, "\alpha"'] \arrow[d, "\varphi"] & \\
K \arrow[r, "\varphi|K"] & M/N \arrow[r] & 0
\end{tikzcd}$$

means $K = K\alpha + (N \cap K)$ and consequently that $K = K\alpha = M\alpha$ is finitely generated. But then K also has a complement in M and so the argument in [**8**] goes through to show K is a direct summand of M. Thus

$$K \xrightarrow{\varphi|_K} M/N \longrightarrow 0$$

is a projective cover as required.

This was announced for projective modules by Jansen [**6**]. We do not know whether an arbitrary projective module is semiregular if every finitely generated submodule has a complement. The answer is affirmative for noetherian rings.

PROPOSITION 1.22. *If R is a left noetherian ring, the following conditions are equivalent for a projective R-module M:*

(1) *M is semiregular.*
(2) *Every finitely generated submodule has a complement in M.*
(3) *Every cyclic submodule has a complement in M.*

Proof. We need only show (3) $\Rightarrow$ (1). If $x \in M$ let K be a complement of Rx in M. Let $\varphi : M \to M/Rx$ be the natural map and, as in the preceding proof, choose $\alpha : M \to K$ such that $\alpha\varphi = \varphi$ and $K = K\alpha = M\alpha$. Then $M = \ker \alpha + K$ and $\ker \alpha \subseteq Rx$ so it remains to show $\ker \alpha \cap K = 0$. Now $\ker \alpha \subseteq \ker \alpha^2 \subseteq \ldots \subseteq Rx$ so, since R is noetherian, let $\ker \alpha^n = \ker \alpha^{n+1}$. If $x \in \ker \alpha \cap K$ then $x \in K\alpha^n$ so let $x = y\alpha^n$, $y \in K$. Then $y \in \ker \alpha^{n+1} = \ker \alpha^n$ and consequently $x = 0$.

2. Semiregular rings. An important application of the results of the previous section is to the study of rings R such that ${}_RR$ is a semiregular module.

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This condition turns out be left-right symmetric and several characterizations of such rings will be given.

An element a of a ring R is said to be *regular* (in the sense of von Neumann) if $aba = a$ for some $b \in R$. If each element of a ring R is regular, R is said to be a *regular ring*. It is clear that an element a in a ring R is regular if and only if it is regular in ${}_RR$ (regular in R_R). The following fact is equally easy to verify.

LEMMA 2.1. *Let a be an element of a ring R. Then a is semiregular in ${}_RR$ if and only if there exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$. An analogous result holds for R_R.*

PROPOSITION 2.2. *The following are equivalent for an element a of a ring R:*
(1) *There exists $e^2 = e \in aR$ such that $(1 - e)a \in J(R)$.*
(2) *There exists $e^2 = e \in Ra$ such that $a(1 - e) \in J(R)$.*
(3) *There exists a regular element $b \in R$ with $a - b \in J(R)$.*
(4) *There exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$.*

Proof. In the presence of Lemma 2.1, Proposition 1.3 gives (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4) is easily verified. The rest follows by symmetry.

Definition. An element a of a ring R is called *semiregular* (in R) if it satisfies these conditions. A ring is a *semiregular ring* if each of its elements is semiregular.

It is clear that every regular ring is semiregular. A ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents can be lifted modulo $J(R)$. Since this is equivalent to ${}_RR$ being a semiperfect module, every semiperfect ring is semiregular. (Another proof of this is given in Theorem 2.9 below).

The next two results are easy consequences of Proposition 2.2.

COROLLARY 2.3. *If R is a semiregular ring so also is every homomorphic image of R and every subring of the form eRe, $e^2 = e$.*

COROLLARY 2.4. *If $a - b \in J(R)$ and b is semiregular, so is a.*

An idempotent e in a ring R is called *primitive* if eRe has no proper idempotents and e is called *local* if eRe is a local ring. The following result is well known for regular and semiperfect rings.

COROLLARY 2.5. *In a semiregular ring every primitive idempotent is local.*

Proof. If $e^2 = e \in R$ is primitive let $a \in eRe$, $a \notin J(eRe)$. Choose $f^2 = f \in aR$ with $a - fa \in J(R)$. Then $f \neq 0$ (since $a \notin J(R)$) and $ef = f$. Since e is primitive it follows that $fe = e$ and consequently that $aR = eR$. Hence a has a right inverse in eRe.

A useful device in the study of regular rings is *McCoy's lemma* which states that an element a in a ring R is regular if $a - aba$ is regular for some $b \in R$. The analog of McCoy's lemma for semiregular rings seems to be the following:

LEMMA 2.6. *Let R be a ring and let $a \in R$. If there exists $e^2 = e \in aR$ such that $(1 - e)a$ is semiregular then a is semiregular.*

Proof. Choose an idempotent $f \in (1 - e)aR$ such that $(1 - f)(1 - e)a \in J(R)$. If $g = e + f - fe$ then $g^2 = g \in aR$ and $(1 - g)a \in J(R)$.

McCoy's lemma can be used to show that the $n \times n$ matrix ring $M_n(R)$ over a regular ring is again regular. The analogous result for semiregular rings is also true.

PROPOSITION 2.7. *If R is semiregular so is $M_n(R)$.*

Proof. It suffices to prove the result for $n = 2$. Write $M = M_2(R)$ and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M$. Choose $e^2 = e = cr$, $r \in R$, such that $c - ec \in J(R)$. Then $E = \begin{bmatrix} 0 & are \\ 0 & e \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & re \\ 0 & 0 \end{bmatrix}$ is an idempotent so, by Lemma 2.6, it suffices to show that $(1 - E)A$ is semiregular. But $(1 - E)A \in \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} + J(M)$ for some $a', b', d' \in R$ so, by Corollary 2.4, we may assume $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. In this case choose $f^2 = f = as$, $s \in R$, such that $a - fa \in J(R)$. Then $F = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$ is an idempotent and $(1 - F)A \in \begin{bmatrix} 0 & b' \\ 0 & d' \end{bmatrix} + J(M)$ for some $b', d' \in R$ so we may assume $A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$. The process clearly continues.

If R is semiregular so is eRe for any $e^2 = e \in R$. This combined with Proposition 2.7 gives:

COROLLARY 2.8. *Semiregularity is a Morita invariant.*

The equivalence of (2), (4) and (5) in the next result was first proved by Oberst and Schneider ([**11**, Satz 1.2]) who called these semiregular rings *F-semiperfect*. Conditions (2) and (4) show that these rings generalize the semiperfect rings in a natural way. A module M is said to be *finitely related* if there is an exact sequence $0 \to K \to F \to M \to 0$ with F free and both F and K finitely generated.

THEOREM 2.9. *The following statements (and their left-right analogs) are equivalent for a ring R:*

(1) *R is semiregular.*
(2) *$R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$.*
(3) *Every finitely generated (cyclic) left ideal lies over a direct summand.*
(4) *Every finitely related (finitely related and cyclic) left R-module has a projective cover.*
(5) *Every finitely generated (cyclic) left ideal has a complement in R.*

Proof. It is clear that (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (5) by Proposition 1.21. If R is semiregular every free R-module is semiregular by Theorem 1.10 so (1) $\Leftrightarrow$ (4). Since (2) $\Rightarrow$ (1) by Proposition 1.20, it remains to prove (1) $\Rightarrow$ (2). If R is semiregular then $R/J(R)$ is regular by Proposition 2.2. Let $a \in R$, $a^2 - a \in J(R)$ and choose $e^2 = e \in aR$ with $a - ea \in J(R)$. If $f = e + ea(1 - e)$ then $f^2 = f$ and (since $e - ae \in J(R)$) $f - a \in J(R)$.

If L is a left ideal of R, it is easy to see that L lies over a summand of R if and only if there exists an idempotent $e \in L$ and a left ideal $M \subseteq J(R)$ such that $L = Re \oplus M$. Furthermore, it is clear from (2) of Theorem 2.9 that a semiregular ring is semiperfect if and only if it contains no infinite family of orthogonal idempotents.

COROLLARY 2.10. *A ring is semiperfect if and only if every countably generated left ideal lies over a direct summand.*

Proof. If R satisfies the condition, let $e_1, e_2, \ldots$ be orthogonal idempotents. Put $L = Re_1 + Re_2 + \ldots$ and let $L = Re + M$, $e^2 = e$, $M \subseteq J(R)$. If $e = r_1e_1 + \ldots + r_ne_n$ then $ee_k = 0$ for all $k > n$ and so $e_k - e_ke$ is an idempotent in $J(R)$. Consequently $e_k = 0$ for all $k >$ n and the result follows.

The following theorem is due to Warfield and is included here for completeness.

THEOREM 2.11. (Warfield [**14**, Theorem 3]). *If R is a semiregular ring, then every projective module is isomorphic to a direct sum of left ideals of the form Re, $e^2 = e$.*

The cases when R is local or regular are due to Kaplansky ([**7**, Theorems 2 and 4]) and the semiperfect case is due to Mueller ([**10**, Theorem 3]). The case when R is semiregular and right coherent was proved by Oberst and Schneider (2.4 of [**11**]).

It is worth noting here that, if $e^2 = e \in R$, the left ideal Re is a semiregular module if and only if each element of Re is a semiregular element of R (Proposition 2.2).

A module M is *local* if it is projective and $M = Rx$ for each $x \in M - \text{rad } M$. (Recall that $M \neq \text{rad } M$ for a projective module $M \neq 0$ by Proposition 2.7 of Bass [**2**]). The equivalence of (2) and (3) in the next result was given by Ware ([**13**, Theorem 4.2]) but the present proof is shorter. Denote the endomorphism ring of a module M by end M.

PROPOSITION 2.12. *The following are equivalent for a module $M \neq$* rad *M.*

(1) *M is semiregular and indecomposable.*

(2) *M is projective and* end *M is a local ring.*

(3) *M is local.*

Proof. (1) $\Rightarrow$ (2). Let $N \subseteq M$ be finitely generated. Then N lies over a summand so $N \subseteq \text{rad } M$ or $N = M$. If follows that M is cyclic (since $M \neq$

rad M) and hence projective. But then each element outside the ideal $A = \{\alpha \in \text{end } M | M\alpha \subseteq \text{rad } M\}$ has a left inverse, proving (2).

(2) $\Rightarrow$ (3). If $x \notin$ rad M let $x \notin K$ where K is maximal in M. If $\varphi : M \rightarrow M/K$ is natural then $\varphi : Rx \rightarrow M/K$ is an epimorphism so there exists $\alpha : M \rightarrow Rx$ such that $\alpha\varphi = \varphi$. Since $\varphi \neq 0$ it follows that $\alpha \notin J(\text{end } M)$. But then α is a unit by (2) and so $Rx = M$.

(3) $\Rightarrow$ (1). This is immediate since rad M is small.

The next result is immediate and provides an alternative proof of Corollary 2.5.

COROLLARY 2.13. *Over a semiregular ring every indecomposable projective module is local.*

A well known theorem of Auslander, McLaughlin and Connell asserts that, if G is a group and R is a ring, the group ring RG is regular if and only if R is regular, G is locally finite and each element of G is a unit in R. An extension of this to semiregular rings is impossible as the next example shows.

Example 2.14. Woods [**15**] has shown that, if R is the ring of rational numbers with denominator prime to 7 and G is the group of order three, then RG is not semiperfect. Since G is (locally) finite it follows that $J(R)G \subseteq J(AG)$ and this is equality in this case since $RG/J(R)G \cong [R/J(R)]G$ is semisimple. Hence $RG/J(RG)$ is semisimple and it follows that RG is not semiregular.

If L is a left ideal of a ring R the *idealizer* of L is defined by

$$I(L) = \{a \in R | La \subseteq L\}.$$

It is known [**1**] that, if R is semiperfect and L is maximal then $I(L)$ is again semiperfect. However $I(L)$ can fail to be semiregular even when L is maximal and R is regular.

Example 2.15. Let F be a field, $M = M_2(F)$, $K = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. Then K is a maximal left ideal in M and the idealizer $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is not regular. Let $R = \{(x_1, \ldots, x_n, x, x, \ldots) | x_i, x \in M, n \geqq 1\}$. With component-wise operations this is a regular ring and $L = \{(x_1, \ldots, x_n, x, x, \ldots) | x \in K\}$ is a maximal left ideal. It is easily verified that $I(L) = \{(x_1, \ldots, x_n, x, \ldots) | x \in I\}$ and $J[I(L)] = 0$ since each non-zero ideal of $I(L)$ contains a non-zero idempotent. Hence the fact that $I(L)$ is not regular (it has I as homomorphic image) shows it is not semiregular.

3. Endomorphism rings. If M is a module, denote the endomorphism ring of M by end M. The module M will be called *direct-projective* if, given any direct summand P of M with projection $\pi : M \rightarrow P$ and any epimorphism $\alpha : M \rightarrow P$, there exists $\beta \in$ end M such that $\beta\alpha = \pi$. Dually, M is *direct-injective* if, given any direct summand P of M with inclusion $i : P \rightarrow M$ and

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any monomorphism $\alpha : P \to M$, there exists $\beta \in \text{end } M$ such that $\alpha\beta = i$. It is clear that projective (injective) modules are direct-projective (direct-injective). In addition, it is not difficult to verify that a module M is direct-projective (direct-injective) provided $\ker \alpha$ (respectively $M\alpha$) is a direct summand for each $\alpha \in \text{end } M$. In particular, any module M for which end M is a regular ring is both direct-projective and direct-injective.

Let $N \subseteq M$ be modules. N is said to be *large* in M if $N \cap K \neq 0$ for each non-zero submodule K of M. N is said to *lie under a direct summand* of M if there exists a decomposition $M = P \oplus Q$ where $N \subseteq P$ and N is large in P. If M is injective it is well known that every submodule lies under a direct summand of M.

THEOREM 3.1. *Let M be a module. Write $E =$ end M and put $A = \{\alpha \in E | M\alpha$ is small in $M\}$ and $B = \{\alpha \in E | \ker \alpha$ is large in $M\}$.*

(1) *If M is direct-projective then $A \subseteq J(E)$. Moreover E is semiregular and $A = J(E)$ if and only if $M\alpha$ lies over a direct summand of M for all $\alpha \in E$.*

(2) *If M is direct-injective then $B \subseteq J(E)$. Moreover E is semiregular and $B = J(E)$ if and only if* $\ker \alpha$ *lies under a direct summand of M for each $\alpha \in E$.*

Proof. We prove only (1) as the proof of (2) is dual. If $\alpha \in A$ then $M(1 - \alpha) = M$ (since $M = M\alpha + M(1 - \alpha)$) and so $1 - \alpha$ has a left inverse (since M is direct-projective). The fact that A is an ideal of E implies $A \subseteq J(E)$.

Suppose now that $M\alpha$ lies over a direct summand of M for each $\alpha \in E$. Let $M = P \oplus Q$ where $P \subseteq M\alpha$ and $M\alpha \cap Q$ is small in M. Let $\pi^2 = \pi \in E$ be such that $M\pi = P$ and $\ker \pi = Q$. Then $M \xrightarrow{\alpha\pi} P$ is an epimorphism so there exists $\beta \in E$ such that $\beta\alpha\pi = \pi$. Define $\tau = \pi\beta\alpha$. Then $\tau^2 = \tau \in E\alpha$, $M\tau \subseteq M\alpha$ and $\ker \tau = \ker \pi = Q$. Hence $M(\alpha - \alpha\tau) \subseteq M\alpha \cap Q$ so that $\alpha - \alpha\tau \in A$. But this implies E/A is regular and so $A = J(E)$. Thus E is semiregular by Proposition 2.2.

Conversely, assume E is semiregular and $A = J(E)$. Given $\alpha \in E$ there exists $\pi^2 = \pi \in E\alpha$ such that $\alpha - \alpha\pi \in J(E) = A$. Then $M = M\pi \oplus M(1 - \pi)$ where $M\pi \subseteq M\alpha$ and $M\alpha \cap M(1 - \pi) = M\alpha(1 - \pi)$ is small in M.

If M is projective (injective) it is well known that, in the notation of the theorem, $J(E) = A$ (respectively $J(E) = B$). Hence:

COROLLARY 3.2. *If M is injective then* end *M is semiregular.*

COROLLARY 3.3. *If M is projective,* end *M is semiregular if and only if $M\alpha$ lies over a direct summand for all $\alpha \in E(M)$. In particular, if M is semiperfect then* end *M is semiregular.*

A module is called *finite-dimensional* if it contains no infinite direct sum of non-zero submodules. In this case it is clear that end M has no infinite family of orthogonal idempotents so:

COROLLARY 3.4. *If a finite-dimensional module M is either injective or semi-perfect,* end M *is a semiperfect ring.*

The following result (announced by Jansen [**6**]) is immediate from Corollary 3.3 and Theorem 1.6.

COROLLARY 3.5. *If M is semiregular and finitely generated then* end M *is a semiregular ring.*

This includes the result of Ware ([**13**, Theorem 3.6)] that end M is regular if M is regular and finitely generated. It also provides another proof that, if R is a semiregular ring, so also is $M_n(R)$.

LEMMA 3.6. *If M is a projective module and* end M *is a semiregular ring then* rad M *is small in M.*

Proof. If rad $M + K = M$ and $\varphi : M \to M/K$ is the natural map there exists $\alpha : M \to$ rad M with $\alpha\varphi = \varphi$. Choose $\pi^2 = \pi \in$ (end $M)\alpha$ with $\alpha - \alpha\pi \in J($end $M)$. Then $M\pi \subseteq M\alpha \subseteq$ rad M so $M\pi = 0$ by Proposition 2.7 of Bass [**2**]. But then $\alpha \in J($end $M)$ and so $\varphi = 0$.

COROLLARY 3.7. *If R is a semiperfect ring, the following are equivalent for a projective R-module M:*

(1) *M is semiperfect.*

(2) end *M is semiregular.*

(3) rad *M is small.*

Proof. $M/$rad M is semisimple since R is semiperfect and M is semiregular since R is semiregular. Hence (3) $\Rightarrow$ (1) by Proposition 1.19.

The endomorphism ring of a regular projective module need not be a semi-regular ring. Indeed:

Example 3.8. Ware ([**13**, Example 3.4]) gives an example of a regular ring R and a projective regular module $M = P \oplus Q$ such that end $P \cong R \cong$ end Q but end M is not regular. Since $J(R) = 0$ it follows that rad $M = 0$ and consequently that $J($end $M) = 0$. This means end M is not semiregular.

If M is a finitely generated projective module and end M is a semiregular ring, we do not know whether M is necessarily a semiregular module. In this connection, it is known that end M is semiperfect if and only if M is semiperfect and finitely generated (see Ware [**13**, Proposition 5.1]).

An ideal A in a ring R is *left T-nilpotent* if, given elements $a_1, a_2, \ldots$ from A, there exists an integer n such that $a_1a_2 \ldots a_n = 0$. A ring R is *left perfect* if $R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent or, equivalently, if every left R-module has a projective cover. These are just the rings for which endomorphism rings of projective modules are semiregular.

THEOREM 3.9. *The following are equivalent for a ring R:*

(1) *R is left perfect.*

(2) *Every projective left R-module is semiperfect.*

(3) end *M is semiregular for all projective left R-modules M.*

(4) end *M is semiregular for a countably generated free left R-module M.*

Proof. (1) $\Rightarrow$ (2) by Lemma 1.2, (2) $\Rightarrow$ (3) by Corollary 3.3 and (3) $\Rightarrow$ (4) is obvious. Given (4) let F be free with basis $x_1, x_2, \ldots$. Then rad F is small in F by Lemma 3.6 and it is well known that this implies $J(R)$ is left T-nilpotent. Let $L = Ra_1 + Ra_2 + \ldots$ be a countably generated left ideal of R and define $\alpha : F \rightarrow F$ by $(\sum r_i x_i)\alpha = (\sum r_i a_i)x_1$. Then $F\alpha$ lies over a direct summand of F (by Corollary 3.3) and hence lies over a direct summand of Rx_1. But the isomorphism $Rx_1 \rightarrow R$ given by $rx_1 \mapsto r$ carries $F\alpha$ to L so R is semiperfect by Corollary 2.10. Hence (4) $\Rightarrow$ (1).

COROLLARY 3.10. *A ring is semisimple if and only if some countably generated free module has regular endomorphism ring.*

Proof. If M is free and $E =$ end M is regular then $R \cong \pi E \pi$ for some $\pi^2 = \pi \in E$ so $J(R) = 0$.

Corollary 3.10 was proved by Cukerman [**4**], then by Shanny [**12**], and finally by Ware [**13**]. The proof of Theorem 3.9 is modeled on the proof of Shanny.

A ring R, possibly with no identity, is called *semiregular* if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. The proof of (1) $\Leftrightarrow$ (2) in Theorem 2.9 shows that this is equivalent to each element of R having the equivalent properties in Proposition 2.2.

THEOREM 3.11. *Let M be a semiregular module such that every epimorphism in* end *M has a left inverse. Write $G = \{\alpha \in$ end $M | M\alpha$ is noetherian$\}$. Then G is a semiregular ring and is an ideal of* end *M.*

Proof. Write $E =$ end M. It is clear that G is an ideal of E so let $\alpha \in G$. Since $M\alpha$ is finitely generated, write $M = P \oplus Q$ where $P \subseteq M\alpha$ is projective and $M\alpha \cap Q$ is small in M. Let π be the projection with $M\pi = P$ and ker $\pi = Q$. There exists $\beta : P \rightarrow M$ such that $\beta\alpha = \pi$ (see diagram). If we set $Q\beta = 0$

$$\begin{tikzcd}
& P \arrow[ld, dashed, "\beta"'] \arrow[d, "\pi|_P"] & \\
M \arrow[r, "\alpha"] & M\alpha \arrow[r] & 0
\end{tikzcd}$$

then $\beta \in G$ because $P \subseteq M\alpha$ is noetherian. Moreover $M(\alpha - \alpha\pi) \subseteq M\alpha \cap$ ker $\pi = M\alpha \cap Q$ so $\alpha - \alpha\pi \in A \cap G$ where $A = \{\gamma \in E | M\gamma$ is small in $M\}$. This A is an ideal of E and we claim that $J(G) = A \cap G$. We have shown that $G/(A \cap G)$ is regular so $J(G) \subseteq A \cap G$. On the other hand, the hypothesis

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that epimorphisms in E have left inverses guarantees that $A \subseteq J(E)$ and so $A \cap G \subseteq J(E) \cap G = J(G)$.

If M is regular, Zelmanowitz ([**16**, Theorem 4.3]) has shown that G is a regular ideal of end M without using the condition that epimorphisms of M have left inverses. This lends credence to the conjecture that this hypothesis is unnecessary in Theorem 3.11.

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