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ON THE ZEROES OF PROFILES

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Abstract

The k-profile of an Hadamard matrix of order n is a function defined on the integers $0, 1, \dots, n$. If k is even, k-profiles have been used in investigations of Hadamard equivalence. In this paper it is shown that the k-profile of an Hadamard matrix of order n (k even) has non-zero terms only in every eighth position. If k is divisible by 4, the non-zero positions are those congruent to n (modulo 8).

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1. Introduction

A general discussion of Hadamard matrices can be found in various places, such as [2], [4], [5]. We defined the *profile*, or 4-*profile*, in [1]. Given an Hadamard matrix $H = (h_{ij})$, define

$$P_{ijkl} = \left| \sum_{\alpha} h_{i\alpha} h_{j\alpha} h_{k\alpha} h_{l\alpha} \right|.$$

and define $\pi(t)$ to equal the number of 4-sets $\{i, j, k, l\}$ such that $P_{ijkl} = t$. The function π is the 4-profile of H. More generally one can define 6-profiles, 8-profiles, and k-profiles for any larger even integer k. (The case of odd k, and the case $k = 2$, are not interesting.) These profiles have been used in studying Hadamard equivalence (see [1], [3]).

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It is easy to show (see [1]) that every P_{ijkl} is congruent to n modulo 8; so, for a 4-profile of an $n \times n$ Hadamard matrix, $\pi(t) = 0$ unless $t \equiv n$ (mod 8). Empirical evidence suggests a similar result for k-profiles in general (only one in eight of the integers give a non-zero value of the function) but clearly the proof in [1] does not generalize to k greater than 4; it depends on solving a set of equations, and the number of variables grows more quickly than the number of equations when k increases. Moreover the behavior of the profile function is not easy to predict: the 6-profile of an Hadamard matrix of order 20 has its non-zero entries at $t = 0$, 8 and 16 (not $t \equiv n$), but the 6-profiles of Hadamard matrices of order 24 have zero entries for $t = 0, 8, 16$ and 24 ($t \equiv n$).

Our aim in this paper is to clarify this aspect of the behavior of profiles.

2. Binomial coefficients

We need some well-known preliminary results on binomial coefficients. Proofs are included for completeness; various alternative proofs are known.

LEMMA 1. *For any positive integer t,*

$$\binom{4t}{0} + \binom{4t}{4} + \binom{4t}{8} + \cdots + \binom{4t}{4t}$$

is even.

PROOF. Since $\binom{4t}{0} = \binom{4t}{4t}$, $\binom{4t}{4} = \binom{4t}{4t-4}$, and so on, we can pair off the terms in the sum with equals. When t is odd, this process exhausts the terms, so the sum is even. When t is even, $\binom{4t}{2t}$ is left unpaired; but

$$\binom{4t}{2t} = \binom{4t-1}{2t-1} + \binom{4t-1}{2t} = \binom{4t-1}{2t-1} + \binom{4t-1}{2t-1}$$

which is even.

LEMMA 2. *If t is any positive integer, then*

$$1 + \binom{4t+2}{4} + \binom{4t+2}{8} + \cdots + \binom{4t+2}{4t}$$

is even.

PROOF. Consider

$$S = \binom{4t+2}{0} + \binom{4t+2}{2} + \binom{4t+2}{4} + \cdots + \binom{4t+2}{4t+2}.$$

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Using the rule

$$\binom{4t+2}{i} = \binom{4t+1}{i} + \binom{4t+1}{i-1}$$

we see that

$$S = 1 + \left[\binom{4t+1}{1} + \binom{4t+1}{2}\right] + \left[\binom{4t+1}{3} + \binom{4t+1}{4}\right] + \cdots + \binom{4t+1}{4t+1} = 2^{4t+1}.$$

This is divisible by 4. Now

$$S = \binom{4t+2}{0} + \binom{4t+2}{4} + \cdots + \binom{4t+2}{4t} + \binom{4t+2}{4t+2} + \binom{4t+2}{4t-2} + \cdots + \binom{4t+2}{2}$$

which equals twice the required sum.

3. Symmetric functions

Suppose the m variables $x_1, x_2, \ldots, x_m$ each satisfy $x_i^2 = 1$. Write $X = \{x_1, x_2, \ldots, x_m\}$, and define σ_j to be the jth symmetric function on X:

$$\sigma_j = \sum_Y \prod_{x \in Y} x,$$

where Y ranges through the distinct j=subsets of X. In particular $\sigma_0 = 1$. Then σ_j is the sum of $n_j = \binom{m}{j}$ terms.

LEMMA 3. $\sigma_1\sigma_j = (j+1)\sigma_{j+1} + (m-j+1)\sigma_{j-1}$.

PROOF. Since σ_j contains $\binom{m}{j}$ terms, so does $x_i\sigma_j$; of these $\binom{m-1}{j}$ contain no repeated terms and $\binom{m-1}{j-1}$ contain an x_i^2. So $x_i\sigma_j$ contains $\binom{m-1}{j}$ terms of length $j+1$ and $\binom{m-1}{j-1}$ terms of length $j-1$. Consequently $\sigma_1\sigma_j$ contains $m\binom{m-1}{j}$ terms of length $j+1$ and $m\binom{m-1}{j-1}$ terms of length $j-1$. By symmetry it is clear that $\sigma_1\sigma_j = A\sigma_{j+1} + B\sigma_{j-1}$ for some integers A and B; therefore $An_{j+1} = m\binom{m-1}{j}$, $Bn_{j-1} = m\binom{m-1}{j-1}$, and it follows that $A = j+1$, $B = m-j+1$, giving the lemma.

COROLLARY. *If x_0 is another variable satisfying $x_0^2 = 1$, then*

$$(x_0 + x_1)(x_0 + x_2)\cdots(x_0 + x_m)(x_1 + x_2 + \cdots + x_m) = m(\sigma_0 + x_0\sigma_1 + \sigma_2 + x_0\sigma_3 + \cdots + x_0^m\sigma_m).$$

PROOF. The left-hand side equals $(x_0^m + x_0^{m-1}\sigma_1 + x_0^{m-2}\sigma_2 + \cdots + \sigma_m)\sigma_1$. The result now follows from repeated application of Lemma 3.

4. Profiles

THEOREM. *Suppose H is an Hadamard matrix of side n, $n \geq 4$, and suppose k is even. Then the generalized inner product of k rows,*

$$P_{i_1 i_2 \cdots i_k} = \sum_{j=1}^{n} h_{i_1 j} h_{i_2 j} \cdots h_{i_k j},$$

is congruent to n modulo 8 when 4 divides k, and is congruent to 0 modulo 8 when k is congruent to 2 modulo 4.

PROOF. We write $X_j = \{h_{i_2 j}, h_{i_3 j}, \ldots h_{i_k j}\}$, and denote the ith symmetric function on X_j by σ_{ij}.

Assume $k \geq 4$; consider

$$Q_{jk} = (h_{i_1 j} + h_{i_2 j})(h_{i_1 j} + h_{i_3 j}) \cdots (h_{i_1 j} + h_{i_k j})(h_{i_2 j} + h_{i_3 j} + \cdots + h_{i_k j}).$$

As $k \geq 4$, Q_{jk} is divisible by 8. On the other hand, by the Corollary to Lemma 3,

$$Q_{jk} = (k-1)(\sigma_{0j} + h_{i_1 j}\sigma_{1j} + \sigma_{2j} + h_{i_1 j}\sigma_{3j} + \cdots + h_{i_1 j}\sigma_{k-1,j}).$$

Now

$$\sum_{j=1}^{n} Q_{jk} = (k-1)\left(n + \sum h_{i_1 j}\sigma_{ij} + \sum \sigma_{2j} + \cdots + \sum h_{i_1 j}\sigma_{k-1,j}\right).$$

We now reduce modulo 8. Since each Q_{jk} is zero (mod 8), the sum is zero. Since n is the order of an Hadamard matrix and $n \geq 4$, n is divisible by 4, so $n \equiv -n$ (mod 8). Finally $k-1$ is odd. So

$$\text{(1)} \qquad \sum h_{i_1 j}\sigma_{1j} + \sum \sigma_{2j} + \cdots + \sum h_{i_1 j}\sigma_{k-1,j} \equiv n \pmod 8.$$

We now write $f(k)$ for the residue class of $P_{i_1 i_2 \cdots i_k}$ (mod 8), and define $f(0) \equiv n$. We prove by induction that $f(k)$ is well defined (in other words, the residue class depends only on k), and that

$$\text{(2)} \qquad \sum_{2|\alpha} \binom{k}{\alpha} f(\alpha) \equiv 0 \pmod 8.$$

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When $k = 0$ the result is obvious. In general, if $2t < k$, $\sum h_{i,j}\sigma_{2t-1,j} + \sum \sigma_{2t,j}$ is the sum of $n_{2t-1} + n_{2t}$ terms, each of which equals an inner product of $2t$ rows, so by the hypothesis that $f(2t)$ is well defined the sum is congruent to $(n_{2t-1} + n_{2t})f(2t)$; and

$$n_{2t-1} + n_{2t} = \binom{k-1}{2t-1} + \binom{k-1}{2t} = \binom{k}{2t}.$$

Moreover $P_{i_1 i_2 \cdots i_k} = \sum h_{i,j}\sigma_{k-1,j}$. So (2) yields

$$\binom{k}{2} f(2) + \binom{k}{4} f(4) + \cdots + f(k) \equiv n \pmod 8,$$

which becomes (1) when we observe that $\binom{k}{0} f(0) \equiv n \equiv -n \pmod 8$.

We now show, by induction again, that $f(k) \equiv n$ when 4 divides k and is congruent to 0 otherwise. Suppose this is true of $f(\alpha)$ for all values of α less than k. Then

$$-f(k) \equiv \sum \binom{k}{\alpha} f(\alpha) \equiv \left[\sum\nolimits_1 \binom{k}{\alpha}\right] \cdot n + \left[\sum\nolimits_2 \binom{k}{\alpha}\right] \cdot 0$$

where $\sum_1$ is over α divisible by 4 and $\sum_2$ over α congruent to $2 \pmod 4$, $0 \le \alpha < k$.

If 4 divides k then by Lemma 1, $\sum_1 \binom{4}{\alpha} + \binom{k}{k}$ is even, so $\sum_1 \binom{k}{\alpha}$ is odd and the expression is congruent to n modulo 8. If 4 does not divide k then $\sum_1 \binom{k}{\alpha}$ is even by Lemma 2 and the expression is congruent to zero. The other bracketed term is zero. Since $-n \equiv n \pmod 8$, the negative sign is irrelevant.

COROLLARY. *If* $k \equiv 0 \pmod 4$ *then the k-profile of an Hadamard matrix has non-zero terms only in positions congruent to* $n \pmod 8$. *If* $k \equiv 2 \pmod 4$ *then the k-profile of an Hadamard matrix has non-zero terms only in positions congruent to* $0 \pmod 8$.

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