NORMAL HOLOMORPHIC MAPPINGS AND CLASSICAL THEOREMS OF FUNCTION THEORY

KEN-ICHI FUNAHASHI

§ 0. Introduction

In [7], O. Lehto and K. I. Virtanen introduced the concept of normal meromorphic functions in connection with the study of boundary behaviour of meromorphic functions of one complex variable.

In this paper, we generalize the theory of normal meromorphic functions to the case of holomorphic mappings into higher dimensional complex spaces in connection with the theory of hyperbolic manifolds and Nevanlinna theory.

The main concern of this paper is the generalizations of the big Picard theorem and Lindelöf's theorem which appear in the classical function theory.

§ 1. Definition of normal holomorphic mappings

In this section, we define the concept of normal holomorphic mappings similar to normal meromorphic functions (cf. [7]).

Let M and N be complex analytic spaces. We denote the set of holomorphic mappings from M into N by $\mathrm{Hol}\,(M, N)$. We say a subset $\mathscr{F}$ of $\mathrm{Hol}\,(M, N)$ to be a *normal family* if $\mathscr{F}$ is relatively compact in $\mathrm{Hol}\,(M, N)$ in the sence of compact open topology.

DEFINITION 1. Let D be a homogeneous bounded domain in C^n and N be a complex analytic space. We say that a holomorphic mapping f: $D \to N$ is *normal* if the family

$$\mathscr{F} = \{f \circ g;\, g \in \mathrm{Aut}\, D\}$$

is normal, where $\mathrm{Aut}\, D$ denotes the holomorphic automorphism group of D.

DEFINITION 2. We say that a subset $\mathscr{F}$ of $\mathrm{Hol}\,(D, N)$ is $\mathrm{Aut}\, D$-invariant, if $f \circ g \in \mathscr{F}$ for every $f \in \mathscr{F}$ and every $g \in \mathrm{Aut}\, D$.

We study the condition of normality of holomorphic mappings.

PROPOSITION 1. *Let ds_D^2 be the Bergman metric of a homogeneous bounded domain D in C^n and (N, ds_N^2) be a compact Hermitian manifold. If a holomorphic mapping $f\colon D \to N$ satisfies*

$$f^* ds_N^2 \leqq C \cdot ds_D^2$$

for a finite constant C, then f is a normal holomorphic mapping.

Proof. As ds_D^2 is Aut D-invariant,

$$(f \circ g)^* ds_N^2 = g^* f^* ds_N^2 \leqq C \cdot g^* ds_D^2 = C \cdot ds_D^2$$

for every $g \in \operatorname{Aut} D$. Hence $\mathscr{F} = \{f \circ g; \, g \in \operatorname{Aut} D\}$ is equi-continuous. As N is compact, $\mathscr{F} \subset \operatorname{Hol}(D, N)$ is normal according to Ascoli-Arzelà theorem.   q.e.d.

THEOREM 1. *Let D be a bounded homogeneous domain in C^n and (N, ds_N^2) be a Hermitian complex manifold. If a subset $\mathscr{F}$ of $\operatorname{Hol}(D, N)$ is an Aut D-invariant normal family, then there exists a constant C such that $f^* ds_N^2 \leqq C \cdot ds_D^2$ for every $f \in \mathscr{F}$.*

Proof. Let

$$C(z) = \sup_{\|x\|=1, f \in \mathscr{F}} f^* ds_N^2 / ds_D^2(z, x)$$

for each $z \in D$, and $x \in T_z C^n$, where $\| \ \|$ is the length by a flat metric of C^n. We first prove $C(0) < \infty$ for a point $0 \in D$.

Suppose that $C(0) = \infty$. Then there exist sequences $\{f_n\}_{n=1}^\infty \subset \mathscr{F}$ and $\{x_n \in T_0 C^n, \, \|x_n\| = 1\}$ of holomorphic tangent vectors such that
  (i) $f_n^* ds_N^2(0, x_n) \geqq n^2 \cdot ds_D^2(0, x_n)$,
where we may assume that $\{x_n\}$ converges to $x \in T_0 C^n$. From (i), we see
  (ii) $\|f_{n*} x_n\|_N \geqq n \cdot \alpha$,
where $\| \ \|_N$ is the length measured by ds_N^2 and α is a positive constant. Since $\mathscr{F}$ is relatively compact in $\operatorname{Hol}(D, N)$, some subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}$ exists such that $f_{n_k} \to f \in \operatorname{Hol}(D, N)$ and $f_{n_k}(0) \to p \in N$ as $k \to \infty$. Consequently $f_{n_k*} x_{n_k} \to f_* x$ and $\|f_{n_k*} x_{n_k}\|_N \to \|f_* x\|_N < \infty$ as $k \to \infty$. This is contradictory to (ii) and hence $C(0) < \infty$.

We secondly prove that $C(z)$ is constant on D. From definition of $C(z)$,

$$f^* ds_N^2(z) \leqq C(z) \cdot ds_D^2(z)$$

for every $f \in \mathscr{F}$. As ds_D^2 is Aut D-invariant,

$$(f \circ g)^* ds_N^2(z) \leqq C(g(z)) \cdot ds_D^2(z)$$

for every $g \in \operatorname{Aut} D$. Hence $C(g(z)) \geq C(z)$. Similarly $C(z) \geq C(g(z))$ and so $C(z) = C(g(z))$. As D is homogeneous, $C(z)$ is constant on D and the theorem is proved.   q.e.d.

COROLLARY 1.   *Let D and N be as above.   If $f \colon D \to N$ is a normal holomorphic mapping, then there exists a finite constant C such that $f^* ds_N^2 \leqq C \cdot ds_D^2$.*

§ 2.   Examples

In this section, we give some examples of normal holomorphic mappings.

Example 1.   Let $\varDelta = \{|z| < 1\} \subset \boldsymbol{C}$ be the unit disc.

(i)   We choose arbitrarily distinct q points $p_1, \cdots, p_q$ in the Riemann sphere $\boldsymbol{P}^1$. Let $\mathscr{F}$ be a family of holomorphic mappings from $\varDelta$ into $\boldsymbol{P}^1$ which satisfies the following condition (C).

(C)   Every f in $\mathscr{F}$ takes each value $p_i \in \boldsymbol{P}^1 (i = 1, \cdots, q)$ with multiplicity $\geqq m_i$ or f omits a point p_i and $\{m_i\}_{i=1}^q$ satisfies

$$-2 + \sum_{i=1}^q \left(1 - \frac{1}{m_i}\right) > 0 \,,$$

where we set $m_i = \infty$ in the case f omits p_i.

Then $\mathscr{F}$ is normal (Montel-Valiron, cf. Theorem 8.3 in H. Fujimoto [2]).   Particularly, if a holomorphic map $f \colon \varDelta \to \boldsymbol{P}^1$ satisfies the condition (C), then f is a normal holomorphic mapping, that is, a normal meromorphic function (cf. [7]).

(ii)   Let $T = \boldsymbol{C}/L$ be a complex torus and take a point p of T.   We consider the family $\mathscr{F}$ of all holomorphic mappings from $\varDelta$ into T which omit a value $p \in T$.   Then $\mathscr{F}$ is normal.   Particularly, every f in $\mathscr{F}$ is a normal holomorphic mapping.

(iii)   Let V be a compact Riemann surface of genus ≥ 2.   Then $\operatorname{Hol}(\varDelta, V)$ is normal and hence every holomorphic mapping $f \colon \varDelta \to V$ is normal.

(iv)   Let N be a paracompact connected complex analytic space and M be a *hyperbolically embedded subspace* of N.   Then $\operatorname{Hol}(D, M)$ is relatively compact in $\operatorname{Hol}(D, N)$ (See [5]).   Particularly a holomorphic mapping from D into N is normal if the image of f is contained in M.   For such examples see M. L. Green [3] and [4].

EXAMPLE 2. (i) Let $ds_\Delta^2 = dz d\bar{z}/(1 - |z|^2)^2$ be a Bergman metric of the unit disc Δ. We take homogeneous coordinates (z_0, z_1) of P^1 and let $w = z_1/z_0$ for $z_0 \neq 0$. Then Fubini-Study metric of P^1 is given by

$$ds_{P^1}^2 = \frac{1}{(1+|w|^2)^2} dw d\bar{w} .$$

Consequently a meromorphic function $f \colon \Delta \to P^1$ is normal if and only if

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leqq C \cdot \frac{1}{1 - |z|^2} \qquad (z \in \Delta)$$

for some finite constant C (cf. [7]).

(ii) Let $T = C^n/L$ be an n-dimensional complex torus. Then $f \colon \Delta \to T$ is normal if and only if a lifting $\tilde{f} = (f_1, \cdots, f_n)$ of f to C^n satisfies

$$(*) \qquad \sum_{i=1}^n |f_i'(z)|^2 \leqq C \cdot \frac{1}{(1 - |z|^2)^2}$$

for some finite constant C. A holomorphic function satisfying condition $(*)$ is classically called a *Bloch function*.

§3. The estimate for characteristic functions

In this section, we consider the case that D is the unit ball $B^n = \{\sum_{i=1}^n |z_i|^2 < 1\} \subset C^n$. The Bergman metric of B^n is given by

$$ds_{B^n}^2 = \sum_{i,j=1}^n \frac{1}{(1 - |z|^2)^2} [(1 - |z|^2)\delta_{ij} + z_i \bar{z}_j] dz_i d\bar{z}_j,$$

where $|z|^2 = \sum_{i=1}^n |z_i|^2$. Let (N, ds_N^2) be a Hermitian manifold and let ω_N, ω_{B^n} be the (1, 1)-form associated with ds_N^2, $ds_{B^n}^2$ respectively. We calculate the *characteristic function* of a normal holomorphic mapping $f \colon B^n \to N$.

LEMMA 1. *Let* $f \colon B^n \to N$ *be a holomorphic mapping such that* $f^* ds_N^2 \leqq C \cdot ds_{B^n}^2$, *where* C *is a constant. Set* $\varphi = \sqrt{-1}/2 \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ *and* $B(r) = \{|z| < r\}$. *Then,*

$$\int_{B(r)} f^* \omega_N \wedge \varphi^{n-1} \leqq C \cdot \int_{B(r)} \omega_{B^n} \wedge \varphi^{n-1} .$$

Proof. We set

$$f^* \omega_N = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n g_{ij} dz_i \wedge d\bar{z}_j ,$$

$$\omega_{B^n} = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar{z}_j .$$

Then,

$$f^*\omega_N \wedge \varphi^{n-1} = (n-1)! \cdot \left(\sum_{i=1}^{n} g_{i\bar{i}}\right) \prod_{j=1}^{n} \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} g_{i\bar{i}}\right)\varphi^n .$$

Since a matrix $(g_{i\bar{j}}) - C \cdot (h_{i\bar{j}})$ is of negative semi-definite,

$$g_{i\bar{i}} \leqq C \cdot h_{i\bar{i}} \qquad (i = 1, \cdots, n) .$$

Hence,

$$\int_{B(r)} f^*\omega_N \wedge \varphi^{n-1} = \frac{1}{n}\int_{B(r)} \left(\sum_{i=1}^{n} g_{i\bar{i}}\right)\varphi^n \leqq C \cdot \frac{1}{n}\int_{B(r)} \left(\sum_{i=1}^{n} h_{i\bar{i}}\right)\varphi^n$$

$$= C \cdot \int_{B(r)} \omega_{L^n} \wedge \varphi^{n-1} . \qquad\qquad \text{q.e.d.}$$

LEMMA 2.

$$\frac{1}{\pi^n}\int_0^r \frac{dt}{t^{2n-1}}\left(\int_{B(r)} \omega_{B^n} \wedge \varphi^{n-1}\right) \leqq \frac{1}{2}\log\frac{1}{1-r^2} + (n-1)\cdot C_1$$

$$(0 < r < 1) ,$$

where $C_1 = 2\log 2.$

Proof. Since the area of the boundary $\partial B(r)$ of $B(r)$ is $2\pi^n/(n-1)!\cdot r^{2n-1}$ and

$$\omega_{B^n} \wedge \varphi^{n-1} = \frac{n!}{(1-|z|^2)^2}\left(1 - \frac{n-1}{n}|z|^2\right)\cdot\frac{\varphi^n}{n!} ,$$

$$\int_{B(t)} \omega_{B^n} \wedge \varphi^{n-1} = \int_0^t \frac{n!}{(1-u^2)^2}\left(1 - \frac{n-1}{n}u^2\right)\frac{2\pi^n}{(n-1)!}\cdot u^{2n-1}du$$

$$= 2\pi^n \int_0^t \frac{u^{2n-1}}{(1-u^2)^2}du + 2\pi^n(n-1)\int_0^t \frac{u^{2n-1}}{1-u^2}du .$$

Using the estimates

$$\int_0^t \frac{u^{2n-1}}{(1-u^2)^2}du \leqq t^{2n-2}\int_0^t \frac{udu}{(1-u^2)^2}$$

and

$$\int_0^t \frac{u^{2n-1}}{1-u^2}du \leqq t^{2n-1}\int_0^t \frac{du}{1-u^2} ,$$

we see

$$\int_{B(t)} \omega_{B^n} \wedge \varphi^{n-1} \leqq \pi^n \frac{t^{2n}}{1 - t^2} + (n-1)\pi^n \cdot t^{2n-1}\left[\log \frac{1}{1-t} + \log(1+t)\right].$$

Hence,

$$\frac{1}{\pi^n} \int_0^r \frac{dt}{t^{2n-1}} \left(\int_{B(t)} \omega_{B^n} \wedge \varphi^{n-1}\right)$$

$$\leqq \int_0^r \frac{t\,dt}{1 - t^2} + (n-1) \int_0^r \left[\log \frac{1}{1-t} + \log(1+t)\right]dt$$

$$= \frac{1}{2} \log \frac{1}{1 - r^2} + (n-1)C_1 ,$$

where

$$C_1 = \int_0^1 \left[\log \frac{1}{1-t} + \log(1+t)\right]dt = 2\log 2 . \qquad \text{q.e.d.}$$

DEFINITION 3.   Let B^n, (N, ds_N^2) be as above and $f\colon B^n \to N$ be a holomorphic mapping.   We define the *characteristic function* of f by

$$T(f, r) = \frac{1}{\pi^n} \int_0^r \frac{dt}{t^{2n-1}} \left(\int_{B(t)} f^*\omega_{B^n} \wedge \varphi^{n-1}\right).$$

PROPOSITION 2.   *Let $\mathscr{F}$ be an Aut B^n-invariant normal family of holomorphic mapping from B^n to (N, ds_N^2).   Then there exists a constant $C(r)$ depending only r such that*

$$T(f, r) \leqq C(r) \qquad (0 \leqq r < 1)$$

for every $f \in \mathscr{F}$.

Proof.   Since there exists a constant C such that $f^*ds_N^2 \leqq C \cdot ds_{B^n}^2$ for every $f \in \mathscr{F}$, according to Theorem 1, it is clear by Lemma 2.   We can take

$$C(r) = \frac{C}{2} \log \frac{1}{1 - r^2} + (n-1)C_1 \cdot C . \qquad \text{q.e.d.}$$

COROLLARY.   *If a holomorphic mapping $f\colon B^n \to N$ is normal, then*

$$T(f \circ g, r) = O\left(\log \frac{1}{1 - r}\right) \qquad (r \longrightarrow 1)$$

for every $g \in$ Aut B^n.

Remark 1.   The property of the characteristic function for a normal holomorphic mapping is stronger than the property

$$T(f, r) = O\!\left(\log \frac{1}{1-r}\right) \qquad (r \longrightarrow 1) .$$

Remark 2. If N is the projective space $\boldsymbol{P}^n$, the inverse of the above proposition holds. This is clear by Theorem 5.10 in H. Fujimoto [2], because B^n is a homogeneous domain.

§4. A generalization of the big Picard theorem

We set $\varDelta = \{w \in C, |w| < 1\}$ and $\varDelta^* = \{w \in C, 0 < |w| < 1\}$. Let $\pi\colon \varDelta \to \varDelta^*$ be the universal covering, where $\pi(w) = e^{(w+1)/(w-1)}$.

Definition 3. A holomorphic mapping f from $\varDelta^*$ into a complex analytic space N is said to be *normal* if $f \circ \pi\colon \varDelta \to N$ is a normal holomorphic mapping.

Theorem 2. *Let N be a paracompact complex manifold. If $f\colon \varDelta^* \to N$ is a normal holomorphic mapping, then f can be extended to a holomorphic mapping from $\varDelta$ into N.*

Proof. First of all, we show that there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset \varDelta^*$ converging to the origin 0 such that $\{f(z_n)\}_{n=1}^{\infty}$ converges to some point p_0 in N. Take a sequence $\{z_n\}_{n=1}^{\infty} \subset \varDelta^*$ converging to 0, points $\{w_n\}$ such that $\pi(w_n) = z_n$ and take $g_n \in \mathrm{Aut}\,\varDelta$ such that $g_n(0) = w_n$. Consider the family $\mathscr{F} = \{f \circ \pi \circ g_n\}_{n=1}^{\infty}$. By the assumption, $\mathscr{F}$ is normal. We can take a subsequence $\{f \circ \pi \circ g_{n_k}\}_{n=1}^{\infty}$ of $\{f \circ \pi \circ g_n\}_{n=1}^{\infty}$ such that $\{f \circ \pi \circ g_{n_k}\}_{k=1}^{\infty}$ converges to some $h \in \mathrm{Hol}\,(\varDelta, N)$. Particularly $f \circ \pi \circ g_{n_k}(0) \to h(0) = p_0$. We set $\pi \circ g_{n_k}(0) = z_k$. Then $f(z_k) \to p_0$.

Secondary, we take a hermitian metric ds_N^2 of N. Since $f \circ \pi\colon \varDelta \to N$ is normal, according to Theorem 1 of Section 1

$$(1) \qquad\qquad (f \circ \pi)^* ds_N^2 \leqq C \cdot ds_{\varDelta}^2 ,$$

where C is a constant and $ds_{\varDelta}^2 = dz d\bar{z}/(1 - |z|^2)^2$, (1) implies

$$(2) \qquad\qquad f^* ds_N^2 \leqq C \cdot ds_{\varDelta^*}^2 ,$$

where

$$ds_{\varDelta^*}^2 = \frac{1}{4} \cdot \frac{dz d\bar{z}}{|z|^2 \,(\log 1/|z|)^2} .$$

We set $r_k = |z_k|$. It may be assumed that $\{r_k\}_{k=1}^{\infty}$ is a decreasing sequence. We denote by d_N the distance of N defined by ds_N^2. Then

(3) $$d_N(f(z), p_0) \leqq d_N(f(z), f(z_k)) + d_N(f(z_k), p_0) .$$

When $|z| = r_k$, according to (2)

(4) $$d_N(f(z), f(z_k)) = \frac{\sqrt{C}}{2} \int_{z_k}^{z} \frac{|dz|}{|dz| \log 1/|z|} \leqq \sqrt{C} \cdot \frac{\pi}{\log 1/r_k} .$$

The right hand side converges to 0 as $k \to \infty$. According to (3) and (4), we see that, for every ε-neighbourhood W of p_0, when k_0 is sufficiently large, $f(\{|z| = r_k\})$ is contained in W for $k \geqq k_0$. By the method in the proof of Theorem 3.1 in S. Kobayashi [6], Chapter VI, we can see that, for every neighbourhood U of p_0, if $\delta > 0$ is sufficiently small, $f(\{z \in \Delta^*, |z| < \delta\})$ is contained in U, and so we can define a holomorphic extension of f to Δ by $f(0) = p_0$.   q.e.d.

Remark 3. The above theorem remains valid for a paracompact connected complex analytic space N, because N has a hermitian metric h in the extended sence, which induces the original topology of N, and the discussion of Section 1 and the above proof are available for such (N, h) (see the introduction of P. Kiernan [5]).

Remark 4. Let N be a paracompact connected complex analytic space and M be hyperbolically embedded subspace of N. Then a holomorphic mapping $f: \Delta^* \to N$ with $f(\Delta^*) \subset M$ is normal. Hence, Theorem 2 is a generalization of a big Picard theorem given in S. Kobayashi [6], Chapter V, Theorem 6.1.

§5. Asymptotic values

In this section, we consider a generalization of the Lindelöf's theorem on asymptotic values of bounded holomorphic functions.

DEFINITION 4. Let f be a holomorphic mapping from a simply connected domain $G \subset C$ into a complex analytic space N. We define an *angle* with vertex at $z_0 \in \partial G$ to be a domain A defined as follows: Taking a boundary point $z_1(\neq z_0)$ of G and a positive constant ε, we set

$$A = \{z; \varepsilon < \omega(z) < 1 - \varepsilon\} ,$$

where $\omega(z)$ is the harmonic measure of the arc between z_0 and z_1 with respect to G.

We say that f possesses the *angular limit* p_0 at $z_0 \in \partial G$ if for every

neighbourhood U of p_0 and every angle A with vertex at z_0, a neighbourhood V of z_0 exists such that $f(V \cap A) \subset U$.

The following theorem is a generalization of Theorem 1 in the Lehto-Virtanen [7].

THEOREM 3. *Let f be a holomorphic mapping from the unit disc $\varDelta$ into a complex analytic space N. Suppose that f has an asymptotic value $p_0 \in N$ at a point $z_0 \in \partial\varDelta$ along a Jordan curve $\varGamma$ lying in $\varDelta$ and that f does not possess the angular limit p_0 at the point z_0. Then, there exists an arbitrarily small neighbourhood U of p_0 such that, for every $\varepsilon > 0$, there are a Jordan arc L in $\varDelta$ with end point at z_0 along which $f(z)$ tends to p_0 and a sequence $\{z_n\}_{n=1}^{\infty}$ in $\varDelta$ converging to z_0 satisfying the condition $f(z_n) \in \partial U$ and $d_\varDelta(z, L) < \varepsilon$, where $d_\varDelta$ is the hyperbolic distance of $\varDelta$.*

For the proof, we need the following

LEMMA (An extension of the Lindelöf's theorem). *Let r_0 be greater than or equal to 0 and $D = \{z; |z| > r_0, 0 < \arg z < \alpha\}$. Let $f(z)$ be a bounded holomorphic function on D which is continuous on $D \cup \{r \in R, r > r_0\}$ and $\limsup\limits_{r \to \infty} |f(r)| = 0$. Then for every $\delta > 0$,*

$$\limsup_{\rho \to \infty} \{|f(z)|; \ 0 \leqq \arg z \leqq \alpha - \delta, |z| > \rho\} = 0 .$$

Proof. This proof is suggested by H. Fujimoto.

We use the theory of harmonic measure (cf. [8], III, § 2). We may take $\alpha = \pi$. As the case $r_0 = 0$ is well known, we may take r_0 is greater than 0. We set $\delta' = \delta/\pi$.

Let $\tilde{\omega}(z)$ be a harmonic measure of $\{|z| = r_0, \operatorname{Im} z > 0\}$ with respect to $D = \{z; \operatorname{Im} z > 0, |z| > r_0\}$. Then, by the well-known theorem on solutions of Dirichlet problem (cf. [8], II, § 1, p. 22–23) there exists $r_1 > 0$ such that

$$|\tilde{\omega}(z) < \frac{\delta'}{4} \qquad \text{on } D' = \{z; \ z \in D, |z| \geqq r_1\} .$$

We define harmonic functions ω_1, ω_2 for all r_2 greater than r_1, r_0 as follows.

$\omega_1(z)$: harmonic measure of $\{z; \text{real}, z > r_2\}$ with respect to D,

$\omega_2(z)$: harmonic measure of $\{z; \text{real}, z > r_2\}$ with respect to $\{\operatorname{Im} z > 0\}$.

We see $\omega_2(z) = 1 - \arg(z - r_2)/\pi$. Then,

$$\left\{\omega_2(z) > \frac{\delta'}{2}, \ |z| > r_2\right\} \supset \{\pi - \delta' \cdot \pi > \arg z > 0, |z| > r_2\} .$$

Also we see $\omega_2(z) \leqq \omega_1(z) + \tilde{\omega}(z)$ on D by the maximum principle. As $\omega_1(z) \geqq \omega_2(z) - \tilde{\omega}(z)$.

$$\left\{\omega_2(z) > \frac{\delta'}{2}, \ |z| > r_2\right\} \subset \left\{\omega_1(z) > \frac{\delta'}{4}\right\} \qquad \text{on } D' \ .$$

Hence,

$$\left\{\pi - \delta > \arg z > 0, \ |z| > r_2\right\} \subset \left\{\omega_1(z) > \frac{\delta'}{4}\right\} \qquad \text{on } D' \ .$$

We define $m(r_2)$, $m^*(r_2)$ as follows.

$$m(r_2) = \sup\left\{|f(r)|; \ r \text{ is real}, \ r \geqq r_2\right\},$$
$$m^*(r_2) = \sup\left\{|f(z)|; \ \pi - \delta > \arg z \geqq 0, \ |z| \geqq r_2\right\} \ .$$

If $|f(z)| \leqq M$, then by two-constants theorem (cf. [8], III, § 2, p. 41–42)

$$m^*(r_2) \leqq m(r_2)^{\delta'/4} \cdot M^{1-\delta'/4} \ .$$

When $r_2 \to \infty$, from assumption $m(r_2) \to 0$, and then $m^*(r_2) \to 0$. q.e.d.

Proof of Theorem 3. (This is essentially due to Lehto-Virtanen [7].)

We may suppose that $G = \{z; \ 0 < \arg z < \pi/2\}$ and $z_0 = \infty$, because the hyperbolic distance is conformally invariant. We may suppose Γ starts at $z = 0$ so that Γ divides G into two distinct parts G_1 and G_2. Let G_1 denote the part of G bounded by Γ and the imaginary axis. By assumption, f does not converge uniformly to p_0 on some angle A: $\delta < \arg z < \pi/2 - 2\delta$ ($\delta > 0$). The same is also true at least in one of the intersections $G_1 \cap A$ and $G_2 \cap A$. We assume in the following that it is true in $G_1 \cap A$. Let U be a sufficiently small neighbourhood of p_0 which is biholomorphic to a subvariety V in a bounded domain W in C^m by a map φ and let p_0 be mapped to the origin 0 in C^m.

We map G_1 onto the right angle $0 < \arg w < \pi/2$ and Γ onto positive real axis R^+, keeping 0, ∞ fixed. The image of $G_1 \cap A$ lies in the angle $0 < \arg w < \pi/2 - 2\delta$, as follows immediately if we apply the maximum principle to the harmonic measures of the real axis and Γ, respectively with respect to G and G_1.

Consider $f^{-1}(U)$ and the connected component U' of $f^{-1}(U)$ which is a neighbourhood of $\Gamma \cap \{\infty > |z| > R\}$ for a large R and let U'' be the image of $G_1 \cap U'$. The map $\varphi \circ f$ can be represented as $\varphi \circ f(w) = (f_1(w), \cdots, f_m(w))$ on $f^{-1}(U)$, and each $f_i(w)$ $(i = 1, \cdots, m)$ satisfy $|f_i(w)| < M$, where M is a constant.

For some U, it is not possible that the upper part of the boundary of $U'' \cap \{|z| > R\}$ is always above the line $\{\arg w = \pi/2 - \delta, |w| > R\}$ for some $R > 0$, because otherwise $G_1 \cap A \cap \{|z| > R\}$ is contained in $f^{-1}(U)$ for large R and f has uniform limit p_0 in $G_1 \cap A$ according to the above lemma and it is a contradiction.

We can introduce a countable family of similar triangles $\{T_\lambda\}_{\lambda \in A}$ defined as follow: the base of T_λ lies on the positive real axis, the two other sides are equal and the vertex angle is $\delta/2$, for infinite $\Lambda' \subset \Lambda$, the vertex of T_λ $(\lambda \in \Lambda')$ is in $\{0 < \arg w < \pi/2 - \delta\} \cap \partial U''$ and $\bigcup_{\lambda \in A} T_\lambda$ is a simply connected domain which is contained in U''. We set $D = \bigcup_{\lambda \in A} T_\lambda$.

Let $\omega(w, D)$ be a harmonic measure with respect to D which is equal to 0 on the real axis and is equal to 1 on the rest of the boundary. Since $f(z)$ tends to the limit p_0 along Γ, $f_i(w)$ tends to 0 along the positive real axis.

We consider the level line L: $\omega(w, D) = \lambda$ $(0 < \lambda < 1)$. According to the above lemma, $f_i(w)$ tends to 0 along L, that is $f(w)$ tends to p_0 along L as $w \to \infty$. By the above construction, we can take a sequence $\{z_n\}_{n=1}^\infty \subset \partial D$ and a triangle T_n in $\{T_\lambda\}$ with vertex z_n such that $f(z_n) \in \partial U$. Let $\omega(w, T_n)$ be a harmonic measure with respect to T_n which is equal to 0 on the real axis and equal to 1 on the rest of the boundary. Since T_n is contained in D, $\omega(w, D) < \omega(w, T_n)$ on T_n by the maximum principle of harmonic measure. This implies that the level line $\omega(w, D) = \lambda$ is above the level line $\omega(w, T_n) = \lambda$ on T_n. We set $z_n = u + i \cdot v$. Let $Q_1(u + i \cdot v_1)$ and $Q_2(u + i \cdot v_2)$ be the intersections of the straight line $\mathrm{Re}\, w = u$ and the level curves $\omega(w, D) = \lambda$, $\omega(w, T_n) = \lambda$, respectively. Then $v_2 < v_1$. Since $G_1 \subset G$, we see $d_G < d_{G_1}$, where d_G, d_{G_1} are hyperbolic distance of G, G_1 respectively, and

$$d_G(z_n, Q_1) < d_{G_1}(z_n, Q_1)$$
$$= \int_{Q_1}^{z_n} \frac{1}{2} \cdot \sqrt{\frac{u^2 + v^2}{u^2 v^2}}\, |dw| < \frac{1}{2} \cdot \frac{1}{\sin \delta} \int_{Q_1}^{z_n} \frac{|dv|}{|v|}$$
$$= \frac{1}{2} \cdot \frac{1}{\sin \delta} \log \frac{v}{v_1} < \frac{1}{2} \cdot \frac{1}{\sin \delta} \log \frac{v}{v_2},$$

where $v/v_2 = k(\lambda)$ is independent on z_n. Since $k(\lambda) \to 1$ as $\lambda \to 1$, $\{z_n\}$, L can be chosen such that

$$d_A(z_n, L) < \varepsilon \qquad \text{for any } \varepsilon > 0. \qquad \text{q.e.d.}$$

We give a generalization of a result in Lehto-Virtanen [7].

Theorem 4.   *Let $f: \Delta \to N$ be a normal holomorphic mapping and let
f have an asymptotic value $p_0 \in N$ at a point $z_0 \in \partial\Delta$ along a Jordan curve
J lying in Δ.   Then f possesses the angular limit p_0 at the point z_0.*

Proof.   We repeat the discussion of [7], p. 52 or [9], p. 86–87 for self-containedness of this paper.

Suppose that f does not possess the angular limit p_0 at z_0.   Then,
by Theorem 3, for a small neighbourhood U of p_0 there exists a Jordan
curve L in Δ with end point at z_0 on which $f(z)$ tends to p_0, and a sequence
$\{z_n\}_{n=1}^\infty$ which converges to z_0 and $f(z_n) \in \partial U$ and $d_\Delta(z_n, L) < M < \infty$.   By
compactness of ∂U, we can take $\{z_n\}_{n=1}^\infty$ such that $f(z_n) \to p' \in \partial U$.   Denote
by $z' = S_n(z)$ a function which maps U conformally onto itself and satisfies
the condition $S_n(0) = z_n$.   We denote by K the hyperbolic circle with
center at $z = 0$ and hyperbolic radius $M + 1$.   Since d_Δ is invariant by
S_n, S_n^{-1} maps some arcs of L into K.   We denote by C_n the unions of such
arcs of L for a number n.   Since $\{f \circ S_n\}$ is a normal family, a subsequence
$\{f \circ S_{n_k}\}_{k=1}^\infty$ converges to a holomorphic mapping $\varphi: \Delta \to N$.   As $f(S_n(0)) =$
$f(z_n) = p_n$ converges to $p' (\neq p_0)$, $\varphi(0) = p'$.   While, $C = \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} S_n^{-1}(C_n)}$
$\subset K$ possesses a continuum on which $\varphi(z) = p_0$, and hence $\varphi(z) \equiv p_0$.   We
have thus arrived at a contradiction and the theorem is proved.   q.e.d.

Corollary.   *Let N be a paracompact connected complex analytic space
and let M be a hyperbolically embedded subspace of N.   We consider a holo-
morphic mapping $f: \Delta \to N$ with $f(\Delta) \subset M$.   If f has an asymptotic value
$p_0 \in N$ at $z_0 \in \partial\Delta$ along a Jordan curve J lying in Δ, then f possesses the
angular limit p_0 at z_0.*

Therefore we can see that the Lindelöf's theorem holds for the fol-
lowing mappings (cf. § 2, Example 1, (iv)).

Example 3.   (i)   Let $f: \Delta \to P^n$ be a holomorphic mapping which omits
$D = \bigcup_{i=1}^{2n+1} H_i$, where $\{H_i\}_{i=1}^{2n+1}$ are hyperplanes in general position.

(ii)   Let $f: \Delta \to T = C^n/L$ be a holomorphic mapping which omits D
$\subset T$, where D is a hypersurface of T not containing a proper complex
subtorus.

We give an interesting example of the non-normal holomorphic map-
ping.

Example 4.   We denote (w_0, w_1, w_2) the homogeneous coordinates of P^2.
We define a holomorphic mapping $f: \Delta \to P^2$ by

$$z \longmapsto (2, z + 1, (1 - z)e^{(1+z)/(1-z)})$$

and four complex lines ℓ_0, ℓ_1, ℓ_2, ℓ_3 in P^2 in general position as follows,

$$\ell_0 \colon w_0 = 0 , \qquad \ell_1 \colon w_1 = 0 ,$$
$$\ell_2 \colon w_1 - w_2 = 0 , \qquad \ell_3 \colon w_0 - w_1 - w_2 = 0 .$$

Then we can easily see that f omits $D = \bigcup_{i=0}^{3} \ell_i$. We remark that $\mathrm{Re}\,(1 + z)/(1 - z) = 1$ on $\{z \in \varDelta; |z - 1/2| = 1/2\}$ and $\lim_{r \to 1-0}(1 - r)e^{(1+r)/(1-r)} = \infty$. Hence, we see that $f(z)$ has an asymptotic value $(1, 1, 0) \in P^n$ at $z = 1$ along a curve $\{z \in \varDelta, |z - 1/2| = 1/2\}$, while f has a radial limit $(0, 0, 1) \in P^2$ at $z = 1$. Therefore f is not a normal holomorphic mapping according to Theorem 4. But this mapping f is of *bounded type* (i.e. $\limsup_{r \to 1} T(f, r) < \infty$).

In fact, let $f^* ds_{P_2}^2 = \rho(f(z))^2 \, dz d\bar{z}$, where $ds_{P_2}^2$ is the Fubini-Study metric on P^2, then $\rho(f(z)) = \partial^2/\partial z \partial \bar{z} \log \|f(z)\|^2$, where

$$\|f(z)\|^2 = \sum_{i=0}^{2} |f_i|^2 , \qquad f = (f_0, f_1, f_2) ,$$

and we have

$$T(f, r) = \frac{1}{\pi} \int_0^r \frac{dt}{t} \int_{\varDelta(t)} \rho(f(z))^2 dx dy$$
$$= \frac{1}{2\pi r} \int_0^{2\pi} \log \|f(re^{i\theta})\| \, d\theta - \log \|f(0)\| .$$

Using

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{|1 - re^{i\theta}|^2} d\theta = 1 \qquad (0 \leqq r < 1) ,$$

we easily see $T(f, r) \leqq C$, where C is some finite constant.

§6. Hyperbolic analysis and generalization of the Lindelöf's theorem

Let M be a connected complex analytic space, d_M be its *Kobayashi pseudo-distance* (cf. [6]), and let $A(M)$ be the set of *non-hyperbolic points*, i.e.

$$A(M) = \{p \in M; \text{ there exists a point } q \in M \text{ such that}$$
$$q \neq p \text{ and } d_M(p, q) = 0\} .$$

We remark that $d_M(p, q)$ is continuous with respect to p and q (cf. T. J. Barth [1]).

We give a condition of normality for holomorphic mapping in the term of hyperbolic analysis.

PROPOSITION 3. *Let M be a connected compact complex manifold and let $f: \Delta \to M$ be a holomorphic mapping. We define a global cluster set of f by*

$$C(f) = \{p \in M; \text{ there exists a sequence } \{z_n\}_{n=1}^{\infty} \subset \Delta \text{ such that}$$
$$|z_n| \longrightarrow 1, \ f(z_n) \longrightarrow p\} \ .$$

If $C(f) \cap A(M) = \phi$, then f is a normal holomorphic mapping.

Proof. Suppose that f is not normal. We take a hermitian metric ds_M^2 of M. Then there exists some sequence $\{G_n\}_{n=1}^{\infty} \subset \operatorname{Aut} \Delta$ such that

$$(1) \qquad\qquad (f \circ G_n)^* ds_M^2 / ds_{\Delta}^2 \left(0, \ \frac{\partial}{\partial z}\right) \longrightarrow \infty \qquad (n \longrightarrow \infty)$$

and $|G_n(0)| \to 1$ $(n \to \infty)$ according to Section 1.

We set $g_n = f \circ G_n$ and let $p \in M$ be an accumulated point of the set $\{g_n(0)\}_{n=1}^{\infty} \subset M$. (1) implies

$$(2) \qquad\qquad \left\| g_{n*}\left(\frac{\partial}{\partial z}\right)_{z=0} \right\|_M \longrightarrow \infty \ ,$$

where $\| \ \|_M$ is the length measured by ds_M^2. Let U be a sufficiently small coordinate neighbourhood of p. Using the Cauchy estimate, we see that for any positive integer n and sufficiently large m

$$g_n\left(\Delta\left(\frac{1}{m}\right)\right) \cap \partial U \neq \phi \ ,$$

where $\Delta(1/m)$ is a disc of radius $1/m$. That is, there exists a sequence $\{x_m\}_{m=1}^{\infty} \subset \partial U$ such that $x_m \in g_n(\Delta(1/m))$. We see

$$d_M(g_n(0), \ x_m) \leqq d_{\Delta}\left(0, \ \frac{1}{m}\right) \longrightarrow 0 \qquad (m \longrightarrow \infty) \ .$$

By continuity of $d_M(p, q)$, we see that there exists a point $q \in \partial U$ such that $d_M(p, q) = 0$, that is $p \in A(M)$. This is a contradiction and the theorem is proved.   q.e.d.

We give another type of generalizations of the Lindelöf's theorem.

THEOREM 5. *Let f be a holomorphic mapping from Δ to a connected*

complex analytic space M. Suppose that f has an asymptotic value $p_0 \in M$ at $z_0 \in \partial \Delta$ along a Jordan curve in Δ and that f does not possess the angular limit p_0 at z_0. Then $p_0 \in \overline{A(M)}$.

Proof. We remark that d_M induces the usual topology on $M - \overline{A(M)}$ (cf. Barth [1]). Suppose p_0 is not in $\overline{A(M)}$. Then we can take 2ε-neighbourhood U with respect to d_M which is contained in $M - \overline{A(M)}$. According to Theorem 3, there exist a Jordan curve L in Δ with end point z_0 and a sequence $\{z_n\}_{n=1}^{\infty}$ which converges to z_0 such that f has an asymptotic value p_0 along L and satisfy the condition $f(z_n) \in \partial U$ and $d_\Delta(z_n, L) < \varepsilon$.

Let u_n be points of L which satisfy $d_\Delta(z_n, u_n) < \varepsilon$. Then,

$$d_M(f(z_n), p_0) \leqq d_M(f(z_n), f(u_n)) + d_M(f(u_n), p_0) .$$

For large n,

$$d_M(f(u_n), p_0) > \frac{1}{2}\varepsilon .$$

On the other hand,

$$d_M(f(z_n), f(u_n)) \leqq d_\Delta(z_n, u_n) .$$

Hence,

$$d_M(f(z_n), p_0) < \frac{3}{2}\varepsilon \qquad \text{for large } n .$$

As $d_M(f(z_n), p_0) = 2\varepsilon$, this is a contradiction and hence the theorem is proved. q.e.d.

COROLLARY. *Let M be a hyperbolic complex space and $f: \Delta \to M$ be a holomorphic mapping. If f has an asymptotic value $p_0 \in M$ at a point $z_0 \in \partial \Delta$ along a Jordan curve lying in Δ, then f possesses the angular limit p_0 at z_0.*

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Production Technology Laboratories
Engineering Center
Sharp Corpolation
2613-1 Ichinomoto-cho, Tenri-shi, Nara 632
Japan