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SMALL EMBEDDINGS OF PARTIAL DIRECTED CYCLE SYSTEMS

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In this paper, a generalisation of the Andersen, Hilton, Rodger Theorem for embedding partial idempotent latin squares is proved. This result is then used to prove that a partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order $(2n+1)m$ if m is odd, of order $2nm$ if $m \geqslant 8$ is even, $12n+1$ if $m=6$ and approximately $2n+\sqrt{2n}$ if $m=4$.

1. INTRODUCTION

By a *cyclic triangle* of the complete directed graph D_n (based on the set S) is meant a collection of three directed edges of the form $\{(a,b),(b,c),(c,a)\}$, where a, b, and c are three *distinct* elements of S. We will denote the cyclic triangle $\{(a,b),(b,c),(c,a)\}$ by $(a,b,c),(b,c,a)$ or (c,a,b). A *Mendelsohn triple system* (MTS) is a pair (S,T), where T is a collection of edge disjoint cyclic triangles which partition the edge set of D_n with vertex set S. The number $|S|=n$ is called the *order* (or size) of the Mendelsohn triple system (S,T) and it is (by now) well-known [**9**] that the *spectrum* of MTSs (that is, the set of all n such that a MTS of order n exists) is precisely the set of all $n \equiv 0$ or $1 \pmod 3$, *except* $n=6$ for which no such system exists. It is trivial to see that if (S,T) is a MTS of order n then $|T| = n(n-1)/3$.

A *partial* MTS of order n is a pair (S,P), where P is a collection of edge disjoint cyclic triangles of the edge set of D_n with vertex set S. The difference between a (complete) MTS and a partial MTS is that the edge disjoint cyclic triangles belonging to a partial MTS do not necessarily include all of the edges of D_n.

EXAMPLE 1.1. (S,P) is a partial MTS of order 5, where $S=\{1,2,3,4,5\}$ and $P=\{(1,2,3),(3,2,4),(3,1,5)\}$.

Now, given a partial MTS (S,P) the obvious question of *completion* arises. That is, can we decompose $E(D_n)\backslash E(P)$ into cyclic triangles? The above example shows that this cannot be done in general since $|S|=5$ is *not* the order of a MTS. Since a partial MTS cannot necessarily be completed to a MTS, the problem of whether or not a partial MTS can always be *embedded* in a (complete) MTS is immediate. The partial

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MTS (S,P) is said to be *embedded* in the MTS (S^*,T) provided $S \subseteq S^*$ and $P \subseteq T$. Naturally, we would like $|S^*|$ to be as small as possible.

In 1971 Lindner [4] showed that a partial MTS of order n can be embedded in a MTS of order $\leqslant 2^{2n}$. In 1976 Lindner and Cruse [5] reduced the size of the containing MTS to $12n$. This was subsequently reduced to $6n+3$ [1] in 1982 by Andersen, Hilton and Rodger *and finally* in 1986 to the smallest admissible order $\geqslant 4n$ by Rodger [10]. Although the best possible embedding is the smallest admissible order $\geqslant 2n+1$, the approximately $4n$ embedding by C.A. Rodger is the best to date. This is where the partial MTS embedding problem stands at the moment. (We remark that if the partial system is complete, then the best possible embedding can be obtained. In particular D.G. Hoffman and C.C. Lindner proved in 1981 that a MTS of order n can be embedded in a MTS of order t for every admissible $t \geqslant 2n+1$ [3].)

A *directed* m-cycle of D_n (based on S) is a collection of m directed edges of the form $\{(x_1,x_2), (x_2,x_3), (x_3,x_4), \ldots, (x_{m-1},x_m), (x_m,x_1)\}$ where $x_1, x_2, x_3, \ldots, x_m$ are m *distinct* elements of S. We will denote this directed m-cycle by any cyclic shift of $(x_1, x_2, x_3, \ldots, x_m)$. A *directed m-cycle system* ($mDCS$) is a pair (S,C), where C is a collection of edge disjoint directed m-cycles which partition the edge set of D_n with vertex set S. As with MTSs, the number $|S| = n$ is called the *order* (or *size*) of the $mDCS$ (S,C) and of course $|C| = n(n-1)/m$.

Now a Mendelsohn triple system is a directed 3-cycle system. Since there is nothing particularly sacred about the number 3, we can ask the same questions for directed m-cycle systems that are asked for directed 3-cycle systems. In particular, for a given $m \geqslant 4$, we can ask for the spectrum (that is, the set of all n such that an $mDCS$ of order n exists) of $mDCS$s as well as for an embedding (as small as possible) of partial $mDCS$s. An obvious definition here: a partial $mDCS$ is a pair (S,P), where P is an edge disjoint collection of directed m-cycles.

The obvious necessary conditions for the existence of a $mDCS$ of order n are

$$\begin{cases} (1) & n \geqslant m, \text{ if } n > 1, \text{ and} \\ (2) & n(n-1)/m \text{ is an integer.} \end{cases}$$

For a given m Wilson [13] has shown that these necessary conditions are sufficient for *sufficiently large* n. However the general spectrum problem is far far from being settled [6]. Wilson has also shown that a partial $mDCS$ can be embedded in a $mDCS$ [13]. The containing system however is exponentially large with respect to the partial system. Apart from Mendelsohn triple systems, nothing has been done on the problem of obtaining a "small" embedding for partial $mDCS$s. (The analogous problem for *undirected* cycle systems is addressed by the authors in [7].) The purpose of this paper is to rectify this situation by showing that a partial $mDCS$ of order n can always be embedded in a $mDCS$ of order $(2n+1)m$ when m is odd, *and* order nm when m is even, $m \neq 4$ or 6. For $m = 4$ or 6 the bounds are $4n+1$ and $6n+1$ respectively. Additionally, we give an *ad hoc* argument to reduce the size of the embedding for $m = 4$ to about $2n + \sqrt{2n}$. Since the techniques of proof are completely different for m odd, m even, and $m = 4$, we will handle each case in a separate section.

Finally, the principal ingredient in the embedding for odd m is a generalisation of a theorem due to Andersen, Hilton and Rodger [**1**] from embedding partial idempotent quasigroups to embedding partial idempotent groupoids. So, we will begin our discussion with this generalisation.

2. A GENERALISATION OF THE ANDERSEN, HILTON, RODGER THEOREM

In [**1**] Andersen, Hilton and Rodger proved the following result.

THEOREM 2.1. *A partial idempotent quasigroup of order r (based on $\{1,2,3,\dots,r\}$) can be embedded in an idempotent quasigroup of order t, for all $t \geqslant 2r+1$.*

To obtain a small embedding for partial directed m-cycle systems, we need to generalise this result. But first we need some definitions.

An $r \times s$ *(partial) groupoid* based on $\{1,\dots,t\}$ is an $r \times s$ array R in which each cell is occupied by (at most) one of the symbols from the set $\{1,\dots,t\}$. If the symbols in each row of R are all different then we say that R is *row latin*, and if the symbols in each column of R are all different then we say that R is *column latin*. If $r = s$ then we say that R has *order* r. If R is a (partial) groupoid that is both row latin and column latin then we have the usual definition of a (partial) $r \times s$ latin rectangle. Let $R(i)$ denote the number of times the symbol $i \in \{1,\dots,t\}$ occurs in R. A *hole of order* r is a subset H of the set of cells $\{(i,j) \mid 1 \leqslant i,j \leqslant r\}$. A *patterned hole of order* r based on $\{1,\dots,t\}$ is a triple (H,R,C) where R and C are partial groupoids of order r based on $\{1,\dots,t\}$, and H is a hole of order r, that satisfy:

(a) cell (i,j) of R contains a symbol if and only if cell (i,j) of C contains a symbol,

(b) $H = \{(i,j) \mid$ cell (i,j) of R contains a symbol$\}$,

(c) R is row latin and C is column latin, and

(d) $R(i) = C(i)$ for $1 \leqslant i \leqslant t$ (we say that R and C have the same *frequency*).

In what follows, unless otherwise stated, everything is based on $\{1,\dots,t\}$. We will denote by $P(H)$ a partial latin rectangle in which the empty cells are precisely the cells belonging to the hole H. We will say that the patterned hole (H,R,C) is *embedded* in the partial latin rectangle $P(H)$ if and only if the groupoid $P(R)$ obtained from $P(H)$ by filling in H with R is row latin, and the groupoid $P(C)$ obtained from $P(H)$ by filling in H with C is column latin. Finally, we say that the patterned hole (H, R, C) of order r is *idempotent* if both R and C are idempotent (so cell (i,i) contains symbol i for $1 \leqslant i \leqslant r$).

We shall need the following result of Hall [**2**].

THEOREM 2.2. *Every $r \times t$ latin rectangle based on $\{1,2,3,\dots,t\}$ $(r < t)$ can be embedded in a latin square of order t.*

We are now ready to state and prove the generalisation of Theorem 2.1.

THEOREM 2.3. *An idempotent patterned hole (H, R, C) of order r based on $\{1,\dots,r\}$ can be embedded in a partial idempotent latin square $P(H)$ of order t (based on $\{1,\dots,t\}$) for all $t \geqslant 2r+1$.*

PROOF: Clearly we can assume that $r \geqslant 4$. Suppose that a symbol, say symbol r, occurs exactly once in R. Then either all cells (u,v), $1 \leqslant u,v \leqslant r-1$, are filled with symbols $1,\dots,r-1$, or one such cell (x,y) is empty. In the former case, all cells in rows and columns r of R and C are empty except for the diagonal cell, so the result follows by deleting rows and columns r from R and C and then applying Theorem 2.1. In the latter case, place symbol r in cell (x,y) of both R and C. Continue this process until in the resulting partial groupoids R' and C' we have $R'(i) = C'(i) \geqslant 2$ for $1 \leqslant i \leqslant r$, R' is row latin, and C' is column latin.

If $R'(i) = r$ for $1 \leqslant i \leqslant r$ then clearly the result follows from Theorem 2.1, so we can assume that at least one cell of R' (and so also of C') is empty. Let (y,z) be such an empty cell (so $y \neq z$).

The proof of the theorem has three main steps (the second of which is quite long!).

As a first step, we embed (H', R', C') in a partial $(r+1)\times(r+2)$ latin rectangle L_{r+1} with the following properties:

(1) $L_{r+1}(R')(i) = L_{r+1}(C')(i) \geqslant \begin{cases} 2r+4-t & \text{for } r+1 \leqslant i \leqslant t-2, \\ 2r+3-t & \text{otherwise,} \end{cases}$

(2) cells $(r,r+1), (r,r+2), (r+1,r+1)$ and $(r+1,r+2)$ contain symbols, $t, t-1, t-1$ and t respectively, and

(3) $L_{r+1}(R')$ and $L_{r+1}(C')$ are row and column latin respectively.

To do this when $t \geqslant 2r+2$, let L be an incomplete $(r+1)\times(r+2)$ latin rectangle on the symbols $\{r+1,\dots,t\}$ that satisfies (2) and, if $t = 2r+3$, then the symbol missing from row $r+1$ of L occurs in column $r+2$ of L. L is easy to construct by first filling rows r and $r+1$ as required and then using Hall's Theorem. Fill each cell $(j,k) \notin H$ of L_{r+1} with the symbol in cell (j,k) of L. It is easy to check that $L_{r+1}(R')$ satisfies (1) and that (3) is satisfied.

If $t = 2r+1$ then let L be a latin square of order $r+2$ on the symbols $\{0, r+1, r+2, \dots, 2r+1\}$ (0 is a dummy symbol that does not appear in L_{r+1}) in which cells $(r,r+1),(r,r+2),(r+1,r+1),(r+1,r+2),(r+2,r+1),(r+2,r+2)$ and $(r+1,z)$ contain symbols $2r+1,2r,0,2r+1,2r,0$ and $2r$ respectively (recall cell (y,z) of C' is empty), and for $1 \leqslant i \leqslant r$ cell (i,i) contains symbol 0. Again L is easy to construct by filling rows $r,r+1$ and $r+2$ as required, and then completing L using Hall's Theorem. Fill each cell $(j,k) \notin H$ of L_{r+1} with the symbol in cell (j,k) of L, except that cell $(r+1,r+1)$ is filled with symbol $2r$ and cell $(r+1,z)$ is filled with a symbol from $\{1,\dots,r\}$ that does not occur in column z of C' (such a symbol exists because cell (y,z) of C' is empty). Then each symbol in $\{r+1,\dots,2r-1\}$ occurs in columns $r+1$ and $r+2$ and in row $r+1$, so (1) is satisfied, and clearly (3) is satisfied.

The second step is to embed L_{r+1} in a partial $(t-1)\times t$ latin rectangle L_{t-1} in which cell $(i,i+1)$ for $r+2 \leqslant i \leqslant t-1$ contains symbol $i-1$ (eventually row $r+1$ and columns $r+1$ and $r+2$ are deleted, so these cells "become" diagonal idempotent cells), and so that $L_{t-1}(R')$ and $L_{t-1}(C')$ are row and column latin respectively (see Figure 1). The proof that this can be done follows the proof of Theorem 3.2 in [1], but we present it here since the setting is a little different.

$L_{t-1} =$

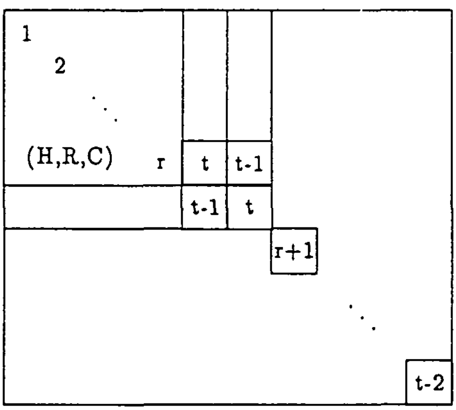

Figure 1. The embedding of L_{r+1} in L_{t-1}.

L_{t-1} is obtained inductively by forming L_x for $r+1 \leqslant x \leqslant t-1$, where L_x is an $x \times (x+1)$ partial latin rectangle, formed by adding a row and a column to L_{x-1} (for $x \geqslant r+2$), that satisfies

$$\text{(4)} \quad L_x(R')(i) = L_x(C')(i) \geqslant \begin{cases} 2x+2-t & \text{for } x+1 \leqslant i \leqslant t-2 \\ 2x+1-t & \text{otherwise,} \end{cases}$$

(5) cell $(x,x+1)$ of L_x contains symbol $x-1$, for $x > r+1$, and

(6) $L_x(R')$ and $L_x(C')$ are row and column latin.

Clearly L_{r+1} satisfies these properties, so we now proceed by induction assuming that L_x satisfies (4), (5) and (6).

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Form two simple bipartite graphs, B_r and B_c as follows. B_r is defined on the vertex sets $\{\rho_1, \dots, \rho_x, \rho^*\}$ and $\{\sigma'_1, \dots, \sigma'_t\}$ by joining σ'_i to ρ_j if and only if symbol i does not occur in row j of $L_x(R')$ and joining ρ^* to σ'_i for $x \leqslant i \leqslant t-2$. B_c is formed on the vertex sets $\{c_1, \dots, c_{x+1}, c^*\}$ and $\{\sigma''_1, \dots, \sigma''_t\}$ by joining σ''_i to c_j if and only if symbol i does not occur in column j of $L_x(C')$ and joining c^* to σ''_i for $x \leqslant i \leqslant t-2$. Since, by (6), each row of $L_x(R')$ contains $x+1$ symbols, $d_{B_r}(\rho_j) = t-x-1, 1 \leqslant j \leqslant x$. Similarly $d_{B_c}(c_j) = t-x$ for $1 \leqslant j \leqslant x+1$. Also, $d_{B_r}(\rho^*) = t-x-1 = d_{B_c}(c^*)$. From (4), $d_{B_r}(\sigma'_i) \leqslant x-(2x+1-t) = t-x-1$ and $d_{B_c}(\sigma''_i) \leqslant (x+1)-(2x+1-t) = t-x$ with equality if and only if symbol i satisfies equality in (4).

For each i with $d_{B_r}(\sigma'_i) = t-x-2$ (and so $d_{B_c}(\sigma''_i) = t-x-1$), add a vertex σ^*_i and join it to σ'_i and to σ''_i, thus forming a single bipartite graph B.

We now want to select a matching M in B with the following properties:

(7) M contains the edges $\rho^*\sigma'_{x-1}$ and $c^*\sigma''_{x-1}$,

(8) ρ_i and c_j are incident with an edge in M, $1 \leqslant i \leqslant x$ and $1 \leqslant j \leqslant x+1$,

(9) if $L_x(R')(i)$ satisfies equality in (4) then both σ'_i and σ''_i are incident with an edge in M, and

(10) if $L_x(R')(i)$ is one less than equality in (4) then either σ'_i or σ''_i is incident with an edge in M that is not also incident with σ^*_i.

If such a matching M is found, then L_{x+1} can be found by filling cell $(j, x+2)$ or $(x+1, k)$ with symbol i if $\rho_j\sigma'_i$ or $c_k\sigma''_i$ is in M respectively, and fill cell $(x+1, x+2)$ with symbol $x-1$. Then clearly (5) is satisfied. (6) is satisfied since M is a matching and by the definition of edges in B. (4) is satisfied because of (9) and (10): symbols satisfying equality in (4) are placed in both the added row and column, while symbols satisfying one less than equality are placed in at least one of the added row and column.

So how do we find M? Begin by giving B a proper edge-colouring with $t-x$ colours, say $1, \dots, t-x$. Since $d_B(c^*) = t-x-1$, let $t-x$ be the colour occurring on no edge incident with c^*. Let S be the set of edges coloured $t-x$ that are incident with c_j $(1 \leqslant j \leqslant x+1)$ or with σ''_i $(1 \leqslant i \leqslant t)$. Give $B-S$ a proper edge-colouring with $t-x-1$ colours so that $c^*\sigma''_{x-1}$ is coloured 1, and replace the edges in S, coloured $t-x$, to obtain a proper edge-colouring of B in which

(11) ρ_j $(1 \leqslant j \leqslant x)$ and c_k $(1 \leqslant k \leqslant x+1)$ are incident with an edge coloured 1,

(12) if $L_x(R)(i)$ satisfies equality in (4) then σ'_i and σ''_i are incident with edges coloured 1, and

(13) if $L_x(R)(i)$ is one less than equality in (4) then since σ'_i is incident with edges coloured $1, \dots, t-x-1, \sigma''_i$ is incident with edges coloured $1, \dots, t-x$ and σ^*_i is incident with at most one edge coloured 1, at least one of σ_i and σ''_i is incident with an edge coloured 1 that is not incident with σ^*_i.

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It looks as if we're nearly done, since the edges coloured 1 satisfy (8), (9) and (10), but possibly not (7) yet. We have assumed that $c^*\sigma''_{x-1}$ is coloured 1, so suppose that $\rho^*\sigma'_{x-1}$ is coloured 2. Let P be the path starting at c^* in which edges are alternately coloured 2 and $t-x$ that stops at the first vertex σ^*_α that it reaches, or if it reaches no such vertex then is maximal. Interchange colours along P (so σ^*_α, if defined, is now incident with two edges coloured 2). Now let Q be a path or cycle that contains the edge $\rho^*\sigma'_{x-1}$, has edges alternately coloured 1 and 2, which (in one direction) stops at the first vertex σ^*_β it comes across for which $\sigma'_\beta\sigma^*_\beta$ is coloured 1, and otherwise is maximal. Since c^* is incident with no edge coloured 2, Q does not contain the edge $c^*\sigma''_{x-1}$, so interchanging colours along Q produces a set of edges coloured 1 that satisfies (7), (8), (9) and (10).

The third and final step is to obtain the required embedding. Delete row $r+1$ and columns $r+1$ and $r+2$ from L_{t-1} to obtain a partial idempotent latin square T of order $t-2$. Notice that

(14) $T(C')(i) = t-3$ for $t-1 \leqslant i \leqslant t$, and
(15) symbols $t-1$ and t are both missing from row r of $T(R')$.

Form a bipartite graph on the vertex sets $\{c_1, \ldots, c_{t-2}\}$ and $\{1, \ldots, t\}$ by joining c_j to i if and only if symbol i is missing from column j of $T(C')$. Each vertex c_j has degree 2, vertices $t-1$ and t have degree 1 (by (14)) and the maximum degree is 2. Give this a proper edge-colouring with two colours, namely $t-1$ and t, making sure that the edges incident with vertices $t-1$ and t receive colours t and $t-1$ respectively; it is easy to see that this is possible. Now add two rows to T by filling cell (x,y) with symbol i if and only if the edge $\{c_y, i\}$ is coloured x. For the resulting partial latin rectangle T', by (15) we have that in $T'(R')$, symbol $t-1$ is missing from rows r and t, while symbol t is missing from rows r and $t-1$. Therefore, using Hall's Theorem to add two columns to $T'(R')$ produces the required embedding of (H, R, C). $\square$

3. Embedding directed odd cycle systems

Let $m = 2k+1$ be odd and let (X, C_1) be a partial $mDCS$ of order n. Define a pair of partial idempotent groupoids R and C of order n as follows:

(i) cell (i,i) is occupied by i for each $i \in X = \{1, 2, 3, \ldots, n\}$ in each of R and C (and so both are idempotent), and
(ii) if the directed edge (a,b) belongs to a directed cycle of C_1 fill in cell (a,b) of R with b and cell (a,b) of C with a.

If $H = \{(i,i) \mid i \in X\} \bigcup \{(a,b) \mid (a,b)$ belongs to a cycle of $C_1\}$, then (H, R, C) is an idempotent patterned hole of order n based on $\{1, 2, 3, \ldots, n\}$ and so by Theorem 2.3 can be embedded in a partial idempotent quasigroup $(P(H), \circ)$ of order t for every

$t \geqslant 2n+1$. Let $(P(H), \circ)$ be based on $P = \{1,2,3,\dots,t\}$ and set $S = P \times Z_m$. Define a collection of directed m-cycles C_2 as follows:

(1) For each $a \in P$, let $(\{a\} \times Z_m, c(a))$ be a $mDCS$ (such a system always exists [**12**]) and place the directed m-cycles of $c(a)$ in C_2.

(2) For each directed cycle $(x_1, x_2, \dots, x_m) \in C_1$ place the m directed m-cycles $((x_1,i),(x_2,i),\dots,(x_m,i))$, $i \in Z_m$, in C_2.

(3) Let $(z_1, z_2, \dots, z_k, z_{k+1})$ be a sequence such that

$$(|z_2, -z_1|, |z_3 - z_2|, \dots, |z_{k+1} - z_k|) = (1,2,3,\dots,k),$$

where $|z| = \min\{z, -z\} \pmod m$. (For example, define $z_i = (-1)^i \lfloor i/2 \rfloor$.) Let

$$I = \{(0, i, 2i, 3i, \dots, (m-1)i), ((m-1)i, (m-2)i, (m-3)i, \dots, 2i, i, 0) \\ (\operatorname{mod} m) \mid i \in \{1,2,3,\dots,k\}\}.$$

For each cycle $c \in C_1$ take a fixed representation $(x_1, x_2, \dots, x_m)$ of c and define a collection of $2km$ directed m-cycles as follows: Set $o(c) = \{((x_1,a_1),(x_m,a_2),\ (x_{m-1},a_3),\dots,(x_2,a_m)) \mid (a_1,a_2,\dots,a_m) \in I$ and i is ODD $\}$ and $e(c) = \{((x_1,b_1),\ (x_2,b_2),(x_3,b_3),\dots,(x_m,b_m)) \mid (b_1,b_2,\dots,b_m) \in I$ and i is EVEN$\}$. Place the $2km$ cycles $o(c)+j$ and $e(c)+j$ $(j \in Z_m)$ in C_2, where $o(c)+j$ and $e(c)+j$ are the directed m-cycles obtained from $o(c)$ and $e(c)$ by adding $j \pmod m$ to the second coordinates.

(4) If $x \neq y \in P$ and the directed edge (x,y) is NOT in a directed cycle of C_1, $x \circ y$ is *defined* in $(P(H), \circ)$. Let

$$c(x,y) = ((x,z_1),\ (y,z_1),\ (x,z_2),\ (y,z_3),\ \dots, \\ (x,z_k), (x \circ y, z_{k+1}),\ (y,k),\ (x,z_{k-1}),\ (y,z_{k-2}),\ \dots,\ (x,z_3),\ (y,z_2))$$

if k is EVEN, and

$$c(x,y) = ((x,z_1),\ (y,z_1),\ (x,z_2),\ (y,z_3),\ \dots, \\ (y,z_k),\ (x \circ y, z_{k+1}),\ (x,z_k),\ (y,z_{k-1}),\ (x,z_{k-2}),\ \dots,\ (x,z_3),\ (y,z_2))$$

if k is ODD. ($z_1, z_2, z_3, \dots, z_{k+1}$ are defined in (3).) Now place the m directed m-cycles $c(x,y)+j$, $j \in Z_m$, in C_2 where $c(x,y)+j$ is the m-cycle obtained from $c(x,y)$ by adding $j \pmod m$ to the second coordinates.

CLAIM. (S, C_2) is a directed m-cycle system of order tm containing m disjoint copies of the partial $mDCS(X, C_1)$.

PROOF: A counting argument shows that $|C_2|$ contains the correct number of m-cycles and so in order to prove that (S, C_2) is a $mDCS$ it remains to show that each edge in D_{tm} (based on S) belongs to a directed m-cycle of C_2. So let $((x,i),(y,j))$ be an edge. There are several cases to consider.

(i) $x = y$. Then $((x,i),(y,j)) \in (\{x\} \times Z_m, c(x))$ and so belongs to a directed m-cycle of $c(x)$ and therefore to a directed m-cycle of C_2.

(ii) $x \neq y$ *and* $i = j$. If (x,y) belongs to a cycle of C_1, say $(x, y, x_3, x_4, \ldots, x_m)$, then $((x,i),(y,i),(x_3,i),(x_4,i),\ldots,(x_m,i)) \in C_2$ (part (2) of the construction). If (x,y) does not belong to a cycle of C_1 let $z_1 + j = i$. Then $c(x,y) + j \in C_2$ (part (4) of the construction).

(iii) $x \neq y$ *and* $i \neq j$. One of two things is true: $|i-j|$ is ODD or EVEN.

$|i-j|$ *is ODD*. If (y,x) belongs to a directed m-cycle in C_1, then $c = (\ldots,(x,a_r),(y,a_{r+1}),\ldots) \in C_2$ where $|a_r - a_{r+1}| = i - j$ (the other cases are similar). Let $a_r + d = i$ and $a_{r+1} + d = j$. Then the edge $((x,i),(y,j))$ belongs to the directed m-cycle $c + d$, where $c + d$ is obtained from c by adding d (mod m) to the second coordinates of c. If (y,x) does *not* belong to a directed m-cycle in C_1 *and* $|i-j| \neq k+1$ we can assume $z_r - z_{r-1} = i - j$. Let $z_r + d = j$ and $z_{r-1} + d = i$. Then $((x,i),(y,j))$ belongs to the cycle $c(y,x) + d$. If (y,x) does *not* belong to a directed m-cycle in C_1 *and* $|i-j| = k+1$, then $|z_{k+1} - z_k| = k+1 = |i-j|$. We can assume $a_k + d = i$ and $z_{k+1} + d = j$ (the other cases are similar). In $(P(C), \circ)$, $z \circ x = y$ for some $z \in P$. Now $(z,x) \notin H$, since if so $z \circ x = z = y$ implies that $(y,x) \in H$; that is, the directed edge (y,x) belongs to a cycle of C_1. Hence $((x,i),(y,j))$ belongs to the cycle $c(z,x) + d$ (a type (4) cycle).

$|i=j|$ *is EVEN*. An argument similar to the odd case handles this case. ∎

Combining all of the above cases shows that (S, C_2) is a $mDCS$. The type (2) directed m-cycles show that m disjoint copies of (X, C_1) are embedded in (S, C_2). We now have the following theorem.

THEOREM 3.1. *If m is ODD, a partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order tm for every $t \geqslant 2n+1$.*

4. EMBEDDING DIRECTED EVEN CYCLE SYSTEMS

The techniques for embedding partial directed even cycle systems are in stark contrast to the directed odd cycle case.

Let $m = 2k \neq 4$ or 6 be EVEN and let (X, C_1) be a partial $mDCS$ of order n. Let X^* be a set of size $t \geqslant 2n$ such that $X \subseteq X^*$ and t is even. Further, let

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$H = \{h_1, h_2, \dots, h_{t/2}\}$ be a partition of X^* into "holes" of size 2 (2-element subsets) such that the elements of X belong to different holes of X^*. Let $S = X^* \times Z_k$ and define a collection of directed m-cycles C_2 as follows:

(1) For each $h_i \in H$, let $(h_i \times Z_k, c(h_i))$ be a $mDCS$ (such a system always exists [**12**]) and place the directed m-cycles of $c(h_i)$ in C_2.

(2) For each cycle $(x_1, x_2, \dots, x_m) \in C_1$ place the k directed m-cycles $((x_1, i), (x_2, i), \dots, (x_m, i))$, $i \in Z_k$, in C_2.

(3) For each directed m-cycle $(x_1, x_2, \dots, x_m) \in C_1$ let $c = ((x_1, 0), (x_m, 1), (x_{m-1}, 2), \dots, (x_2, k-1))$ and put the k directed m-cycles $c + i, i = 1, 2, \dots, k-1$, in C_2 where $c + i$ is obtained from c by adding i (mod k) to the second coordinates.

(4) If x and y belong to different holes and (x, y) does not belong to a cycle of C_1, place the directed m-cycle $((x, 0), (y, 0), (x, 1), (y, 1), \dots, (x, k-1), (y, k-1))$ in C_2.

(5) If x and y belong to different holes place the $k-2$ directed m-cycles $c(x, y) + i$, $i = 0, 1, 2, \dots, k-3$, in C_2, where $c(x, y) = ((x, 0), (y, 2), (x, 1), (y, 3), (x, 2), (y, 4), \dots, (x, k-1), (y, 1))$ and $c(x, y) + i$ is obtained from $c(x, y)$ by adding i (mod k) to the second coordinates of the ordered pairs with first coordinate y.

CLAIM. (S, C_2) is a $mDCS$ of order $kt = mt/2$ containing k disjoint copies of the partial $mDCS\,(X, C_1)$.

PROOF: As with the odd cycle case, a counting argument shows that $|C_2|$ contains the correct number of directed m-cycles and so it remains to show that each directed edge in D_{kt} (based on S) belongs to a directed m-cycle of C_2. So let $((x, i), (y, j))$ be an edge. Again there are several cases to consider.

(i) *x and $y \in h_i \in H$.* Then $((x, i), (y, j)) \in (h_i \times Z_k, c(h_i))$ and so belongs to a directed m-cycle of $c(h_i)$ and therefore to a directed m-cycle of C_2. All of the remaining cases are predicated on x and y belonging to different holes of H.

(ii) *$x \neq y$ and $i = j$.* If (x, y) belongs to a cycle of C_1, then the edge $((x, i), (y, j))$ belongs to a cycle of type (2). If (x, y) does not belong to a cycle of C_1, then $((x, i), (y, j))$ belongs to a cycle of type (4).

(iii) *$x \neq y$ and $i \neq j$.* If $|i - j| = 1$, one of two things is true: $j = i + 1$ or $i = j + 1$. If $j = i + 1$ and (y, x) is in a directed m-cycle of C_1, then $((x, i), (y, j))$ is in a directed m-cycle of type (3). If (y, x) is not in a directed m-cycle of C_1, then $((x, i), (y, j))$ is in a directed m-cycle of type (4). A similar argument takes case of the case where $i = j + 1$. If $|i - j| \geqslant 2$, write $j = i + 2 + d$ (mod k). Then the edge $((x, i), (y, j))$

belongs to the type (5) directed m-cycle $c(x,y)+d$.

Combining the above cases shows that (S,C_2) is a $mDCS$. The type (2) directed m-cycles show that k disjoint copies of (X,C_1) are embedded in (S,C_2). $\square$

We have the following theorem (replacing t in the previous discussion with $t/2$).

Theorem 4.1. *If m is EVEN and $m \neq 4$ or 6, a partial directed m-cycle system of order n can be embedded in a directed m-cycle system of order tm for every even $t \geqslant n$. If $m = 4$ or 6 the containing system has size $tm+1$ for every even $t \geqslant 8$ or 12 respectively.*

Proof: In the construction set $S = \{\infty\} \bigcup (X^* \times Z_k)$ and replace (1) by: for each $h_i \in H$, let $\{\infty\} \bigcup (h_i \times Z_k, c(h_i))$ be a $mDCS$ of order $m+1 = 5$ or 7 as the case may be (such a system exists [8]) and place the directed m-cycles of $c(h_i)$ in C_2. $\square$

5. Embedding directed 4-cycle systems

In the case where $m = 4$ we can obtain a smaller embedding that that of Theorem 4.1. The proof of Theorem 5.2 closely follows the proof of the related result for undirected cycles [8]. But we shall also need the following theorem due to Dominique Sotteau:

Theorem 5.1. [11] *The complete directed bipartite graph $D_{x,y}$ can be decomposed into directed $2k$-cycles if and only if $x \geqslant k$, $y \geqslant k$ and k divides xy.*

Theorem 5.2. *Let (N,C) be a partial directed 4-cycle system of order n, and let x be the smallest odd number such that $\binom{x}{2} \geqslant n$. Then (N,C) can be embedded in a directed 4-cycle system of order x^2.*

Proof: Let $N = \{1,\ldots,n\}$, let $S = \{1,\ldots,\binom{x}{2}\}$ and let $X = \{\binom{x}{2}+1,\ldots,\binom{x}{2}+x\}$.

Form a directed 4-cycle system (V,C_1) of order x^2, with $V = (S \times \{1,2\}) \cup X$ as follows:

(1) For each $(a,b,c,d) \in C$, and for $1 \leqslant i \leqslant 2$, place the directed 4-cycles $((a,i),(b,i),(c,i),(d,i))$, $((a,i),(d,i+1),(c,i),(b,i+1))$ in C_1 (reducing the second component modulo 2).

(2) If $\{a,b\} \subseteq S$ and (a,b) does not occur in any directed 4-cycle in C, then $((a,1),(b,1),(a,2),(b,2)) \in C_1$.

(3) Let $T(x)$ be the set of 2-element subsets of X and let $\phi : S \to T(x)$ be any bijection. For each $s \in S$ place $((s,1),(s,2),t,u)$ and $((s,1),u,t,(s,2))$ in C_1, where $\phi(s) = \{t,u\}, t < u$.

(4) For each $t \in X$ let $D(t)$ be the set of $(s,i) \in S \times \{1,2\}$ such that $((s,i),t)$ is in a directed 4-cycle defined in (3). (Notice that if $(s(i),t)$

is replaced with $(t, s(i))$ in this definition then $D(t)$ remains the same.) Clearly $|D(t)| = x - 1$. Also $\{D(t) \mid t \in X\}$ is a partition of $S \times \{1,2\}$. By Sotteau's Theorem, there exists a directed 4-cycle system (V_t, C_t) of the complete directed bipartite graph $D_{x-1,x-1}$ on the vertex set $V(t) = D(t) \bigcup (X \backslash \{t\})$. For each $t \in X$ let $C_t \subset C_1$.

It is easy to check that (V, C_1) is a directed 4-cycle system, and from (1) it contains 2 disjoint copies of (N, C). ∎

Note that x^2 is approximately $2n + \sqrt{2n}$.

References

[1] L.D. Andersen, A.J.W. Hilton and C.A. Rodger, 'A solution to the embedding problem for partial idempotent latin squares', *J. London Math. Soc.* **26** (1982), 21–27.

[2] M. Hall, 'An existence theorem for latin squares', *Bull. Amer. Math. Soc.* **51** (1945), 387–388.

[3] D.G. Hoffman and C.C. Lindner, 'Embeddings of Mendelsohn triple systems', *Ars Combin.* **11** (1981), 265–269.

[4] C.C. Lindner, 'Finite partial cyclic triple systems can be finitely embedded', *Algebra Universalis* **1** (1971), 93–96.

[5] C.C. Lindner and A.B. Cruse, 'Small embeddings for partial semisymmetric and totally symmetric quasigroups', *J. London Math. Soc.* **12** (1976), 479–484.

[6] C.C. Lindner and C.A. Rodger, *Decompositions into cycles II: cycle systems, selected surveys in combinatorial design theory* (Wiley, to appear).

[7] C.C. Lindner and C.A. Rodger, 'A partial $m = (2k+1)$-cycle system of order n can be embedded in an m-cycle system of order $(2n+1)m$', *Discrete Math.* (to appear).

[8] C.C. Lindner, C.A. Rodger and J.D. Horton, 'A small embedding for partial 4-cycle systems', *J. Combin. Math. Combin. Comput.* **5** (1989), 23–26.

[9] N.S. Mendelsohn, 'A natural generalisation of Steiner triple systems', in *Computers in Number Theory*, Editors A.O.L. Atkin and B.J. Birch, pp. 323–338 (Academic Press, London, 1971).

[10] C.A. Rodger, 'Embedding partial Mendelsohn triple systems', *Discrete Math.* **65** (1987), 187–196.

[11] D. Sotteau, 'Decompositions of $K_{m,n}\left(K^*_{m,n}\right)$ into cycles (circuits) of length $2k$', *J. Combin. Theory Series B* **29** (1981), 75–81.

[12] T.W. Tillson, 'A Hamilton decomposition of $K^*_{2m}, 2m \geqslant 8$', *J. Combin. Theory Series B* **29** (1980), 68–74.

[13] R.M. Wilson, 'Construction and uses of pairwise balanced designs', *Maths. Centre Tracts* **55** (1974), 18–41.

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