The discrete version of the Aharonov–Bohm potential corresponding to the magnetic flux Ψ is defined as

$$\mathcal{A}_\Psi(k) = -i\,(A_1(k), A_2(k)) = -i\left(1 - e^{2\pi i \Psi \phi_1(k)}, 1 - e^{2\pi i \Psi \phi_2(k)}\right).$$

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The main result is Theorem 1.1 of [91]:

Theorem 5.19 *For all functions $u\colon \mathbb{Z}^2 \to \mathbb{C}$ decaying sufficiently fast,*

$$\sum_{k\in\mathbb{Z}^2}\sum_{j=1,2} \left|u(k+e_j) - u(k) + iA_j(k)u(k)\right|^2 \geq 4\sin^2\left(\pi\,\frac{dist(\Psi,\mathbb{Z})}{8}\right) \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{|u(k)|^2}{|k|_\infty^2}. \tag{5.6.6}$$

Since $\mathrm{dist}(\Psi, \mathbb{Z}) \leq 1/2$, we have

$$4\sin^2\left(\pi\,\frac{\mathrm{dist}(\Psi,\mathbb{Z})}{8}\right) \geq 4\left[\pi\,\frac{\mathrm{dist}(\Psi,\mathbb{Z})}{8}\right]^2 \frac{\sin^2(\frac{\pi}{16})}{\left(\frac{\pi}{16}\right)^2} = 16\sin^2\left(\frac{\pi}{16}\right)\min_{l\in\mathbb{Z}} |l-\Psi|^2,$$

and (5.6.6) implies

Corollary 5.20 *For all functions $u\colon \mathbb{Z}^2 \to \mathbb{C}$ decaying sufficiently fast,*

$$\sum_{k\in\mathbb{Z}^2}\sum_{j=1,2} \left|u(k+e_j) - u(k) + iA_j(k)u(k)\right|^2 \geq 16\sin^2\left(\frac{\pi}{16}\right)\min_{k\in\mathbb{Z}^2} |l-\Psi|^2 \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{|u(k)|^2}{|k|_\infty^2}. \tag{5.6.7}$$

Note that $16\sin^2\left(\frac{\pi}{16}\right) = 4\left(2-\sqrt{2+\sqrt{2}}\right) \sim 0.50896....$

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6

Fractional Analogues

6.1 Special Cases and Consequences

6.1.1 Fractional Hardy Inequalities on $\mathbb{R}^n$ and $\mathbb{R}^n_+$

The fractional Hardy inequality on a domain $\Omega \subset \mathbb{R}^n$ with non-empty boundary $\partial\Omega$ has the form

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dx \geq C(s,p,\Omega)\int_{\Omega} \frac{|u(x)|^p}{\delta(x)^s}\,dx, \ \ u \in C_0^\infty(\Omega), \quad (6.1.1)$$

where $1 < p < \infty$, $0 < s < 1$, $\delta(x) := \inf\{|x-y| : y \in \mathbb{R}^n \setminus \Omega\}$ and $C(s,p,\Omega)$ is a positive constant which is independent of u. The expression on the left-hand side of (6.1.1) is $[u]^p_{s,p,\Omega}$, where $[u]_{s,p,\Omega}$ is the Gagliardo seminorm of u defined in Section 3.1.

We begin our investigation of these inequalities with important special cases on the half-space $\mathbb{R}^n_+ = \{x\colon x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ x_n > 0\}$ and $\mathbb{R}^n$, and examine significant implications in the latter case for results on the limiting behaviour of fractional inequalities from [23] and [134] discussed in Section 3.2. The first theorem was proved by Bogdan and Dyda in [22] in the case $p = 2$ and extended to all other values of p in [83]. We denote by $\overset{0}{W}{}^s_p(\mathbb{R}^n_+)$ the completion of $C_0^\infty(\mathbb{R}^n_+)$ with respect to the $W^s_p(\mathbb{R}^n_+)$ norm; for $ps < 1$ this coincides with the completion of $C_0^\infty(\overline{\mathbb{R}^n_+})$.

Theorem 6.1 *Let* $n \geq 1$, $1 \leq p < \infty$ *and* $0 < s < 1$ *with* $ps \neq 1$. *Then, for all* $u \in \overset{0}{W}{}^s_p(\mathbb{R}^n_+)$,

$$\int_{\mathbb{R}^n_+\times\mathbb{R}^n_+} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dx \geq D_{n,s,p}\int_{\mathbb{R}^n_+} \frac{|u(x)|^p}{|x|^{ps}}\,dx, \quad (6.1.2)$$

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with sharp constant

$$D_{n,p,s} := 2\pi^{(n-1)/2} \frac{\Gamma\left((1+ps)/2\right)}{\Gamma\left((n+ps)/2\right)} \int_0^1 |1 - r^{(ps-1)/2}|^p \frac{dr}{(1-r)^{1+ps}}. \qquad (6.1.3)$$

If $p = 1$ and $n = 1$, equality holds if and only if u is proportional to a non-increasing function. If $p > 1$ or if $p = 1$ and $n \geq 2$, the inequality is strict for any non-trivial function $u \in \overset{0}{W}{}^s_p(\mathbb{R}^n_+)$.

The theorem follows from the special case $\Omega = \mathbb{R}^n_+$ of an abstract Hardy inequality in [83], Proposition 2.2,

$$E[u] := \int_\Omega \int_\Omega |u(x) - u(y)|^p k(x,y)\, dx\, dy \geq \int_\Omega V(x)|u(x)|^p\, dx, \qquad (6.1.4)$$

on compactly supported functions u on $\Omega \subset \mathbb{R}^n$, under the following assumptions. There exists a family of measurable functions k_ε $(\varepsilon > 0)$ on $\Omega \times \Omega$ satisfying $k_\varepsilon(x,y) = k_\varepsilon(y,x),\ 0 \leq k_\varepsilon(x,y) \leq k(x,y)$ and $\lim_{\varepsilon\to 0} k_\varepsilon(x,y) = k(x,y)$ for a.e. $x, y \in \Omega$. Moreover, with w a positive, measurable function on Ω, the integrals

$$V_\varepsilon(x) := 2w(x)^{-p+1} \int_\Omega (w(x) - w(y))\, |w(x) - w(y)|^{p-2} k_\varepsilon(x,y)\, dy$$

are absolutely convergent for a.e. x, belong to $L_{1,loc}(\Omega)$, and $\int V_\varepsilon \phi\, dx \to \int V\phi\, dx$ for any bounded ϕ with compact support in Ω. For the proof of Theorem 6.1, $\Omega = \mathbb{R}^n_+$ and setting $\alpha := (1-ps)/p$, the following choices are made:

$$w(x) = x_n^{-\alpha},\ k(x,y) = |x-y|^{-n-ps},\ k_\varepsilon(x,y) = |x-y|^{-n-ps}\chi_{|x_n - y_n|}.$$

Then [82], Lemma 3.1 gives that $V(x) = D_{n,p,s} x_n^{-ps}$ and hence

$$2\lim_{\varepsilon\to 0} \int_{||x|-|y||>\varepsilon} (w(x) - w(y))\, |w(x) - w(y)|^{p-2} k(x,y)\, dy = \frac{D_{n,p,s}}{|x|^{ps}} w(x)^{p-1}. \qquad (6.1.5)$$

Therefore (6.1.2) is established. We refer to the proof of Theorem 1.1 in [83] for showing that the constant $D_{n,p,s}$ in (6.1.3) is optimal and also for details on the remainder of Theorem 6.1.

The approach sketched above for establishing Theorem 6.1 in [83], based on (6.1.4) with $\Omega = \mathbb{R}^n_+$, is used for $\Omega = \mathbb{R}^n$ in [82], and, in fact, will be used for a general domain Ω in Section 6.2. The choices

$$w(x) = |x|^{-\alpha},\ k(x,y) = |x-y|^{-n-ps},\ V(x) = C(n,s,p)|x|^{-ps}$$

yield the following modification in [82]:

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Theorem 6.2 *Let $n \geq 1$ and $0 < s < 1$. Then for all $u \in \overset{0}{W}{}^s_p(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$ (see (3.2.2)) if $1 \leq p < n/s$, and for all $u \in \overset{0}{W}{}^s_p(\mathbb{R}^n \setminus \{0\})$ if $p > n/s$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dx \geq C(n,s,p)\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{ps}}\,dx, \tag{6.1.6}$$

where

$$C_{n,s,p} := 2\int_0^1 r^{ps-1}|1-r^{(n-ps)/p}|^p\,\Phi_{n,s,p}(r)\,dr, \tag{6.1.7}$$

and

$$\Phi_{n,s,p}(r) := \omega_{n-2}\int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{(1-2rt+r^2)^{(n+ps)/2}}\,dt,\ \ n \geq 2,$$

$$\Phi_{1,s,p}(r) := \left(\frac{1}{(1-r)^{1+ps}} + \frac{1}{(1+r)^{1+ps}}\right),\ \ n-1.$$

The constant $C_{n,s,p}$ is optimal. If $p = 1$, equality holds if and only if u is proportional to a symmetric decreasing function. If $p > 1$, the inequality is strict for any non-trivial function $u \in W^s_p(\mathbb{R}^n)$ or $\overset{0}{W}{}^s_p(\mathbb{R}^n \setminus \{0\})$, respectively.

6.1.2 The Limiting Cases of $s \to 0+$ and $s \to 1-$

It is proved in [134], Theorem 2, that for $n \geq 1,\ 0 < s < 1,\ 1 \leq p < n/s$ and $u \in W^s_p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \geq c(n,p)\frac{(n-sp)^p}{s(1-s)}\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{ps}}\,dx \tag{6.1.8}$$

for some constant $c(n,p)$ which depends only on n and p. Since $C(n,s,p)$ in Theorem 6.2 is optimal, it follows that

$$c(n,p)\frac{(n-sp)^p}{s(1-s)} \leq C(n,s,p). \tag{6.1.9}$$

There are related consequences of Corollary 3.2.19, where we saw that for $p \in (1,\infty)$ and $u \in W^s_p(\mathbb{R}^n)$, there exists a positive constant $K(p,n)$ such that

$$\lim_{s\to 1-}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = \frac{K(p,n)}{p}\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx$$

and hence by Hardy's inequality,

$$\lim_{s\to 1-}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \geq \frac{K(p,n)}{p}\left(\frac{p-1}{p}\right)^p\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p}\,dx. \tag{6.1.10}$$

Also, for $u \in \bigcup_{0<s<1} W_p^s(\mathbb{R}^n)$, there exists a positive constant $C'(n,p) \approx p^{-1}n$ such that

$$\lim_{s\to 0+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = C'(n,p)\int_{\mathbb{R}^n} |u(x)|^p dx; \tag{6.1.11}$$

see Remark 3.2.20.

It is fitting to recall here that for $ps < 1$, the Sobolev embedding theorem asserts that $W_p^s(\mathbb{R}^n) \hookrightarrow L_{p^*}(\mathbb{R}^n)$, where $p^* = np/(n-ps)$, and

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p} \geq S_{n,s,p}\int_{\mathbb{R}^n} |u(x)|^{p*} dx. \tag{6.1.12}$$

The optimal values of the constants $S_{n,s,p}$ are not known. Estimates are given in [23] which reflect the correct behaviour as s tends to 1; in [134], Theorem 1, the sharp constant is shown to satisfy

$$S_{n,s,p} \geq c(n,p)\frac{(n-ps)^{p-1}}{s(1-s)} \tag{6.1.13}$$

for some positive constant $c(n,p)$, and for $u \in \bigcup_{0<s<1} W_p^s(\mathbb{R}^n)$, the asymptotic result

$$\lim_{s\to 0+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = 2p^{-1}\omega_{n-1}\int_{\mathbb{R}^n} |u(x)|^p\,dx \tag{6.1.14}$$

is proved in [134] Theorem 3. The following asymptotic result when $s \to 1-$ is established in [150] and [30] for all $u \in C_0^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ convex:

$$\lim_{s\to 1-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = \alpha_{n,p}\int_{\Omega} |\nabla u(x)|^p\,dx, \tag{6.1.15}$$

where, with $\mathbf{e_1} = (1,0,0,...,0)$,

$$\alpha_{n,p} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} |\langle\sigma\cdot\mathbf{e_1}\rangle|^p d\sigma = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n+p}{2}\right)}.$$

Thus an extension of Corollary 3.20 is achieved which allows for the inequality on a convex subset Ω of $\mathbb{R}^n$.

In [145], Peetre proved that the standard Sobolev embedding $W_p^s(\mathbb{R}^n) \hookrightarrow L_{p*}(\mathbb{R}^n)$ can be refined to $W_p^s(\mathbb{R}^n) \hookrightarrow L_{p*,p}(\mathbb{R}^n)$. The following sharp inequality associated with this embedding is given in [82], Theorem 4.1:

Theorem 6.3 *Let $n \in \mathbb{N}$, $0 < s < 1$, $1 \leq p < n/s$ and $p^* = np/(n-ps)$. Then $W_p^s(\mathbb{R}^n) \hookrightarrow L_{p^*,p}(\mathbb{R}^n)$ and*

$$\|u\|_{p^*,p} \leq \left(\frac{n}{\omega_{n-1}}\right)^{s/n} C_{n,p,s}^{-1/p}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p} \tag{6.1.16}$$

for any $u \in W_p^s(\mathbb{R}^n)$ *with* $C_{n,p,s}$ *from (6.1.7). The constant is sharp. For* $p = 1$*, equality holds if* u *is proportional to a non-negative function* v *such that the level sets* $\{v > \tau\}$ *are balls for a.e.* τ*. For* $p > 1$ *the inequality is strict for any non-trivial* u*.*

In [21], Proposition 4.2, it is shown that if $1 \le p < r$ and $0 < q \le r \le \infty$, then

$$\|u\|_{q,r} \le \left(\frac{q}{p}\right)^{\frac{(r-p)}{rp}} \|u\|_{q,p}.$$

It follows that we have

Corollary 6.4 *Let* $n \ge 1,\ 0 < s < 1,\ 1 \le p < n/s,\ p^* = np/(n-ps) \le r \le \infty$ *and* $p < r$*. Then* $W_p^s(\mathbb{R}^n) \hookrightarrow L_{p^*,r}(\mathbb{R}^n)$ *and*

$$\|u\|_{p^*,r} \le \left(\frac{p*}{p}\right)^{\frac{r-p}{rp}} \left(\frac{n}{\omega_{n-1}}\right)^{s/n} C_{n,p,s}^{-1/p} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p}. \tag{6.1.17}$$

The choice $r = p^*$ gives the Sobolev inequality

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \ge S_{n,s,p}\|u\|_{p^*}^p$$

where

$$S_{n,s,p} = \left(\frac{p}{p^*}\right)^{ps/n} \left(\frac{\omega_{n-1}}{n}\right)^{ps/n} C_{n,s,p}. \tag{6.1.18}$$

The asymptotic inequality (6.1.11) is then recovered from (6.1.9).

It is proved in [82], Lemma 4.3, that for $0 < s \le 1$ and $1 \le p < n/s$, any non-negative symmetric decreasing function u on $\mathbb{R}^n$ satisfies

$$\|u\|_{p^*,p} = \left(\frac{n}{\omega_{n-1}}\right)^{s/n} \left(\int_{\mathbb{R}^n} \frac{u(x)^p}{|x|^{ps}}\,dx\right)^{1/p}. \tag{6.1.19}$$

This establishes the link between Theorem 6.3 and the sharp inequality (6.1.6).

The ‘local’ analogue of (6.1.16) with $s = 1$ is

$$\|u\|_{p^*,p} \le \left(\frac{n}{\omega_{n-1}}\right)^{1/n} \frac{p}{n-p} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx\right)^{1/p} \tag{6.1.20}$$

for $n \ge 2,\ 1 \le p < n$ and $p^* = np/(n-p)$. This is the inequality (5.1.4) with the optimal constant proved by Alvino in [10].

A fractional Hardy inequality which focuses on the dependence of the constant on s is featured in the following theorem in [25] in which the correct asymptotic behaviour with respect to s as $s \to 0+$ and $s \to 1-$ is exhibited in accordance with (6.1.14) and (6.1.15). Note that the inequality (6.1.21) is not of the type (6.1.1) since the integration on the left-hand side of (6.1.21) is over the whole of $\mathbb{R}^n \times \mathbb{R}^n$ rather than $\Omega \times \Omega$, as in (6.1.1). The result should be compared with Theorem 6.8, which has the same domain of integration on both sides, gives the right asymptotic behaviour as $s \to 1-$ and $s \to 0+$, and has the best constant, but at the expense of requiring the condition $1/p < s < 1$.

Theorem 6.5 *Let $1 < p < \infty$ and $0 < s < 1$, and let Ω be a proper open convex subset of $\mathbb{R}^n$, $n \geq 1$. Then for all $u \in C_0^\infty(\Omega)$ and with $\delta(x) := \inf\{|x - y| : y \notin \Omega\}$, there exists a positive constant $C(n, p)$ such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \geq C(n, p)\frac{1}{s(1 - s)}\int_\Omega \frac{|u(x)|^p}{\delta(x)^{ps}}\, dx. \qquad (6.1.21)$$

This is a nonlocal analogue of the inequality in Section 5.3 for the local Hardy inequality on convex domains. The proof is based on the property that for $0 < s < 1$ and $1 < p < \infty$, δ^s is (s, p)-*locally weakly superharmonic* in the following weak sense:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\delta(x)^s - \delta(y)^s|^{p-2}\,(\delta(x)^s - \delta(y)^s)\,(\phi(x) - \phi(y))}{|x - y|^{n+sp}}\, dx\, dy \geq 0 \quad (6.1.22)$$

for all non-negative $\phi \in W_s^p(\Omega)$ with compact support in Ω. This property is written as $(-\Delta_p)^s\delta^s \geq 0$, where $(-\Delta_p)^s$ is the fractional p-Laplacian of order s; see (3.4.8) for more details on the significance of this property. The pivotal result in [25], Proposition 3.2, is that if Ω is an open, bounded, convex subset of $\mathbb{R}^n$, then δ^s is locally weakly (s, p)-superharmonic. An interesting preliminary result from [102] (and from [38] when $p = 2$) is that for $\Omega := \mathbb{R}^n_+$, δ^s is locally weakly (s, p)-harmonic, i.e., δ^s and $-\delta^s$ are both locally weakly (s, p)-superharmonic.

The proof of Theorem 5.1.5 uses a Moser-type argument which is similar to that in Section 5.3, this time choosing

$$\phi = \frac{|u|^p}{(\delta^s + \varepsilon)^{p-1}}$$

where $u \in C_0^\infty(\Omega)$ and $\varepsilon > 0$; in [25], Lemma 2.4, this choice is proved to be admissible, i.e., $\phi \in W_s^p(\Omega)$. Another important role in the proof is played by the following fractional counterpart from [25], Proposition 2.5, of the property that $|\nabla\delta| = 1$ a.e. in Ω.

Proposition 6.6 *Let $1 < p < \infty,\ 0 < s < 1$, and let Ω be an open, bounded, convex subset of $\mathbb{R}^n$. Then*

$$\int_{y\in\Omega:\ \delta(y)\leq\delta(x)} \frac{|\delta(x) - \delta(y)|^p}{|x - y|^{n+sp}}\, dy \geq \frac{C_1}{1 - s}\delta(x)^{p(1-s)}$$

for a.e. $x \in \Omega$, *where*

$$C_1 = C_1(n,p) = \frac{1}{p} \sup_{0<\sigma<1\rangle} \left[\sigma^p \mathcal{H}^{n-1}\left(\{\omega \in \mathbb{S}^{n-1} \colon \langle \omega, e_1 \rangle > \sigma\}\right)\right];$$

$n-1$ *is the Hausdorff dimension of the exhibited set and* $\mathcal{H}^{n-1}(\cdot)$ *is its* $(n-1)$*-dimensional Hausdorff measure.*

An analysis of the dependence of (6.1.1) on Ω, and the permissible values of p and s, is given in the following theorem from [55]:

Theorem 6.7 *The inequality (6.1.1) holds in each of the following cases:*

1. Ω *is a bounded Lipschitz domain and* $ps > 1$*;*
2. $\Omega = \mathbb{R}^n \setminus K$*, where* K *is a bounded Lipschitz domain,* $ps \neq 1,\ ps \neq n$*;*
3. Ω *is a domain above the graph of a Lipschitz function* $\mathbb{R}^{n-1} \to \mathbb{R}^n$ *and* $ps \neq 1$*;*
4. Ω *is the complement of a point and* $ps \neq n$*.*

Furthermore, (6.1.1) does not hold if

1. Ω *is a bounded Lipschitz domain and* $ps \leq 1,\ s < 1$*;*
2. Ω *is the complement of a compact set and* $n = ps,\ s < p$*.*

A consequence is that in the integral on the left-hand side of (6.1.21), $\mathbb{R}^n \times \mathbb{R}^n$ cannot be replaced by $\Omega \times \Omega$ whenever Ω is bounded if $ps \leq 1$. It is observed in [55] (see also [44]) that if Ω is a bounded Lipschitz domain and $ps \leq 1$,

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{ps}}\,dx \leq C\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx + \int_\Omega |u(x)|^p\,dx\right) \tag{6.1.23}$$

for all $u \in C_0^\infty(\Omega)$, the inequality being false without the final term even for domains Ω with a C^∞ boundary.

In [23], Theorem 1, it is proved for Ω a cube in $\mathbb{R}^n$ $(n > 1)$, $0 < s < 1,\ p > 1,\ ps < n,\ \frac{1}{q} = \frac{1}{p} - \frac{s}{n}$ and $u \in W_s^p(\Omega)$, that

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx \geq C(n)(n/q)^p(1-s)^{-1} \int_\Omega |u(x)-u_\Omega|^q\,dx, \tag{6.1.24}$$

where $u_\Omega := (1/|\Omega|)\int_\Omega u(x)\,dx$ and $C(n)$ depends only on n. Related work on inequalities of fractional Hardy and Poincaré type may be found in [58], [62], [99], [100] and [101].

6.2 General Domains

This section is mainly devoted to the fractional analogue of the local Hardy-type inequality in Theorem 5.10 on a general domain Ω in $\mathbb{R}^n (n \geq 2)$ with non-empty boundary, involving a mean distance function $M_{p,\Omega}$. The appropriate mean distance function now depends on $s \in (0, 1)$ and is defined by

$$\frac{1}{M_{s,p}(x)^{ps}} := \frac{\pi^{1/2}\Gamma\left(\frac{n+ps}{2}\right)}{\Gamma\left(\frac{1+ps}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^{ps}(x)}\, d\omega(\nu), \tag{6.2.1}$$

where $1/p < s < 1$,

$$\tau_\nu(x) = \min\{t > 0 \colon x + t\nu \notin \Omega\}, \;\; \delta_\nu(x) = \min\{\tau_\nu(x), \tau_{-\nu}(x)\}$$

and the surface measure ω on $\mathbb{S}^{n-1}$ is normalised, i.e., $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$. If Ω is convex, then, as in Theorem 5.12, $M_{s,p}(x) \leq \delta(x)$.

Loss and Sloane show in [130] that Theorem 6.1 continues to hold with the same sharp constant for any convex domain Ω. Their proof makes use of a mean distance function and it is this which is the basis of this section.

Theorem 6.8 *Let Ω be an open subset of $\mathbb{R}^n$ with non-empty boundary, let $p \in (1, \infty)$ and $s \in (1/p, 1)$. Then for all $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}}\, dx\, dy \geq D_{n,p,ps} \int_\Omega \frac{|f(x)|^p}{M_{s,p}(x)^{ps}}\, dx, \tag{6.2.2}$$

where

$$D_{n,p,ps} := \frac{\pi^{(n-1)/2}\Gamma\left(\frac{1+ps}{2}\right)}{\Gamma\left(\frac{n+ps}{2}\right)} D_{1,p,ps} \tag{6.2.3}$$

and

$$D_{1,p,ps} := 2 \int_0^1 \frac{|1 - r^{(ps-1)/p}|}{(1 - r)^{1+ps}}\, dr. \tag{6.2.4}$$

For Ω convex,

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}}\, dx\, dy \geq D_{n,p,ps} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{ps}}\, dx. \tag{6.2.5}$$

The inequality (6.2.5) with some constant $C(n, p)$ continues to hold for $0 < p \leq 1$, but the optimal value of the constant is not known, see [55].

The proof depends on the following one-dimensional inequality.

Lemma 6.9 *Let $1 < p < \infty$, $1 < p < s < 1$ and $f \in C_0^\infty(a, b)$. Then*

$$\int_a^b \int_a^b \frac{|f(x) - f(y)|^p}{|x - y|^{1+ps}}\, dx\, dy \geq D_{1,p,ps} \int_a^b \frac{|f(x)|^p}{\min\{(x - a), (b - x)\}^{ps}}\, dx. \tag{6.2.6}$$

Moreover, if $J \subset (a,b)$ is an open set and $f \in C_0^\infty(J)$, then for all $f \in C_0^\infty(J)$,

$$\int_J\int_J \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy \geq D_{1,p,ps}\int_J \frac{|f(x)|^p}{\delta_J(x)^{ps}}\,dx; \tag{6.2.7}$$

J is a countable union of disjoint intervals I_k, and so for $x \in J$ there is a unique interval I_k containing x and then $\delta_J(x) = \delta_{I_k(x)} := \inf\{|t| : x+t \notin I_k\}$.

Proof From [83], Proposition 2.2 and Lemma 2.4,

$$\int_a^b\int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha}}\,dx\,dy \geq \int_a^b V(x)|f(x)|^p dx,$$

where, with $w(x) = \delta_{(a,b)}(x)^{-(1-\alpha)/p}$, $\alpha = ps$,

$$V(x) = \frac{2}{w(x)^{p-1}}\int_a^\infty (w(x)-w(y))\,|w(x)-w(y)|^{p-2}\frac{dy}{|x-y|^{1+\alpha}}. \tag{6.2.8}$$

It is necessary to prove that $V(x) \geq \frac{1}{\delta(x)^\alpha}\mathcal{D}_{1,p,\alpha}$.

Let

$$\begin{aligned} I(x) :=&\ 2\textstyle\int_0^\infty (w(x)-w(y))\,|w(x)-w(y)|^{p-2}\,\frac{dy}{|x-y|^{1+\alpha}} \\ =&\ 2\left(\textstyle\int_0^x + \int_x^\infty\right)(w(x)-w(y))\,|w(x)-w(y)|^{p-2}\,\frac{dy}{|x-y|^{1+\alpha}} \\ =:&\ 2\left(I_1(x)+I_2(x)\right), \end{aligned} \tag{6.2.9}$$

where

$$w(x) = x^{-(1-\alpha)/p},\ 1 < p < \infty,\ 1 < \alpha < 2.$$

Note that, to be precise, the integrals in (6.2.8), like the integral in (6.2.7), are principal values, being over $(0,\varepsilon)$ and (ε,∞) and the limit as $\varepsilon \to 0+$ taken. Then, on putting $y = tx$,

$$\begin{aligned} I_1(x) &= x^{\frac{(\alpha-1)}{p}(p-1)-\alpha}\int_0^1 \left(1-t^{\frac{\alpha-1}{p}}\right)\left|\left(1-t^{\frac{\alpha-1}{p}}\right)\right|^{p-2}\frac{dt}{|1-t|^{1+\alpha}} \\ &= x^{\frac{(\alpha-1)}{p}(p-1)-\alpha}\int_0^1 \left|\left(1-t^{\frac{\alpha-1}{p}}\right)\right|^{p-1}\frac{dt}{|1-t|^{1+\alpha}}. \end{aligned}$$

On putting $y = x/t$,

$$\begin{aligned} I_2(x) &= -x^{\frac{(\alpha-1)}{p}(p-1)-\alpha}\int_1^0 \left(1-t^{-(\alpha-1)/p}\right)\left|1-t^{-(\alpha-1)/p}\right|^{p-2}\frac{dt}{t^2\,|1-1/t|^{1+\alpha}} \\ &= -x^{\frac{(\alpha-1)}{p}(p-1)-\alpha}\int_1^0 \left(t^{(\alpha-1)/p}-1\right)\left|t^{(\alpha-1)/p}-1\right|^{p-2}\frac{t^{(\alpha-1)/p}dt}{|1-t|^{1+\alpha}} \\ &= -x^{\frac{(\alpha-1)}{p}(p-1)-\alpha}\int_0^1 \left|1-t^{\frac{\alpha-1}{p}}\right|^{p-1}\frac{t^{(\alpha-1)/p}dt}{|1-t|^{1+\alpha}}. \end{aligned}$$

Thus

$$I(x) = 2x^{\frac{(\alpha-1)}{p}(p-1)-\alpha} \int_0^1 \left|1 - t^{(\alpha-1)/p}\right|^{p-1} \left(1 - t^{(\alpha-1)/p}\right) \frac{dt}{|1-t|^{1+\alpha}}$$
$$= 2x^{\frac{(\alpha-1)}{p}(p-1)-\alpha} \int_0^1 \left|1 - t^{(\alpha-1)/p}\right|^{p} \frac{dt}{|1-t|^{1+\alpha}}$$
$$= x^{\frac{(\alpha-1)}{p}(p-1)-\alpha} \mathcal{D}_{1,p,\alpha}; \tag{6.2.10}$$

also

$$I_1(x) \geq x^{\frac{(\alpha-1)}{p}(p-1)-\alpha} \mathcal{D}_{1,p,\alpha}. \tag{6.2.11}$$

From (6.2.7) with $w(x) = \delta_{(a,b)}(x)^{(\alpha-1)/p}$ and $c = (1/2)(a+b)$,

$$w(x)^{p-1} V(x) = 2 \int_a^c (w(x) - w(y)) \, |w(x) - w(y)|^{p-2} \frac{dy}{|x-y|^{(1+\alpha)}} \tag{6.2.12}$$

for $a < x < c$ and

$$w(x)^{p-1} V(x) = 2 \int_c^b (w(x) - w(y)) \, |w(x) - w(y)|^{p-2} \frac{dy}{|x-y|^{(1+\alpha)}} \tag{6.2.13}$$

for $c < x < b$. Similar calculations to those which yield (6.2.9) and (6.2.10) now give the following: for $a < x < c$,

$$\frac{2}{w(x)^{p-1}} \int_a^\infty (w(x) - w(y)) \, |w(x) - w(y)|^{p-2} \frac{dy}{|x-y|^{(1+\alpha)}}$$
$$= \frac{1}{(x-a)^\alpha} \mathcal{D}_{1,p,\alpha} \tag{6.2.14}$$

and

$$V(x) \geq \frac{1}{(x-a)^\alpha} \mathcal{D}_{1,p,\alpha}, \tag{6.2.15}$$

while for $c < x < b$,

$$\frac{2}{w(x)^{p-1}} \int_{-\infty}^b (w(x) - w(y)) \, |w(x) - w(y)|^{p-2} \frac{dy}{|x-y|^{(1+\alpha)}}$$
$$= \frac{1}{(b-x)^\alpha} \mathcal{D}_{1,p,\alpha} \tag{6.2.16}$$

and

$$V(x) \geq \frac{1}{(b-x)^\alpha} \mathcal{D}_{1,p,\alpha}. \tag{6.2.17}$$

Therefore, $V(x) \geq \frac{1}{\delta(x)^\alpha} \mathcal{D}_{1,p,\alpha}$ and (6.2.5) is proved.

Since an open subset J is the countable union of disjoint intervals I_k, we have from (6.2.5)

$$\int_J\int_J \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy \geq \sum_{k=1}^{\infty}\int_{I_k}\int_{I_k}\frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy$$
$$\geq \sum_{k=1}^{\infty} D_{1,p,ps}\int_{I_k}\frac{|f(x)|^p}{\delta_{I_k}(x)^{ps}}\,dx$$
$$\geq D_{1,p,ps}\int_J \frac{|f(x)|^p}{\delta_J(x)^{ps}}\,dx.$$

□

We now quote Lemma 2.4 in [130] which leads to the application of Lemma 6.9 and proof of Theorem 6.8.

Lemma 6.10 *Let Ω be a domain in $\mathbb{R}^n$. Then for all $f \in C_0^\infty(\Omega)$,*

$$\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy$$
$$= \frac{\omega_{n-1}}{2}\int_{\mathbb{S}^{n-1}}d\omega\int_{x:\,x\cdot\omega=0}dL_\omega(x)\int_{x+s\omega\in\Omega}ds\int_{x+t\omega\in\Omega}\frac{|f(x+s\omega)-f(x+t\omega)|^p}{|s-t|^{1+ps}}\,dt \tag{6.2.18}$$

where $\mathcal{L}_\omega$ denotes the $(n-1)$-dimensional Lebesgue measure on the plane $x\cdot\omega = 0$; recall that the measure $d\omega$ on $\mathbb{S}^{n-1}$ is normalised.

Proof Let

$$I_\Omega(f) := \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{1+ps}}\,dx\,dy = \int_\Omega dx\int_{x+z\in\Omega}\frac{|f(x)-f(x+z)|^p}{|z|^{n+ps}}\,dz.$$

On using polar co-ordinates $z = r\omega$, we obtain

$$I_\Omega(f) = \omega_{n-1}\int_\Omega dx\int_{\mathbb{S}^{n-1}}d\omega\int_{x+r\omega\in\Omega,\,r>0}\frac{|f(x)-f(x+r\omega)|^p}{r^{1+ps}}\,dr$$
$$= \frac{1}{2}\omega_{n-1}\int_{\mathbb{S}^{n-1}}d\omega\int_\Omega dx\int_{x+h\omega\in\Omega}\frac{|f(x)-f(x+h\omega)|^p}{|h|^{1+ps}}\,dh.$$

The domain of integration $\{x + h\omega \in \Omega\}$ in the innermost integral is the line $x + h\omega$ intersected with Ω. On splitting the variable x into components perpendicular to ω and parallel to ω, i.e., replacing x by $x + l\omega$, where $x\cdot\omega = 0$, we derive

$$\frac{\omega_{n-1}}{2}\int_{\mathbb{S}^{n-1}}d\omega\int_{x:\,x\cdot\omega=0}d\mathcal{L}_\omega(x)\int_{x+l\omega\in\Omega}dl\int_{x+(l+h)\omega\in\Omega}\frac{|f(x+l\omega)-f(x+(l+h)\omega)|^p}{|h|^{1+ps}}\,dh.$$

The lemma follows by the variable change $t = l + h$.

□

Proof of Theorem 6.8 By Lemma 6.10 and (6.2.6),

$$
\begin{aligned}
&\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{1+ps}} \, dx \, dy \\
&= \frac{1}{2}\omega_{n-1} \int_{\mathbb{S}^{n-1}} d\omega \int_{x:\, x\cdot\omega=0} dL_\omega(x) \int_{x+l\omega\in\Omega} ds \int_{x+t\omega\in\Omega} \frac{|f(x+s\omega) - f(x+t\omega)|^p}{|s-t|^{1+ps}} \, dt \\
&\geq \frac{1}{2}\omega_{n-1} D_{1,p,ps} \int_{\mathbb{S}^{n-1}} d\omega \int_{x:\, x\cdot\omega=0} d\mathcal{L}_\omega(x) \int_{x+l\omega\in\Omega} \frac{|f(x+l\omega)|^p}{\delta_\omega(x+l\omega)^{ps}} \, dl \\
&= \frac{1}{2}\omega_{n-1} D_{1,p,ps} \int_{\mathbb{S}^{n-1}} d\omega \int_\Omega \frac{|f(x)|^p}{\delta_\omega(x)^{ps}} \, dx \\
&= D_{n,p,ps} \int_\Omega \frac{|f(x)|^p}{M_{s,p}(x)^{ps}} \, dx,
\end{aligned}
$$

since

$$
\frac{\omega_{n-1}}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1+ps}{2})}{\sqrt{\pi}\,\Gamma(\frac{n+ps}{2})} D_{1,p,ps} = D_{n,p,ps}.
$$

In the case $\Omega = \mathbb{R}^n_+$, the constant $D_{n,p,ps}$ was proved to be best possible in [22] for $p = 2$ and in [83] for the other values of p, by constructing a sequence of trial functions. For a general convex Ω, these trial functions are transplanted to Ω near a tangent hyperplane, following the proof of Theorem 5 in [132].

The theorem is therefore proved. □

Remark 6.11

In the case $p = 2$, it is proved in the appendix of [22] that

$$
D_{1,2,2s} = \frac{1}{s}\left\{\frac{2^{-2s}}{\sqrt{\pi}}\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1-s) - 1\right\}. \tag{6.2.19}
$$

Also, for $p = 2$ an improvement of (6.2.2) is established in [130], Theorem 1.1, namely,

$$
\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \geq 2\kappa_{n,2s} \int_\Omega \frac{|f(x)|^2}{M_{s,2}(x)^{2s}} \, dx, \tag{6.2.20}
$$

where $\kappa_{n,2s}$ is the sharp constant

$$
\kappa_{n,2s} := \frac{\pi^{(n-1)/2}\Gamma\left(\frac{1+ps}{2}\right)}{\Gamma\left(\frac{n+ps}{2}\right)} \frac{1}{s}\left\{\frac{2^{-2s}}{\sqrt{\pi}}\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1-s) - \frac{1}{2}\right\}. \tag{6.2.21}
$$

This refinement is achieved through the use of the one-dimensional inequality in [130], Theorem 2.1, to replace Lemma 6.9 above, that for $1/2 < s < 1$ and $f \in C_0^\infty(a,b)$,

$$\int_a^b \int_a^b \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}}\, dx\, dy \geq 2\kappa_{1,2s} \int_a^b |f(x)|^2 \left(\frac{1}{x-a} + \frac{1}{b-x} \right)^{2s} dx, \tag{6.2.22}$$

which has the corollary that for any open set $J \subset \mathbb{R}$, $1/2 < s < 1$ and $f \in C_0^\infty(J)$,

$$\int_J \int_J \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}}\, dx\, dy \geq 2\kappa_{1,2s} \int_J |f(x)|^2 \left(\frac{1}{\delta_J(x)} + \frac{1}{d_J(x)} \right)^{2s} dx, \tag{6.2.23}$$

where J is a countable union of disjoint intervals I_k and for $x \in J$, $d_J(x) = d_{I_k}(x) = \sup\{|t| : x + t \notin I_k\}$, where I_k is the unique interval containing x.

By the Sobolev inequality, the left-hand side of (6.2.2) dominates the L_q norm of f for $q = np/(n-ps)$. Dyda and Frank prove in [57] that this remains true even if the right-hand side of (6.2.2) is subtracted from the left; their result is the fractional Hardy–Sobolev–Maz'ya inequality in

Theorem 6.12 *Let $n \geq 2$, $2 \leq p < \infty$, $0 < s < 1$ and $1 < ps < n$. Then there exists a constant $k_{n,p,s} > 0$ such that for $q = np/(n-ps)$,*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}}\, dx\, dy - D_{n,p,ps} \int_\Omega \frac{|f(x)|^p}{M_{s,p}(x)^{ps}}\, dx \geq k_{n,p,s} \left(\int_\Omega |f(x)|^q dx \right)^{p/q} \tag{6.2.24}$$

for all open $\Omega \subsetneq \mathbb{R}^n$ and all $f \in \overset{0}{W}{}_p^s(\Omega)$.

This is the fractional analogue of (5.5.6). In the case of $p = 2$ and $\Omega = \mathbb{R}^n_+$, a proof of (6.2.24) was given in [160]. A variant of (6.2.24) for a half-space, and more general John domains, is given in [59]; for $\mathbb{R}^n_+ = \{x : x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ x_n > 0\}$, the integral on the right-hand side has a weight x_n^{-bq}, where $b = n(1/q - 1/p) + s$.

6.3 Fractional Hardy Inequality with a Remainder Term

Dyda [56] proved the following refinement of (6.2.20):

Theorem 6.13 *Let $1/2 < s < 1$ and Ω a bounded domain in $\mathbb{R}^n$. Then for all $u \in C_0^\infty(\Omega)$,*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geq \kappa_{n,2s}\int_\Omega \frac{|f(x)|^2}{M_{s,2}(x)^{2s}}\,dx + \frac{\lambda_{n,ps}}{\text{diam }(\Omega)}\int_\Omega \frac{|u(x)|^2}{M_{s-1/2}(x)^{2s-1}}\,dx, \tag{6.3.1}$$

where

$$\lambda_{n,ps} = \pi^{(n-1)/2}\Gamma\left(\frac{2s}{2}\right)\frac{4-2^{3-2s}}{2s\Gamma\left(\frac{n+2s-1}{2}\right)}. \tag{6.3.2}$$

The constant $\kappa_{n,2s}$ *cannot be replaced by a larger constant in (6.13).*

The proof in [56] is based on the method developed in [130] to prove (6.2.20), but with the inequality in the following proposition used instead of (6.2.22).

Proposition 6.14 *Let* $1 < \alpha < 2$ *and* $-\infty < a < b < \infty$. *Then for all* $u \in C_0^\infty(a,b)$,

$$\frac{1}{2}\int_a^b\int_a^b \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy \geq \kappa_{1,\alpha}\int_a^b u(x)^2\left(\frac{1}{x-a}+\frac{1}{b-x}\right)^{2s}dx + \frac{4-2^{3-\alpha}}{\alpha(b-a)}\int_a^b u(x)^2\left(\frac{1}{x-a}+\frac{1}{b-x}\right)^{\alpha-1}dx. \tag{6.3.3}$$

The constant $\kappa_{1,\alpha}$ *cannot be replaced by a larger one.*

Proof An important first step is the calculation of

$$Lu(x) := \lim_{\varepsilon\to 0+}\int_{(-1,1)\cap\{y:\,|x-y|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{1+\alpha}}\,dy.$$

Let $q > -1,\ 0 < \alpha < 2$ and $u_q(x) = (1-x^2)^q$. Then on setting $t = y^2$ and integration by parts

$$\begin{aligned}
Lu_q(0) &= 2\lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{(1-y^2)^q-1}{y^{1+\alpha}}\,dy\\
&= 2\left(\frac{1}{2}\int_{\varepsilon^2}^1 (1-t)^q t^{-1-\alpha/2}[(1-t)+t]\,dt - \int_\varepsilon^1 y^{-1-\alpha}\,dy\right)\\
&= 2\lim_{\varepsilon\to 0+}\left(\frac{1}{\alpha}(1-\varepsilon^2)^{q+1}\varepsilon^{-\alpha} - \frac{q+1}{\alpha}\int_{\varepsilon^2}(1-t)^q t^{-\alpha/2}\,dt\right)\\
&\quad + 2\lim_{\varepsilon\to 0+}\left(\frac{1}{2}\int_{\varepsilon^2}(1-t)^q t^{-\alpha/2}\,dt + \frac{1}{\alpha} - \frac{\varepsilon^{-\alpha}}{\alpha}\right)\\
&= \frac{2}{\alpha}[1-(q+1-\alpha/2)B(q+1,1-\alpha/2)]
\end{aligned}$$

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since

$$\lim_{\varepsilon\to 0+}\left(\frac{1}{\alpha}(1-\varepsilon^2)^{q+1}\varepsilon^{-\alpha}-\frac{\varepsilon^{-\alpha}}{\alpha}\right)=\lim_{\varepsilon\to 0+}\frac{\varepsilon^{2-\alpha}}{\alpha}\left(\frac{(1-\varepsilon^2)^{q+1}-1}{\varepsilon^2}\right)=0;$$

B denotes the Euler beta function: $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(1/2)}$.

For $x_0\in(-1,1)$,

$$Lu_q(x_0)=pv\int_{-1}^{1}\frac{(1-y^2)^q-(1-x_0^2)^q}{|y-x_0|^{1+\alpha}}\,dy$$

where pv denotes the principal value. The variable change

$$t=\varphi(y)=\frac{x_0-y}{1-x_0y}$$

transforms Lu_p into

$$\begin{aligned}Lu_q(x_0)&=(1-x_0^2)^{q-\alpha}\int_{-1}^{1}\frac{(1-t^2)^q-(1-tx_0)^{2q}}{|t|^{1+\alpha}}(1-tx_0)^{\alpha-1-2q}\,dt\\&=(1-x_0^2)^{q-\alpha}\left[Lu_q(0)-pv\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{|t|^{1+\alpha}}\,dt\right]\\&\quad+(1-x_0^2)^{q-\alpha}\left[pv\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1-2q}-1}{|t|^{1+\alpha}}(1-t^2)^q\,dt\right].\end{aligned}$$

Let

$$I:=pv\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{|t|^{1+\alpha}}\,dt=\lim_{\varepsilon\to 0+}\left(J_\varepsilon(x_0)+J_\varepsilon(-x_0)\right),$$

where

$$\begin{aligned}J_\varepsilon(x_0)&=\int_{\varepsilon}^{1}\frac{(1-tx_0)^{\alpha-1}-1}{|t|^{1+\alpha}}\,dt=\int_{\varepsilon}^{1}\left(\frac{1}{t}-x_0\right)^{\alpha-1}\frac{dt}{t^2}=\frac{\varepsilon^{-\alpha}-1}{\alpha}\\&=\frac{1}{\alpha}\left(\frac{1}{\varepsilon}-x_0\right)^{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}-\frac{\varepsilon^{-\alpha}-1}{\alpha}\\&=\frac{1}{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}+\frac{(1-\varepsilon x_0)^{\alpha}-1}{\alpha\varepsilon^{\alpha}}.\end{aligned}$$

By l'Hôpital's rule

$$I=\frac{2}{\alpha}-\frac{1}{\alpha}(1-x_0)^{\alpha}-\frac{1}{\alpha}(1+x_0)^{\alpha}$$

and so

$$\begin{aligned}Lu_q(x)&=\frac{(1-x^2)^{q-\alpha}}{\alpha}\left\{(1-x)^{\alpha}+(1+x^{\alpha})\right\}\\&\quad-\frac{(1-x^2)^{q-\alpha}}{\alpha}\left\{(2q+2-\alpha)B(q+1,1-\alpha/2)+\alpha I(q)\right\},\qquad(6.3.4)\end{aligned}$$

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where

$$I(q) := pv\int_{-1}^{1}\frac{(1-tx_0)^{\alpha-1-2q}-1}{|t|^{1+\alpha}}(1-t^2)^q dt.$$

We also have

$$\begin{aligned}
I(\alpha/2) &= pv\int_{-1}^{1}\frac{(1-tx)^{-1}-1}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}dt\\
&= \int_{-1}^{1}\frac{\sum_{k=2}^{\infty}(tx)^k}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}dt\\
&= 2\int_{-1}^{1}\frac{\sum_{k=2}^{\infty}(tx)^k}{|t|^{1+\alpha}}(1-t^2)^{\alpha/2}dt\\
&= \sum_{k=1}^{\infty}B(k-\alpha/2,1+\alpha/2)x^{2k}\\
&= \Gamma(1+\alpha/2)\Gamma(-\alpha/2)\left(\sum_{k=0}^{\infty}\frac{x^{2k}\Gamma(k-\alpha/2)}{\Gamma(-\alpha/2)k!}-1\right)\\
&= \frac{2B(1+\alpha/2,1-\alpha/2)}{\alpha}(1-(1-x^2)^{\alpha/2}).
\end{aligned}$$

It can also be shown that $I(\frac{\alpha-1}{2}) = I(\frac{\alpha-2}{2}) = 0$ and, if $1 < \alpha < 2$, that $I(\frac{\alpha-3}{2}) = x^2B(\frac{\alpha-1}{2}, 1-\frac{\alpha}{2})$.

The next step in the proof of the proposition is the application of a result which is analogous to the ground state representation for half-spaces and $\mathbb{R}^n \setminus \{0\}$ in [82] and [83], and may be considered as a special case of Proposition 2.3 in [82]. The result is that with $0 < \alpha < 2$, $w(x) = (1-x^2)^{(\alpha-1)/2}$ and $u \in C_0((-1,1))$,

$$\begin{aligned}
&\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}}\,dx\,dy\\
&\quad= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{u(x)}{w(x)}-\frac{u(y)}{w(y)}\right)^2\frac{w(x)w(y)}{|x-y|^{1+\alpha}}\,dx\,dy\\
&\quad+ 2^{\alpha}\kappa_{1,\alpha}\int_{-1}^{1}u(x)^2(1-x^2)^{-\alpha}\,dx\\
&\quad+ \frac{1}{\alpha}\int_{-1}^{1}u(x)^2\left[2^{\alpha}-(1+x)^{\alpha}-(1-x)^{\alpha}\right](1-x^2)^{-\alpha}\,dx. \qquad (6.3.5)
\end{aligned}$$

We are now equipped to complete the proof of Proposition 6.14. By scaling we may and shall assume that $a=-1$, $b=1$. By (6.3.5), we require that

$$2^{\alpha}-(1+x)^{\alpha}-(1-x)^{\alpha} \geq (2^{\alpha}-2)(1-x^2),\ 1\leq\alpha\leq 2,\ 0\leq x\leq 1. \quad (6.3.6)$$

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On substituting $u = x^2$, it suffices to prove that

$$g(u) := (2^\alpha - 2)u - (1 - \sqrt{u})^\alpha - (1 + \sqrt{u})^\alpha + 2$$

is concave, or

$$g'(u) = 2^\alpha - 2 + \frac{\alpha}{2\sqrt{u}}\left((1 - \sqrt{u})^{\alpha-1} - (1 + \sqrt{u})^{\alpha-1}\right)$$

is decreasing. Setting $u = t^2,\ h(t) = (1 - t)^{\alpha-1} - (1 + t)^{\alpha-1}$, we have that

$$\frac{(1 - t)^{\alpha-1} - (1 + t)^{\alpha-1}}{t} = \frac{h(t) - h(0)}{t}.$$

Since h is concave, the function $t \mapsto \frac{h(t)-h(0)}{t}$ is decreasing and hence so is g'. Therefore (6.3.6) is proved and the proposition follows. The sharpness of $\kappa_{1,\alpha}$ is already established in [130]. □

Proof of Theorem 6.13 This follows by using Proposition 6.14 with $\alpha = 2s$ instead of (6.2.22) in the proof of the case $p = 2$ of Theorem 6.8.